Course File
2022-23

ADVANCED SOLID MECHANICS

(GR22D5002)

M.Tech. (Structural Enginerring)

I Year -I Semester

Instructor: Dr. V Srinivasa Reddy

Department of Civil Engineering

GOKARAJU RANGARAJU
Institute of Engineering and Technology

Vision and Mission

Gokaraju Rangaraju Institute of Engineering and Technology (GRIET) is established in 1997 by Dr. G Gangaraju as a self-financed institute under the aegis of Gokaraju Rangaraju Educational Society. GRIET is approved by AICTE, New Delhi, permanently affiliated to and autonomous under JNTUH, Hyderabad. GRIET is committed to quality education and is known for its innovative teaching practices.

Vision
To be among the best of the institutions for engineers and technologists with attitudes, skills and knowledge and to become an epicentre of creative solutions.

Mission
To achieve and impart quality education with an emphasis on practical skills and social relevance.

Department of Civil Engineering

Vision
To become a pioneering centre in Civil Engineering and technology with attitudes, skills and knowledge.

Mission
- To produce well qualified and talented engineers by imparting quality education.
- To enhance the skills of entrepreneurship, innovativeness, management and lifelong learning in young engineers.
- To inculcate professional ethics and make socially responsible engineers.

M.Tech PEOs and POs

M.Tech Programme Educational Objectives (PEOs)

PEO 1: Graduates of the program will equip with professional expertise on the theories, process, methods and techniques for building high-quality structures in a cost-effective manner.

PEO 2: Graduates of the program will be able to design structural components using contemporary softwares and professional tools with quality practices of international standards.

PEO 3: Graduates of the program will be effective as both an individual contributor and a member of a development team with professional, ethical and social responsibilities.

PEO 4: Graduates of the program will grow professionally through continuing education, training, research, and adapting to the rapidly changing technological trends globally in structural engineering.

M.Tech Programme Outcomes (POs)

Graduates of the Civil Engineering program will be able to:

PO 1: An ability to independently carry out research / investigation and development to solve practical problems.

PO2: An ability to write and present a substantial technical report / document.

PO 3: Students should be able to demonstrate a degree of mastery over the area as per the specialization of the program. The mastery should be at a level higher than the requirements in the appropriate bachelor's.

PO 4: Possesses critical thinking skills and solves core, complex and multidisciplinary structural engineering problems.

PO 5: Assess the impact of professional engineering solutions in an environmental context along with societal, health, safety, legal, ethical and cultural issues and the need for sustainable development.

PO 6: Recognize the need for life-long learning to improve knowledge and competence.

COURSE FILE Enclosures

The following are to be filed in each Course File:

1. Get a new file from college store for each course and file each sheet of these formats as and when it is completed.
2. Time Table
3. Syllabus copy for your course.
4. Course Plan
5. Unit Plan and
6. Lesson Plan
7. List of Program Objectives & Outcomes;
8. Course Objectives & Outcomes
9. List of various Mappings/Matrix for your Course
   a. Mapping between Course Objectives and Course Outcomes
   b. Mapping between Course Objectives and Program Outcomes(POs)
   c. Mapping between Course Outcomes and Mandatory/Program Outcomes(POs)(a–k)
   d. Mapping between Courses with titles & codes and Mandatory/Program Outcomes(POs)(a – k)
   e. Mapping between the PEOs and Course Outcomes
   f. Mapping between POs and Assignments and Assessments Methods
   g. Mapping between the Assessment Methods and PEOs
10. List of Assessments, Assignments/Seminar Topics, Projects, Experiments, etc. you have given to students and the Criteria used for evaluation
11. Assignment sheets,
12. Tutorial Sheets, and
13. Course Schedules
14. At least 1 to 3 Assessment Rubrics for your course
15. Evaluation Strategy
16. Guidelines to study the course
17. Students Roll list
18. Attach the Marks list of the students in respect of CAE -I (Continuous Assessment Exam), CAE-II, etc. and Final Exam for this Course in your course File.
19. Photocopy of the best, average and the worst answer sheets for CAE-I, & CAE-II be included in the Course File.
20. Model question papers if any, which you have distributed to the students in the beginning of the Semester for the Course may be included in the Course File.
21. Any Teaching/Learning Aids, additional resources like OHP transparencies, LCD Projection material, Soft & Hard Copies of handouts used may also be filed in it.
22. Course Completion Status
23. Grading Sheet of the Course for all students

Assessment Procedure

S. No	Component of Assessment	Marks Allotted	Type of Assessment	Scheme of Examinations
1	Theory	40	Internal Examination & Continuous Evaluation	1) Two mid semester examination shall be conducted for 30 marks each for a duration of 120 minutes. Average of the two mid exams shall be considered **i) Subjective – 20 marks** **ii) Objective – 10 marks** 2) Continuous Evaluation is by conducting Assignments and Quiz exams at the end of each unit **i) Assignment – 5 marks** **ii) Quiz/Subject Viva-voce/PPT/Poster Presentation/ Case Study on a topic in the concerned subject – 5 marks**
		60	Semester end examination	The semester-end examination is for a duration of 3 hours

I YEAR - I SEMESTER

Sl. No	Group	Course Code	Subject	Credits L	Credits T	Credits P	Total Credits	Hours L	Hours T	Hours P	Total Hours	Int. Marks	Ext. Marks	Total Marks
1	PC	GR22D5001	Matrix methods in structural analysis	3	0	0	3	3	0	0	3	40	60	100
2	PC	GR22D5002	Advanced Solid Mechanics	3	0	0	3	3	0	0	3	40	60	100
3	PE I	GR22D5003	1.Theory and Application ofCement Composites	3	0	0	3	3	0	0	3	40	60	100
		GR22D5004	2.Advanced Concrete Technology											
		GR22D5005	3. Theory of Structural Stability											
4	PE II	GR22D5006	1. Analytical and Numerical Methods for Structural Engineering	3	0	0	3	3	0	0	3	40	60	100
		GR22D5007	2.Structural Health Monitoring											
		GR22D5008	3. Structural Optimization											
5	PC	GR22D5009	Structural Design Lab	0	0	2	2	0	0	4	4	40	60	100
6	PC	GR22D5010	Advanced Concrete Technology Lab	0	0	2	2	0	0	4	4	40	60	100
7	PC	GR22D5011	Research Methodology and IPR	2	0	0	2	2	0	0	2	40	60	100
			Total	**14**	**0**	**4**	**18**	**14**	**0**	**8**	**22**	**280**	**420**	**700**
8	AC		Audit Course I	0	0	0	0	2	0	0	2	40	60	100

M.Tech regular students With effect from the academic year 2022-23
GR22 Regulations

The performance of a student in every **subject/course** (including **practicals** and **ProjectStage – I & II**) will be evaluated for 100 marks each, with

- 40 marks allotted for CIE (Continuous Internal Evaluation) and
- 60 marks for SEE (Semester End-Examination).

Theory Courses

In CIE, for theory courses, during a semester, there shall be two mid-term examinations. Each Mid-Term examination consists of two parts

i) Part – A for 10 marks,

Objective/quiz paper for 10 marks. (The objective/quiz paper is set with **multiple choice, fill-in the blanks** and match the following type of questions for a total of 10 marks).

4 bits from Unit-I, 4 bits from Unit-II and 2 bits from Unit-III

ii) **Part – B for 20 marks** with a total duration of 2 hours as follows:

Descriptive paper for 20 marks (The descriptive paper shall **contain 6 full questions** out of which, the student has to **answer 4 questions**, **each carrying 5 marks**.)

2 questions from Unit-I, 2 questions from Unit-II, 1 question from Unit-I and III and 1 question from Unit-II and III.

iii) The remaining 10 marks of Continuous Internal Evaluation are distributed as

- **Assignment for 5 marks**. (Average of all Assignments each for 5 marks)
- **Subject Viva-Voce/PPT/Poster Presentation/ Case Study** on a topic in the concerned subject for **5 marks**.

Mid Term Examination for 30 marks, Assignment for 5 marks and 5 marks for Subject Viva-Voce/PPT/Poster Presentation/ Case Study

In each subject, shall have to earn

✓ 40% of marks (i.e. **16 marks out of 40 marks** in Continuous **Internal**

Evaluation,

- ✓ 40% of marks (i.e. **24 marks out of 60**) in Semester **End-Examination** and
- ✓ Over all 50% of marks (i.e. **50 marks out of 100 marks**) both Continuous Internal Evaluation and Semester **End-Examination** marks put together.

The **Semester End Examinations** (SEE), for theory subjects, will be conducted for **60** marks consisting of two parts viz.

i) **Part- A for 10 marks** (Part-A is a compulsory question which **consists of ten sub-questions** from all units carrying **equal marks**)

ii) **Part - B for 50 marks** (Part-B consists of **five questions** (numbered from 2 to 6) carrying **10 marks** each. Each of these questions is **from each unit** and may contain sub-questions. For each question there will be an "either" "or" choice, which means that **there will be two questions from each unit and the student should answer either of the two questions)**

Practical Courses

For Practical courses there shall be a Continuous Internal Evaluation (CIE) during the semester for **40 marks and 60 marks** for semester end examination.

The 40 marks for internal evaluation:

i) Internal Exam-10 marks
ii) Viva voce – 10 marks
iii) Continuous Assessment- 10 marks
iv) G-Lab on Board(G-LOB) (Case study inter threading of all experiments of lab)/ Laboratory Project/Prototype Presentation/App Development -10 marks

Semester End Examination shall be conducted with an external examiner and the laboratory teacher. The external examiner shall be appointed from the cluster / other colleges which will be decided by the examination branch of the University.

In the Semester End Examination held for **3 hours, total 60 marks** are divided and allocated as shown below:

i) write-up (algorithm/flowchart/procedure) as per the task/experiment/program - 10 marks
ii) task/experiment/program-15 marks

iii) evaluation of results -15 marks

iv) write-up (algorithm/flowchart/procedure) for another task/experiment/program-10 marks

v) viva-voce on concerned laboratory course - 10 marks

S.No	No:	Student Name (As Per SSC)	Student Phone	Email	Phone
1	22241D2001	ADDAGATLA MAHESHKUMAR	9652205718	maheaddagatla@gmail.com	9652205718
2	22241D2002	AHMED ABDUL AZEEM	9553214459	abdulazeem17458@gmail.com	9948123715
3	22241D2003	BAIRAPAKA BHARATH	9010976868	Bairapakabharath7@gmail.com	9182443387
4	22241D2004	BARLAPUDI ACHSAH KEERTHAI	6302131589	achsahkeerthana.b@gmail.com	9553242425
5	22241D2005	CHAKALI SOWMYA	6300048204	Ch.sowmya.1311@gmail.com	9032366043
6	22241D2006	CHAPPIDI NARESH	9398916604	Chappidi.naresh88@gmail.com	9398993443
7	22241D2007	DANTHALA HARIDEEPKUMAR	6303321256	harideepdanthala@gmail.com	9618714550
8	22241D2008	DEVIREDDY ANISH	6309845262	anishdevireddy07@gmail.com	8179118516
9	22241D2009	DHARAVATH NAGENDAR	7673952028	nagendar.d99@gmail.com	8919995124
10	22241D2010	GANGAPURAM SUSHANTH REDI	9502059919	shushanthshush@gmail.com	9440054520
11	22241D2011	JEREPOTHULA RAVALIKA	9676681445	ravalikajerepothula@gmail.com	9346496095
12	22241D2012	KADABOHINA SAIPAVAN	9030300863	kadabohinasaipavan4536@gmail.com	9966358815
13	22241D2013	KASUMURU BHARATH KUMAR	9494066112	Bharathkumarkasumuru@gmail.com	9105222000
14	22241D2014	MACHARLA SRINIVAS	9959766792	macharlasrinivas111@gmail.com	9959766792
15	22241D2015	MALLI SREENIVASULU	6309432349	mallisreenu145@gmail.com	7075081569
16	22241D2016	SHAIK ABDUL MUQEED	7569656490	abdulmuqeed321@gmail.com	9515323031
17	22241D2017	SHAIK ZABI ULLAH	9640330682	shaikzabiullah2000@gmail.com	9849493634
18	22241D2018	SONWANE SAHIL SHIVAJIRAO	8328109850	sahilsss29@gmail.com	9440783546
19	22241D2019	LINGAM LAKSHMI NARAYANA	9392138942	lingam_ln@yahoo.com	9392138942

Gokaraju Rangaraju Institute of Engineering & Technology

Bachupally, Hyderabad-500090

M.Tech Structural Engg. I Yr-I Sem- GR20 2021 -22

S.No	Reg No	Student Name
1	21241D2001	ATKAPURAM PRASHANTH
2	21241D2002	BANDI SRI RAM GOPAL
3	21241D2003	CHALLA MADHAVI
4	21241D2004	PAMMI DIVYA
5	21241D2005	DUMMA UMESH KUMAR
6	21241D2006	K LATHASREE
7	21241D2007	MARIYALA VAISHNAVI
8	21241D2008	MAVOORI PRANAV
9	21241D2009	MITTAPALLI NAGA ASHWINI
10	21241D2010	R VENKATA SURAJ REDDY
11	21241D2011	REPATI MOHAN BABU
12	21241D2012	SANDHYA CHERUKU
13	21241D2013	SHAIK FEROZ
14	21241D2014	SK SAI CHANDRA
15	21241D2015	THOTA HARSHAVARDHAN
16	21241D2016	VARIKUPPALA LALITHA
17	21241D2017	Y RAMA GNANENDRA SAI
18	21241D2018	YENUMALA DEVESH GOUD
19	21241D2019	S PRASHANTH KUMAR
20	21241D2020	B THARUN TEJA
21	21241D2021	G NITISH KUMAR

22241D2001	mahesh22241d2001@grietcollege.com
22241D2002	abdul22241d2002@grietcollege.com
22241D2003	bharat22241d2003@grietcollege.com
22241D2004	keerthana22241d2004@grietcollege.com
22241D2005	sowmya22241d2005@grietcollege.com
22241D2006	naresh22241d2006@grietcollege.com
22241D2007	harideep22241d2007@grietcollege.com
22241D2008	anish22241d2008@grietcollege.com
22241D2009	nagendar22241d2009@grietcollege.com
22241D2010	sushanthreddy22241d2010@grietcollege.com
22241D2011	ravalika22241d2011@grietcollege.com
22241D2012	saipavan22241d2012@grietcollege.com
22241D2013	bharat22241d2013@grietcollege.com
22241D2014	srinivas22241d2014@grietcollege.com
22241D2015	sreenivasulu22241d2015@grietcollege.com
22241D2016	abdul22241d2016@grietcollege.com
22241D2017	zabi22241d2017@grietcollege.com
22241D2018	shivaji22241d2018@grietcollege.com
22241D2019	narayana22241d2019@grietcollege.com

	M.TECH. CIVIL (STE) 2022 Admitted		
	ROLL NO.	**STUDENT NAME**	**JOINING DATE**
1.	22241D2001	ADDAGATLA MAHESHKUMAR scholarship	26-10-2022
2.	22241D2002	AHMED ABDUL AZEEM	26-10-2022
3.	22241D2003	BAIRAPAKA BHARATH scholarship	19-11-2022
4.	22241D2004	BARLAPUDI ACHSAHKEERTHANA	26-10-2022
5.	22241D2005	CHAKALI SOWMYA scholarship	26-10-2022
6.	22241D2006	CHAPPIDI NARESH scholarship	03-11-2022
7.	22241D2007	DANTHALA HARIDEEPKUMAR scholarship	03-11-2022
8.	22241D2008	DEVIREDDY ANISH scholarship	26-10-2022
9.	22241D2009	DHARAVATH NAGENDAR scholarship	19-11-2022
10.	22241D2010	GANGAPURAM SUSHANTH REDDY*	26-10-2022
11.	22241D2011	JEREPOTHULA RAVALIKA scholarship	03-11-2022
12.	22241D2012	KADABOHINA SAIPAVAN scholarship	03-11-2022
13.	22241D2013	KASUMURU BHARATH KUMAR*	26-10-2022
14.	22241D2014	MACHARLA SRINIVAS	03-11-2022
15.	22241D2015	MALLI SREENIVASULU*	26-10-2022
16.	22241D2016	SHAIK ABDUL MUQEED scholarship	03-11-2022
17.	22241D2017	SHAIK ZABI ULLAH scholarship	26-10-2022
18.	22241D2018	SONWANE SAHILSHIVAJIRAO	03-11-2022
19	22241D2019	LINGAM LAKSHMI NARAYANA*	26-10-2022

*Management Quota
REDMARKED STUDENTS HAS ATTENDANCE BETWEEN 65 to 75%

Classes commenced from: 26-10-2022
Counselling Round 1: 12-10-2022 to 15-10-2022
Counselling Round 2: 31-10-2022 to 03-11-2022
Special Round: 15-11-2022 to 19-11-2022

DEPARTMENT OF CIVIL ENGINEERING (STRUCTURAL ENGINEERING)

I M. Tech (GR-22) - I Semester **AY: 2022-23** **wef 26-10-2022**

Day/Hour	09:00-10:00	10:00-11:00	11:00-12:00	12:00-01:00	01:00-02:00	02:00-03:00	03:00-04:00	Room No.	
MONDAY	ARDC	ARDC	ASM	LUNCH	SE LAB			Theory/ Tutorial	4203
TUESDAY	ARDC	ERPW	ERPW		CONM	CONM	TEP	Lab	4205 (CAD Lab/SE Lab)
WEDNESDAY	ASM	ARDC	TEP		CAD LAB				
THURSDAY	ASM	CONM	CONM		CAD LAB			**M.Tech Co-ordinator**	
FRIDAY	TEP	TEP	CONM		SE LAB				
SATURDAY	RM&IPR	RM&IPR	TEP		ASM	ASM	ARDC	**Dr. V Srinivasa Reddy (1117)**	

Sub. Code	**Subjects**	**Faculty Name**	**Almanac**	
	Advanced Structural Mechanics	Dr. G V V Satyanarayana (842)	1st Spell of Instruction	26-10-2022 to 22-12-2022
	Theory of Elasticity and Plasticity	Dr.V.Srinivas Reddy (Dr.VSR-1117)	1st Mid-term Examinations	23-12-2022 to 29-12-2022
	Advanced Reinforced Concrete Design	Dr.V.Mallikarjun Reddy (Dr.VMR-807)	2nd Spell of Instruction	30-12-2022 to 28-02-2023
	Computer Oriented Numerical Methods	Mr.V.Naresh Kumar Varma (1359)	2nd Mid-term Examinations	01-03-2023 to 07-03-2023
	Computer Aided Design Laboratory	Mr.C.Vanadeep (Mr.CV-1645)/Mr.C.Vivek Kumar(1500)/Mrs.P.Sirisha (Mrs.PS-1524)	Preparation	08-03-2023 to 14-03-2023
	Structural Engineering Laboratory	Mr.Kusuma Veera Babu (Mr.KVB-1650)/Mr.V.Ramesh(1646)/Mr.PVVSSR Krishna (Mr.PVVSSRK-1562)	End Semester Examinations/ (Theory/ Practicals) Regular/Supplementary	15-03-2023 to 01-04-2023
	Research Methodology and IPR	Dr. Mohammed Hussain(Dr.Mohd.H-861)		
	English for Research Paper Writing	Dr.R.Lakshmi Kanthi (Dr.LRK-718)		

Coordinator
Dr. V Srinivasa Reddy

Mr.Rathod Ravinder
Time Table Coordinator

Dr.C.Lavanya
HOD-CE

Gokaraju Rangaraju Institute of Engineering and Technology
(Autonomous)
Bachupally, Kukatpally, Hyderabad – 500 090. (040) 6686 4440

COURSE OBJECTIVES

Academic **Y**ear : 2022-23

Semester : I

Name of the Program: M.TECH. STRUCTURAL ENGINEERING

Course/Subject: ADVANCED SOLID MECHANICS Course Code: GR22D5002

Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING

Designation: PROFESSOR.

On completion of this Subject/Course the student shall be able to:

S.No	**Objectives**
	1. To explain the theory, concepts and principles of Elasticity 2. To generalize the equations of elasticity for two-dimensional problems of elasticity in terms of Cartesian and polar coordinates. 3. To demonstrate the equations of elasticity for two-dimensional problems of elasticity in terms of Cartesian and polar coordinates 4. To apply principles of elasticity to analyze the torsion and bending in prismatic bars 5. To extend the principles of stress/strain for plastic deformation to study the modes of failure

Signature of HOD Signature of faculty

Date: Date:

Gokaraju Rangaraju Institute of Engineering and Technology
(Autonomous)
Bachupally, Kukatpally, Hyderabad – 500 090. (040) 6686 4440

COURSE OUTCOMES

Academic **Y**ear : 2022-23

Semester : I

Name of the Program: M.TECH. STRUCTURAL ENGINEERING

Course/Subject: ADVANCED SOLID MECHANICS Course Code: GR22D5002

Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING

Designation: PROFESSOR.

The expected outcomes of the Course/Subject are:

S.No	Outcomes
CO1.	Have a good understanding of the theory, concepts, principles and governing equations of Elasticity principles
CO2.	Develop equations of equilibrium and draw relations among stress, strain and displacement and utilize the equilibrium equations, compatibility equations and various boundary conditions to analyze elastic problems.
CO3.	Gain the understating of three-dimensional problems of elasticity in Cartesian coordinates system ad able to determine principal stresses and planes of 3D problems
CO4.	Apply the principles of elasticity to solve torsional problems in prismatic bars and tubes
CO5.	Use the concepts of stresses and strains for plastic deformation to comprehend the yield criteria of materials

Signature of HOD

Date:

Signature of faculty

Date

Gokaraju Rangaraju Institute of Engineering and Technology
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Bachupally, Kukatpally, Hyderabad – 500 090. (040) 6686 4440

COURSE OUTCOMES

Academic **Y**ear : 2017-18

Semester : I

Name of the Program: M.TECH. STRUCTURAL ENGINEERING

Course/Subject: THOERY OF ELASTICITY AND PLASTICITY Course Code: GR17D5152

Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING

Designation: PROFESSOR

The expected outcomes of the Course/Subject are:

S.No	Outcomes

After completion of this course students will be able to

1. Explain the basic concepts of stress-strain relations in theory of elasticity
2. Analyse and interpret stresses and strains in 2-D and 3-D problems of elasticity in Cartesian coordinate system.
3. Analyse and interpret stresses and strains in 2-D and 3-D problems of elasticity in polar coordinate system.
4. Apply general theorems to find solutions to problems of elasticity.
5. Find the solutions to torsional problems using principles of elasticity
6. Find the solutions to bending problems using soap film method
7. Explain various theories of failures in plasticity.

Signature of HOD Signature of faculty

Date: Date

Gokaraju Rangaraju Institute of Engineering and Technology
(Autonomous)
Bachupally, Kukatpally, Hyderabad – 500 090, India

GRIET/DAA/1H/G/22-23 25 Oct 2022

Academic Calendar
Academic Year 2022-23

I M.Tech – First Semester

S. No.	EVENT	PERIOD	DURATION
1	Orientation Programme	26-10-2022	
2	I Spell of Instructions	26-10-2022 to 22-12-2022	8 Weeks
3	I Mid-term Examinations	23-12-2022 to 29-12-2022	1 Week
4	II Spell of Instructions	30-12-2022 to 28-02-2023	9 Weeks
5	II Mid-term Examinations	01-03-2023 to 07-03-2023	1 Week
6	Preparation / Break	08-03-2023 to 14-03-2023	1 Week
7	End Semester Examinations	15-03-2023 to 01-04-2023	3 Weeks
8	Commencement of Second Semester, AY 2022-23	03-04-2023	

I M. Tech – Second Semester

S. No.	EVENT	PERIOD	DURATION
1	Commencement of Second Semester class work	03-04-2023	
2	I Spell of Instructions	03-04-2023 to 29-04-2023	4 Weeks
3	Summer Vacation	01-05-2023 to 13-05-2023	2 Weeks
4	I Spell of Instructions Contd..	15-05-2023 to 17-06-2022	5 Weeks
5	I Mid-term Examinations	19-06-2023 to 24-06-2023	1 Week
6	II Spell of Instructions	26-06-2023 to 26-08-2023	9 Weeks
7	II Mid-term Examinations	28-08-2023 to 02-09-2023	1 Week
8	Preparation / Break	04-09-2023 to 09-09-2023	1 Week
9	End Semester Examinations	11-09-2023 to 25-09-2023	2 Weeks
10	Commencement of Second Year, First Semester, AY 2023-24	26-09-2023	

J. Praveen

Dean Academic Affairs

Copy to Principal, All HoDs, CoE

GOKARAJU RANGARAJU INSTITUTE OF ENGINEERING AND TECHNOLOGY

ADVANCED SOLID MECHANICS

Course Code: GR22D5002 **L/T/P/C: 3/0/0/3**
I Year I Semester

Course Prerequisites: Mathematics and Strength of Materials

Course objectives:
1. To explain the theory, concepts and principles of Elasticity
2. To generalize the equations of elasticity for two-dimensional problems of elasticity in terms of Cartesian and polar coordinates.
3. To demonstrate the equations of elasticity for two-dimensional problems of elasticity in terms of Cartesian and polar coordinates
4. To apply principles of elasticity to analyze the torsion and bending in prismatic bars
5. To extend the principles of stress/strain for plastic deformation to study the modes of failure

Course Outcomes:
1. Have a good understanding of the theory, concepts, principles and governing equations of Elasticity principles.
2. Develop equations of equilibrium and draw relations among stress, strain and displacement and utilize the equilibrium equations, compatibility equations and various boundary conditions to analyze elastic problems.
3. Gain the understating of three-dimensional problems of elasticity in Cartesian coordinates system ad able to determine principal stresses and planes of 3D problems.
4. Apply the principles of elasticity to solve torsional problems in prismatic bars andtubes.
5. Use the concepts of stresses and strains for plastic deformation to comprehend the yield criteria of materials.

UNIT I

Introduction to Elasticity: Notation for forces and stresses - Components of stresses - Components of strain – Hooke’s law, Strain and Stress Fields, Stress and strain at a Point, Stress Components on an Arbitrary Plane, Hydrostatic and Deviatoric Components, Saint-Venant’s principle.

UNIT II

Equations of Elasticity in Two-dimensional problems in rectangular and polar coordinates: Equations of Equilibrium, Stress- Strain relations, Strain –Displacement and Compatibility Relations, Boundary conditions, Plane stress and plane strain analysis - stress function -Two dimensional problems in rectangular coordinates - solution by polynomials.

UNIT III

Analysis of stress and strain in three dimensions in rectangular and polar coordinates - principal stresses - stress ellipsoid-determination of principal stresses - max shear stresses- equations of equilibrium in terms of displacements.

UNIT IV

Torsion of Prismatic Bars: Saint Venant's Method, Prandtl's Membrane Analogy, Torsion of Rectangular Bar, use of soap films in solving torsion problems, Bending of Prismatic Bars: Stress function - bending of cantilever – circular cross section.

UNIT V

Concepts of plasticity, Plastic Deformation, Strain Hardening, Idealized Stress- Strain curve, Yield Criterions, Plastic Stress-Strain Relations.

Text Books:

1. Theory of Elasticity, S.P. Timoshenko and J.N. Goodier, Tata McGraw Hill, 3rd edition, 2017.
2. Advanced Mechanics of Solids, Srinath L.S., Tata McGraw Hill, 2nd edition, 2010.
3. Theory of Elasticity and Plasticity, H. Jane Helena, PHI Learning, 2017

Reference Books:

1. Theory of Elasticity, Sadhu Singh, Khanna Publishers, 2007.
2. Computational Elasticity, Ameen M., Narosa, 2005.
3. Solid Mechanics, Kazimi S. M. A., Tata McGraw Hill,2nd edition,2017.
4. Elasticity, Sadd M.H., Elsevier,3rd edition, 2014.
5. Engineering Solid Mechanics, Ragab A.R., Bayoumi S.E., CRC Press,first edition,1998.
6. Theory of Plasticity, J. Chakrabarty, Butterworth-Heinemann publications,3rd edition, 2006.

GOKARAJU RANGARAJU INSTITUTE OF ENGINEERING AND TECHNOLOGY

ADVANCED SOLID MECHANICS

Course Code: GR20D5002 **L/T/P/C: 3/0/0/3**

I Year I Semester

Course Prerequisites: Mathematics and Strength of Materials

Course objectives:

1. To explain the theory, concepts and principles of Elasticity
2. To generalize the equations of elasticity for two-dimensional problems of elasticity in terms of Cartesian and polar coordinates.
3. To demonstrate the equations of elasticity for two-dimensional problems of elasticity in terms of Cartesian and polar coordinates
4. To apply principles of elasticity to analyze the torsion and bending in prismatic bars
5. To extend the principles of stress/strain for plastic deformation to study the modes of failure

Course Outcomes: At the end of the course, the student will be able to

1. Have a good understanding of the theory, concepts, principles and governing equations of Elasticity principles.
2. Develop equations of equilibrium and draw relations among stress, strain and displacement and utilize the equilibrium equations, compatibility equations and various boundary conditions to analyze elastic problems.
3. Gain the understating of three-dimensional problems of elasticity in Cartesian coordinates system ad able to determine principal stresses and planes of 3D problems.
4. Apply the principles of elasticity to solve torsional problems in prismatic bars and tubes.
5. Use the concepts of stresses and strains for plastic deformation to comprehend the yield criteria of materials.

UNIT I:

Introduction to Elasticity : Notation for forces and stresses - Components of stresses - Components of strain – Hooke's law, Strain and Stress Fields, Stress and strain at a Point, Stress Components on an Arbitrary Plane, Hydrostatic and Deviatoric Components, Saint-Venant's principle.

UNIT II:

Equations of Elasticity in Two-dimensional problems in rectangular and polar coordinates: Equations of Equilibrium, Stress- Strain relations, Strain –Displacement and Compatibility Relations, Boundary conditions, Plane stress and plane strain analysis - stress function -Two dimensional problems in rectangular coordinates - solution by polynomials.

UNIT III:

Analysis of stress and strain in three dimensions in rectangular and polar coordinates - principal stresses - stress ellipsoid-determination of principal stresses - max shear stresses- equations of equilibrium in terms of displacements.

UNIT IV:

Torsion of Prismatic Bars: Saint Venant's Method, Prandtl's Membrane Analogy, Torsion of Rectangular Bar, use of soap films in solving torsion problems, Bending of Prismatic Bars: Stress function - bending of cantilever – circular cross section.

UNIT V:

Concepts of plasticity, Plastic Deformation, Strain Hardening, Idealized Stress- Strain curve, Yield Criterions, Plastic Stress-Strain Relations.

References:

1. Theory of Elasticity, Timoshenko S. and GoodierJ. N., McGraw Hill, 1961.
2. Elasticity, Sadd M.H.,Elsevier,2005.
3. Engineering Solid Mechanics, RagabA.R., BayoumiS.E., CRC Press,1999.
4. Computational Elasticity, AmeenM., Narosa, 2005.
5. Solid Mechanics, KazimiS. M. A., Tata McGraw Hill,1994.
6. Advanced Mechanics of Solids, SrinathL.S., Tata McGraw Hill,2000.

1. Program Educational Objectives (PEOs) – Vision/Mission Matrix

Vision/Mission / PEOs	Vision of the Institute	Mission of the Institute	Mission of the Program
1	H	H	H
2	H	H	H
3	H	H	H
4	H	H	H

2. Program Educational Objectives(PEOs)-Program Outcomes(POs) Relationship Matrix

P-Outcomes / PEOs	PO1	PO2	PO3	PO4	PO5	PO6
1	H	H	H	M	H	H
2	H	H	H	M	H	H
3	H	H	H	H	H	M
4	H	M	H	H	H	H

3. Course Objectives-Course Outcomes Relationship Matrix

Course-Outcomes / Course-Objectives	1	2	3	4	5
1	**H**				
2		**H**			
3			**H**		
4				**H**	
5					**H**

4. Course Objectives-Program Outcomes(POs) Relationship Matrix

P-Outcomes / C-Objectives	PO1	PO2	PO3	PO4	PO5	PO6
1	**H**	**M**		**M**	**M**	**M**
2	**H**	**M**		**M**	**M**	**M**
3	**H**	**H**	**H**	**H**	**H**	**M**
4	**H**	**H**	**H**	**H**	**H**	**M**
5	**H**	**H**	**H**	**H**	**H**	**M**

5. Course Outcomes-Program Outcomes(POs) Relationship Matrix

P-Outcomes / C-Outcomes	PO1	PO2	PO3	PO4	PO5	PO6
1	**M**	**M**	**M**	**H**	**H**	**H**
2	**M**	**M**	**M**	**H**	**H**	**H**
3	**H**	**H**	**H**	**H**	**H**	**H**
4	**H**	**H**	**H**	**H**	**H**	**H**
5	**H**	**H**	**M**	**H**	**H**	**H**

6. Courses (with title & code)-Program Outcomes (POs) Relationship Matrix

P-Outcomes / Courses	PO1	PO2	PO3	PO4	PO5	PO6
GR22D5002 ASM	**H**	**M**	**H**	**H**	**H**	**H**

7. Program Educational Objectives (PEOs)-Course Outcomes Relationship Matrix

P-Objectives (PEOs) / Course-Outcomes	1	2	3	4
1	**H**	**H**	**H**	**H**
2	**H**	**H**	**H**	**H**
3	**H**	**H**	**H**	**H**
4	**H**	**H**	**H**	**H**
5	**H**	**H**	**H**	**H**

M.TECH –I SEM (2022-23) **SESSION PLAN**

COURSE: ADVANCED SOLID MECHANICS **INSTRUCTOR: DR. V SRINIVASA REDDY**

LECTURES: 50 *Date of commencement of classes:*

SNo	UNIT NO	DATE	TOPICS
1.	1	26-10-2022	Introduction to Elasticity
2.	1	28-10-2022	Notation for forces and stresses
3.	1	29-10-2022	Components of stresses
4.	1	02-11-2022	Components of strain
5.	1	04-11-2022	Hooke’s law
6.	1	05-11-2022	Strain and Stress Fields
7.	1	09-11-2022	Stress and strain at a Point
8.	1	11-11-2022	Stress Components on an Arbitrary Plane
9.	1	12-11-2022	Hydrostatic and Deviatoric Components
10.	1	16-11-2022	Saint- Venant’s principle.
11.	2	18-11-2022	Equations of Elasticity in Two-dimensional problems in rectangular coordinates
12.	2	19-11-2022	Equations of Elasticity in Two-dimensional problems in polar coordinates
13.	2	23-11-2022	Equations of Equilibrium
14.	2	25-11-2022	Stress- Strain relations
15.	2	26-11-2022	Strain –Displacement and Compatibility Relations
16.	2	30-11-2022	Boundary conditions
17.	2	02-12-2022	Plane stress and plane strain analysis
18.	2	03-12-2022	stress function
19.	2	07-12-2022	Two dimensional problems in rectangular coordinates
20.	2	09-12-2022	solution by polynomials.
21.	2	10-12-2022	solution by polynomials.
22.	3	14-12-2022	Analysis of stress and strain in three dimensions in rectangular coordinates
23.	3	16-12-2022	Analysis of stress and strain in three dimensions in polar coordinates
24.	3	17-12-2022	principal stresses
25.	3	21-12-2022	Worked out example on Principal stresses
26.	3	30-12-2022	stress ellipsoid
27.	3	31-12-2022	determination of principal stresses
28.	3	04-01-2023	max shear stresses
29.	3	06-01-2023	equations of equilibrium in terms of displacements
30.	3	07-01-2023	equations of equilibrium in terms of displacements
31.	4	11-01-2023	Torsion of Prismatic Bars
32.	4	13-01-2023	Saint Venant’s Method
33.	4	18-01-2023	Prandtl’s Membrane Analogy
34.	4	20-01-2023	Torsion of Rectangular Bar
35.	4	21-01-2023	Use of soap films in solving torsion problems
36.	4	25-01-2023	Use of soap films in solving torsion problems
37.	4	27-01-2023	Bending of Prismatic Bars: Stress function.
38.	4	28-01-2023	bending of cantilever
39.	4	01-02-2023	circular cross section
40.	4	03-02-2023	circular cross section
41.	5	04-02-2023	Concepts of plasticity
42.	5	08-02-2023	Concepts of plasticity
43.	5	10-02-2023	Concepts of plasticity
44.	5	11-02-2023	Plastic Deformation
45.	5	15-02-2023	Strain Hardening
46.	5	17-02-2023	Idealized Stress- Strain curve
47.	5	22-02-2023	Yield Criterions
48.	5	24-02-2023	Plastic Stress-Strain Relations
49.	5	25-02-2023	Failure theories

CO - PI - PO Mapping Table

STRUCTURAL DYNAMICS (GR22)	**Program Outcomes**											
Course Outcomes0	**1 (5)**		**2 (4)**		**3 (7)**		**4 (7)**		**5 (7)**		**6 (6)**	
1. Comprehend and model the systems subjected to vibrations and dynamic loads	1.1.1	H	2.1.1	H	3.1.1	H	4.1.1	M	5.1.1	M	6.1.1	H
	1.1.2		2.1.2		-		4.1.2		5.2.2		6.2.1	
	1.2.1		2.2.1		-		4.2.2		5.3.3		6.2.2	
	1.2.2		2.2.2		3.2.1		4.3.1				6.3.1	
					3.2.2		4.3.3				6.3.2	
	1.2.3				3.3.1		-				-	
					3.3.2		-					
2. Analyze and obtain dynamics response of single degree freedom system using fundamental Theory and equations of motion.	1.1.1	M	2.1.1	H	3.1.1	H	4.1.1	L	5.1.1	H	6.1.1	H
	1.1.2		2.1.2		3.2.1		4.1.2		5.1.2		6.2.1	
	1.2.3		2.2.1		3.2.2		4.1.3		5.1.3		6.2.2	
	-		2.2.2		-		-		5.2.1		6.3.1	
					-		4.2.2		5.2.2		6.3.2	
	-				3.3.1		4.3.2		-		-	
					3.3.2		-		-			
3. Analyze and obtain dynamics response of Multi degree of freedom system idealized as lumped mass systems. Analyze and obtain dynamics response of Multi degree of freedom system idealized as distributed mass systems.	1.1.1	H	2.1.1	H	3.1.1	H	4.1.1	H	5.1.1	H	6.1.1	H
	1.1.2		2.1.2		3.2.1		4.1.2		5.1.2		6.2.1	
	-		2.2.1		3.2.2		4.1.3		5.1.3		6.2.2	
	1.2.3				3.3.1		-		5.2.1		6.3.1	
	-				3.3.2		4.2.2		5.2.2		6.3.2	
			2.2.2		-		4.3.2		-		-	
					-		-		-			
4. Obtain dynamics response of systems using numerical methods.	1.1.1	H	2.2.1	L	3.2.1	L	4.2.2	L	-	-	6.3.1	L
	1.1.2		-		3.2.2		-		-		6.3.2	
	-		-		-		-		-		-	
	1.2.3 -				-		-		-		-	
					-		-		-		-	
	-		-		-		-		-		-	
					-		-		-			
5. To explain the dynamic effects of Wind Loads, Moving Loads and Vibrations caused by Traffic, Blasting and Pile Driving.	1.1.1	H	2.1.1	H	3.1.1	H	4.1.1	H	5.1.1	L	6.1.1	H
	1.1.2		2.1.2		-		4.1.2		5.2.2		6.2.1	
	1.2.1		2.2.1		-		4.2.2		-		6.2.2	
	1.2.2				3.2.1		4.3.1		-		6.3.1	
					3.2.2		4.3.3		-		6.3.2	
	1.2.3		2.2.2		3.3.1		-		-		-	
					3.3.2							

Note:

1. If more than 67% of PIs match with CO, then CO-PO mapping is HIGH (H)
2. If the number of PIs matching with CO is between 34% & 67%, then CO-PO mapping is MEDIUM (M)
3. If the number of PIs matching with CO is less than 34%, then CO-PO mapping is LOW (L)

M.Tech Structural Engineering Program Program Outcomes – List of Competencies – Associated Performance Indicators	
PO 1: Conduct Investigations of Complex Problems: An ability to independently carry out research /investigation and development to solve practical problems.	
Competencies	**Performance Indicators (PI)**
1.1 Demonstrate an ability to conduct investigations of technical issues	1.1.1 Define a problem, its scope and importance for purposes of investigation 1.1.2 Use appropriate procedures, tools and techniques to conduct experiments and arrive at solution.
1.2 Demonstrate an ability to design experiments to solve open-ended problems	1.2.1 Design and develop an experimental approach, specify appropriate equipment and procedures 1.2.2 Choose an appropriate experimental design plan based on the study objectives. 1.2.3 Analyze data for trends and correlations, stating possible errors and limitations
PO 2: Technical Communication: An ability to write and present a substantial technical report/document.	
Competencies	**Performance Indicators (PI)**
2.1 Demonstrate an ability to comprehend technical literature and document project work	2.1.1 Read, understand and interpret technical and non-technical information 2.1.2 Produce clear, well-constructed, and well-supported written engineering documents with a logical progression of ideas.
2.2 Demonstrate an ability to integrate different modes of communication	2.2.1 Create engineering-standard figures, reports and drawings to complement writing and presentations 2.2.2 Use a variety of media effectively to convey a message in a document or a presentation
PO 3: Modern Engineering Tools and Project Management: Students should be able to demonstrate a degree of mastery over the area as per thespecialization of the program. The mastery should be at a level higher than the requirements in the appropriate bachelor's program.	
Competencies	**Performance Indicators (PI)**
3.1 Demonstrate an ability to evaluate the economic and financial performance of an engineering activity and plan/manage an engineering activity within time and budget constraints	3.1.1 Identify the tasks required to complete an engineering activity, and the resources required to complete the tasks. 3.1.2 Analyze and select the most appropriate engineering project based on economic and financial considerations. 3.1.3 Use project management tools to schedule an engineering project, so as to complete on time and within budget.
3.2 Demonstrate an ability to identify/ create modern	3.2.1 Identify/create/adapt/modify/extend tools such as STAAD Pro, ETABS, MIDAS, SAP 2000, ANSYS and techniques to solve structural engineering problems.

engineering tools, techniques and resources.	3.2.2 Demonstrate proficiency in using Structural engineering-specific tools and verify the credibility of results from tool use with reference to theaccuracy and limitations.
3.3 Demonstrate an ability to formulate and interpret a model.	3.3.1 Combine scientific principles and engineering concepts to formulate model(s) of a system or process that is appropriate in terms of applicability and required accuracy. 3.3.2 Apply engineering mathematics and computations to solve mathematical models.

PO 4: Solutions to Multidisciplinary Problems:
Possess critical thinking skills and solve core, complex and multidisciplinary structural engineering problems.

Competencies	Performance Indicators (PI)
4.1 Demonstrate an ability to identify and formulate a methodology and find solution to core and complex engineering problems	4.1.1 Articulate problem statements and identify objectives 4.1.2 Reframe complex problems into interconnected sub-problems 4.1.3 Identify existing processes/ methods for solving the problem, including forming justified approximations and assumptions
4.2 Demonstrate an ability to analyze data and reach a valid conclusion	4.2.1 Represent data (in tabular and/or graphical forms) so as to facilitate analysis and explanation of the data, and drawing of conclusions 4.2.2 Synthesize information and knowledge about the problem from the raw data to reach appropriate conclusions
4.3 Demonstrate an ability to advance a multidisciplinary engineering design to defined end state	4.3.1 Refine a conceptual design into a detailed design within the existing constraints (of the resources) 4.3.2 Generate information through appropriate tests to improve or revise the design

PO 5: Ethics, Environment and Sustainability:
Assess the impact of professional engineering solutions in an environmental context along with societal, health, safety, legal, ethical and cultural issues and the need for sustainabledevelopment.

Competencies	Performance Indicators (PI)
5.1 Demonstrate an understanding of the impact of engineering and industrial practices on the society and environment.	5.1.1 Identify risks/impacts in the life-cycle of an engineering product or activity related to design and construction of structures. 5.1.2 Understand the relationship between the technical, societal, health, safety and cultural issues.
5.2 Demonstrate an ability to apply principles of sustainable design and development.	5.2.1 Describe management techniques for sustainable development 5.2.2 Apply principles of sustainable development to an engineering activity or product relevant to the discipline.
5.3 Demonstrate an ability to apply the code of ethics and understanding of professional engineering regulations,	5.3.1 Identify situations of unethical professional conduct and propose ethical alternatives as per ICE(I), ECI, NSPE. 5.3.2 Examine and apply moral & ethical principles to known case studies 5.3.3 Interpret legislation, regulations, codes, and standards such as

legislation and standards.	ASCE, ASTM, BIS, ISO etc. which are relevant to Structural Engineering and its contribution to the protection of the public.

PO 6: Lifelong Learning: Recognize the need for life-long learning to improve knowledge and competence.	
Competencies	**Performance Indicators (PI)**
6.1 Demonstrate an ability to identify gaps in knowledge and a strategy to close these gaps	6.1.1 Describe the rationale for the requirement for continuing professional development 6.1.2 Identify deficiencies or gaps in knowledge and demonstrate an ability to source information to close this gap
6.2 Demonstrate an ability to identify changing trends in engineering knowledge and practice	6.2.1 Identify historic points of technological advance in engineering that required practitioners to seek education in order to stay current. 6.2.2 Recognize the need and be able to clearly explain why it is vitally important to keep current regarding new developments in the field of structural Engineering.
6.3 Demonstrate an ability to identify and access sources for new information.	6.3.1 Comprehend technical literature and other credible sources of information. 6.3.2 Analyze sourced technical and popular information for feasibility, viability, sustainability, etc.

Gokaraju Rangaraju Institute of Engineering and Technology
(Autonomous)
Bachupally, Kukatpally, Hyderabad – 500 090. (040) 6686 4440

Students Rubric

Academic Year : 2021-22
Semester : I
Name of the Program: M.Tech Structural Engineering Year: I
Course/Subject: Advance Solid Mechanics Course Code: GR20D5002
Name of the Faculty: Dr.V Srinivasa Reddy. Dept.: Civil engineering
Designation: Professor

		Beginning	Developing	Reflecting Development	Accomplished	Exemplary	Score
Name of the Student	**Performance Criteria**	**1**	**2**	**3**	**4**	**5**	
MARIYALA VAISHNAVI	Level of knowledge on Fundamentals of Stresses , Strains and Displacements					5	14
	Level of knowledge on 2 D and 3D Elasticity principles					5	
	Level of knowledge on Principles of plasticity					4	
SHAIK FEROZ	Level of knowledge on Fundamentals of Stresses , Strains and Displacements				4		10
	Level of knowledge on 2 D and 3D Elasticity principles			3			
	Level of knowledge on Principles of plasticity			3			
G NITISH KUMAR	Level of knowledge on Fundamentals of Stresses , Strains and Displacements		2				5
	Level of knowledge on 2 D and 3D Elasticity principles		2				
	Level of knowledge on Principles of plasticity	1					

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Students Rubric

Academic Year : 2022-23
Semester : I
Name of the Program: M.Tech Structural Engineering Year: I
Course/Subject: Advance Solid Mechanics Course Code: GR22D5002
Name of the Faculty: Dr.V Srinivasa Reddy. Dept.: Civil engineering
Designation: Professor

		Beginning	Developing	Reflecting Development	Accomplished	Exemplary	Score
Name of the Student	**Performance Criteria**	**1**	**2**	**3**	**4**	**5**	
22241D2018	Level of knowledge on Fundamentals of Stresses , Strains and Displacements					5	15
	Level of knowledge on 2 D and 3D Elasticity principles					5	
	Level of knowledge on Principles of plasticity					5	
22241D2005	Level of knowledge on Fundamentals of Stresses , Strains and Displacements				4		8
	Level of knowledge on 2 D and 3D Elasticity principles				4		
	Level of knowledge on Principles of plasticity			3			
22241D2019	Level of knowledge on Fundamentals of Stresses , Strains and Displacements	1					3
	Level of knowledge on 2 D and 3D Elasticity principles	1					
	Level of knowledge on Principles of plasticity	1					

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EVALUATION STRATEGY

Academic **Y**ear : 2022-23
Semester : I
Name of the Program: MTech Structural Engg. Year: I Section: A
Course/Subject: ADVANCED SOLID MECHANICS
Course Code: GR20D5002
Name of the Faculty: DR. V.SRINIVASA REDDY. Dept.: CIVIL ENGINEERING
Designation: PROFESSOR
1. TARGET :(Projected)
Projected Percentage for pass: 100%
Total number of students ENROLLED for this course : 19

2. COURSE PLAN & CONTENT DELIVERY
The course is delivered as Lectures, Lecture with a quiz Tutorials, Assignments, Group Discussion Presentations, Site Visits, Illustrative Videos, and teacher supplied class lecture handouts. In addition to classroom lectures, tutorials are also planned to help the students understand and appreciate the challenges involved in practical implementations and also understand the engineering trade-offs to made while making practical implementations.

- Sixty Two (62) Class room lectures were planned
- Ten (10) Tutorials were planned for discussions on the lectures and various practical implementations.
- Demonstrations are held through various illustrative Videos and Web classrooms
- Assignments and Tutorial work out classes are arranged for every unit of the syllabus

3. METHOD OF EVALUATION
3.1 Continuous Assessment Examinations (MID EXAM-I, MID EXAM-II)
The department follows continuous evaluation system through assignments, projects, Mid exams (2 Nos.) and an end semester examination. The continuous academic quality assessments carried out through a peer (external) review process once in a year. The suitable feedback from Training and Placement cell is also considered. Board of studies of the department includes two external experts (one from Reputed Academic Institute and another from Industry) which advocate areas of skills and knowledge to be improved upon by the students in the context of changing situation.
Continuous Assessment Marks (Best of MID EXAM-I, MID EXAM-II) – 30 Marks
Evaluated mid answer scripts are shown to students by respective subject teachers. Based on marks obtained by the students, remedial classes are conducted by the departments for slow learners.
3.2 Assignments/Seminars
The students' progress is continuously monitored through regular assignments and practice sessions to ensure the achievement of course outcomes. All components in any program of study will be evaluated continuously through internal evaluation and external evaluation component conducted as year-end/ semester-end examination. Internal evaluation includes two components I. Mid Examinations II. Assignments. Assignments improve the continuous learning capacity of student
Five (5) marks are earmarked for assignments. Five (5) marks are earmarked for Assessment.
3.4 Semester/End Examination
The scheme of evaluation for every subject is for 100 marks, out of this, 40 marks are earmarked for continuous internal evaluation. End Semester Exam for 60 Marks
3.5 Others
The improvements, modifications and additions to the curriculum are governed by Board of Studies (BOS) and executed on a continuous basis based on the feedback from the stakeholders and changing societal needs. The meeting of BOS is held and the faculty member will be contributing in the curriculum development along with the experts from the IIT/Industry. The student class committee meets every semester and their views are incorporated in order to improve the curriculum.

Signature of faculty

Date:

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COURSE SCHEDULE

Academic **Y**ear : 2022-23

Semester : I

Name of the Program: M.TECH. STRUCTURAL ENGINEERING

Course/Subject: ADVANCED SOLID MECHANICS Course Code: GR225002

Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING

Designation: PROFESSOR.

The Schedule for the whole Course / Subject is:

Unit. No.	Description	Duration (Date)		Total No. Of Periods
		From	To	
1.	**Introduction to Elasticity**	26-10-2022	16-11-2022	10
2.	**Equations of Elasticity in Two-dimensional problems**	18-11-2022	10-12-2022	11
3.	**Analysis of stress and strain in three dimensions**	14-12-2022	07-01-2023	9
4.	**Torsion of Prismatic Bars**	11-01-2023	03-02-2023	9
5.	**Theory of Plasticity**	04-02-2023	25-02-2023	8

Total No. of Instructional periods available for the course: 49Hours / Periods

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GUIDELINES TO STUDY THE COURSE / SUBJECT

Academic **Y**ear : 2022-23

Semester : I

Name of the Program: M.TECH. STRUCTURAL ENGINEERING

Course/Subject: ADVANCED SOLID MECHANICS Course Code: GR22D5002

Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING

Designation: PROFESSOR.

Guidelines to study the Course/ Subject: ADVANCED SOLID MECHANICS

Course Design and Delivery System (CDD):

- The Course syllabus is written into number of learning objectives and outcomes.
- These learning objectives and outcomes will be achieved through lectures, assessments, assignments, experiments in the laboratory, projects, seminars, presentations, etc.
- Every student will be given an assessment plan, criteria for assessment, scheme of evaluation and grading method.
- The Learning Process will be carried out through assessments of Knowledge, Skills and Attitude by various methods and the students will be given guidance to refer to the text books, reference books, journals, etc.

The faculty be able to –

- Implement principles of Learning
- Comprehend the psychology of students
- Develop instructional objectives for a given topic
- Prepare course, unit and lesson plans
- Demonstrate different methods of teaching and learning
- Use appropriate teaching and learning aids
- Plan and deliver lectures effectively
- Provide feedback to students using various methods of Assessments and tools of Evaluation
- Act as a guide, advisor, counselor, facilitator, motivator and not just as a teacher alone

Signature of HOD Signature of faculty

Date: Date:

Gokaraju Rangaraju Institute of Engineering and Technology
(Autonomous)
Bachupally, Kukatpally, Hyderabad – 500 090. (040) 6686 4440

SCHEDULE OF INSTRUCTIONS
COURSE PLAN

Academic **Y**ear : 2022-23
Semester : I
Name of the Program: M.TECH. STRUCTURAL ENGINEERING
Course/Subject: ADVANCED SOLID MECHANICS Course Code: GR205002
Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING
Designation: PROFESSOR.

Unit	Lesson No.	Date	No of Period	Topics / Sub - Topics	Objective & Outcome Nos.	References (Text Book, Journal…) Page Nos.: ____to ____	Bloom's Knowledge levels
1	1	26-10-2022	1	Introduction to Elasticity	COB-1 CO-1	Lecture Notes	Level 2
	2	28-10-2022	1	Notation for forces and stresses	COB-1 CO-1	Lecture Notes	Level 2
	3	29-10-2022	1	Components of stresses	COB-1 CO-1	Lecture Notes	Level 2
	4	02-11-2022	1	Components of strain	COB-1 CO-1	Lecture Notes	Level 2
	5	04-11-2022	1	Hooke's law	COB-1 CO-1	Lecture Notes	Level 2
	6	05-11-2022	1	Strain and Stress Fields	COB-1 CO-1	Lecture Notes	Level 2
	7	09-11-2022	1	Stress and strain at a Point	COB-1 CO-1	Lecture Notes	Level 2
	8	11-11-2022	1	Stress Components on an Arbitrary Plane	COB-1 CO-1	Lecture Notes	Level 2
	9	12-11-2022	1	Hydrostatic and Deviatoric Components	COb-1 CO-1	Lecture Notes	Levels 2&3
	10	16-11-2022	1	Saint- Venant's principle.	COC-1 CO-1	Lecture Notes	Levels 2&3
	11	18-11-2022	1	Equations of Elasticity in Two-dimensional problems in rectangular coordinates	COB-2 CO-2	Lecture Notes	Levels 2&3
	12	19-11-2022	1	Equations of Elasticity in Two-dimensional problems in polar coordinates	COB-2 CO-2	Lecture Notes	Level 2
	13	23-11-2022	1	Equations of Equilibrium	COB-2 CO-2	Lecture Notes	Level 2

2	14	25-11-2022	1	Stress- Strain relations	COB-2 CO-2	Lecture Notes	Level 2
	15	26-11-2022	1	Strain –Displacement and Compatibility Relations	COB-2 CO-2	Lecture Notes	Level 2
	16	30-11-2022	1	Boundary conditions	COB-2 CO-2	Lecture Notes	Level 3
	17	02-12-2022	1	Plane stress and plane strain analysis	COB-2 CO-2	Lecture Notes	Level 3
	18	03-12-2022	1	stress function	COB-2 CO-2	Lecture Notes	Level 3
3	19	07-12-2022	1	Two dimensional problems in rectangular coordinates	COB-2 CO-2	Lecture Notes	Level 3
	20	09-12-2022	1	solution by polynomials.	COB-2 CO-2	Lecture Notes	Level 3
	21	10-12-2022	1	solution by polynomials.	COB-2 CO-2	Lecture Notes	Level 3
	22	14-12-2022	1	Analysis of stress and strain in three dimensions in rectangular coordinates	COB-3 CO3	Lecture Notes	Level 3
	23	16-12-2022	1	Analysis of stress and strain in three dimensions in polar coordinates	COB-3 CO3	Lecture Notes	Level 3
	24	17-12-2022	1	principal stresses	COB-3 CO3	Lecture Notes	Level 3
	25	21-12-2022	1	Worked out example on Principal stresses	COB-3 CO3	Lecture Notes	Level 3
	26	30-12-2022	1	stress ellipsoid	COB-3 CO3	Lecture Notes	Level 3
4	27	31-12-2022	1	determination of principal stresses	COB-3 CO3	Lecture Notes	Level 3
	28	04-01-2023	1	max shear stresses	COB-3 CO3	Lecture Notes	Level 3
	29	06-01-2023	1	equations of equilibrium in terms of displacements	COB-3 CO3	Lecture Notes	Level 3
	30	07-01-2023	1	equations of equilibrium in terms of displacements	COB-3 CO3	Lecture Notes	Level 3
	31	11-01-2023	1	Torsion of Prismatic Bars	COB-4,CO-4	Lecture Notes	Level 3
	32	13-01-2023	1	Saint Venant’s Method	COB-4,CO-4	Lecture Notes	Level 3
	33	18-01-2023	1	Prandtl’s Membrane Analogy	COB-	Lecture Notes	Level 3

					4,CO-4		
	34	20-01-2023	1	Torsion of Rectangular Bar	COB-4,CO-4	Lecture Notes	Level 3
	35	21-01-2023	1	Use of soap films in solving torsion problems	COB-4,CO-4	Lecture Notes	Level 3
	36	25-01-2023	1	Use of soap films in solving torsion problems	COB-4,CO-4	Lecture Notes	Level 3
5	37	27-01-2023	1	Bending of Prismatic Bars: Stress function.	COB-4,CO-4	Lecture Notes	Level 3
	38	28-01-2023	1	bending of cantilever	COB-4,CO-4	Lecture Notes	Level 3
	39	01-02-2023	1	circular cross section	COB-4,CO-4	Lecture Notes	Level 3
	40	03-02-2023	1	circular cross section	COB-4,CO-4	Lecture Notes	Level 3
	41	04-02-2023	1	Concepts of plasticity	COB-5,CO-5	Lecture Notes	Level 3
	42	08-02-2023	1	Concepts of plasticity	COB-5,CO-5	Lecture Notes	Level 3
	43	10-02-2023	1	Concepts of plasticity	COB-5,CO-5	Lecture Notes	Level 3
	44	11-02-2023	1	Plastic Deformation	COB-5,CO-5	Lecture Notes	Level 3
	45	15-02-2023	1	Strain Hardening	COB-5,CO-5	Lecture Notes	Level 3
	46	17-02-2023	1	Idealized Stress- Strain curve	COB-5,CO-5	Lecture Notes	Level 3
	47	22-02-2023	1	Yield Criterions	COB-5,CO-5	Lecture Notes	Level 3
	48	24-02-2023	1	Plastic Stress-Strain Relations	COB-5,CO-5	Lecture Notes	Level 3
	49	25-02-2023	1	Failure theories	COB-5,CO-5	Lecture Notes	Level 3

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SCHEDULE OF INSTRUCTIONS
COURSE PLAN

Academic **Y**ear : 2017-18

Semester : I

Name of the Program: M.TECH. STRUCTURAL ENGINEERING

Course/Subject: THOERY OF ELASTICITY AND PLASTICITY Course Code: GR17D5152

Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING

Designation: PROFESSOR.

Unit	Lesson No.	Date	No of Period	Topics / Sub - Topics	Objective & Outcome Nos.	References (Text Book, Journal…) Page Nos.: ____to ____	Bloom's Knowledge levels
	1	28-08-2017	1	Introduction: Elasticity	COB-1 CO-1	Lecture Notes	Level 2
	2	29-08-2017	1	Notation for forces and stresses - components of stresses - components of strain	COB-1 CO-1	Lecture Notes	Level 2
	3	1-09-2017	1	Hooks law. Plane stress and plane strain analysis - plane stress - plane strain.	COB-1 CO-1	Lecture Notes	Level 2
	4	4-09-2017	1	differential equations of equilibrium - boundary conditions	COB-1 CO-1	Lecture Notes	Level 2

1	5	5-09-2017	1	compatibility equations– boundary condition	COB-1 CO-1	Lecture Notes	Level 2
	6	8-09-2017	1	stress function	COB-1 CO-1	Lecture Notes	Level 2
	7	11-09-2017	1	Two dimensional problems in rectangular coordinates	COB-1 CO-1	Lecture Notes	Level 2
	8	12-09-2017	1	solution by polynomials	COB-1 CO-1	Lecture Notes	Level 2
	9	15-09-2017	1	Saint- Venant's principle - determination of displacements -	COb-1 CO-1	Lecture Notes	Levels 2&3
	10	18-09-2017	1	bending of simple beams (ss and cantilever)	COC-1 CO-1	Lecture Notes	Levels 2&3
	11	19-09-2017	1	Application of courier series for two dimensional problems - gravity loading.	COB-1 CO-1	Lecture Notes	Levels 2&3
	12	22-09-2017	1	Two dimensional problems in polar coordinates – stress distribution symmetrical about an axis	COB-2 CO-2	Lecture Notes	Level 2
	13	25-09-2017	1	pure bending of curved bars	COB-2 CO-2,3	Lecture Notes	Level 2
2	14	26-09-2017	1	strain components in polar coordinates - displacements for symmetrical stress distributions	COB-2 CO-2,3,4	Lecture Notes	Level 2
	15	29-09-2017	1	simple symmetric and asymmetric problems - general solution of two- dimensional problem in polar coordinates	COB-2 CO-2,3,4	Lecture Notes	Level 2
	16	3-10-2017	1	Application of general solution in polar coordinates.	COB-2 CO-2,3,4	Lecture Notes	Level 3
	17	4-10-2017	1	Analysis of stress and strain in three dimensions - principal stresses	COB-2 CO-2,3,4	Lecture Notes	Level 3
	18	6-10-2017	1	stress ellipsoid - director surface - determination of principal stresses - max shear stresses – Stress tensor and strain tensor-	COB-2 CO-2,3,4	Lecture Notes	Level 3
3		9-10-2017	1	Homogeneous deformation-Principal axes of strain rotation.	COB-3 ,CO-5	Lecture Notes	Level 3
		10-10-2017	1	General Theorems: Differential equations of equilibrium – conditions of compatibility	OB-3 ,OC-5O COB-3,CO-5	Lecture Notes	Level 3
		13-10-2017	1	principle of super position - uniqueness of solution	COB-3,CO-5	Lecture Notes	Level 3
		16-10-2017	1	the reciprocal theorem – Strain energy	COB-3,CO-5	Lecture Notes	Level 3
		17-10-2017	1	determination of displacement - equations of equilibrium in terms of displacements	COB-3,CO-5	Lecture Notes	Level 3
		20-10-2017	1	Torsion of circular shafts - Torsion of Prismatic	COB-3,CO-5	Lecture Notes	Level 3
		23-10-2017	1	Bars – Saint venant's method- torsion of prismatic BARS	COB-3,CO-5	Lecture Notes	Level 3

		25-10-2017	1	bars with elliptical cross sections	COB-3,CO-5	Lecture Notes	Level 3
4		27-10-2017	1	other elementary solution - membrane analogy	OB-4,OCOB-COB-4,CO-6	Lecture Notes	Level 3
		7-11-2017	1	torsion of narrow rectangular bars	- OB-4,OC-6 COB-4,CO-6	Lecture Notes	Level 3
		8-11-2017	1	solution of torsion problems by energy method	- OB-4,OC-COB-4,CO-6	Lecture Notes	Level 3
		10-11-2017	1	use of soap films in solving torsion problems	COB-4,CO-6	Lecture Notes	Level 3
		13-11-2017	1	hydro dynamical analogies	- OB-4,OC-6 COB-4,CO-6	Lecture Notes	Level 3
		17-11-2017	1	torsion of shafts, tubes , bars etc.	COB-4,CO-6	Lecture Notes	Level 3
		20-11-2017	1	Torsion of rolled profile sections.	COB-4,CO-6	Lecture Notes	Level 3
		21-11-2017	1	Bending of Prismatic Bars: Stress function	COB-4,CO-6	Lecture Notes	Level 3
		24-11-2017	1	bending of cantilever – circular cross section	COB-4,CO-6	Lecture Notes	Level 3
		27-11-2017	1	elliptical cross section	COB-5,CO-7	Lecture Notes	Level 3
5		28-11-2017	1	rectangular cross section	COB-5,CO-7	Lecture Notes	Level 3
		1-12-2017	1	bending problems by soap film method	COB-5,CO-7	Lecture Notes	Level 3
		4-12-2017	1	Displacements.	COB-5,CO-7	Lecture Notes	Level 3
		5-12-2017	1	Application to plates with circular holes	COB-5,CO-7	Lecture Notes	Level 3
		8-12-2017	1	Application to plates with circular holes	COB-5,CO-7	Lecture Notes	Level 3
		11-12-2017	1	Application to plates with circular holes	COB-5,CO-7	Lecture Notes	Level 3
		12-12-2017	1	edge dislocations	COB-5,CO-7	Lecture Notes	Level 3
		15-12-2017	1	Rotating Disk	COB-5,CO-7	Lecture Notes	Level 3
		18-12-2017	1	Rotating Disk	COB-5,CO-7	Lecture Notes	Level 3
		19-12-2017	1	Theory of Plasticity: Introduction	COB-5,CO-7	Lecture Notes	Level 3
		22-12-2017	1	Theory of Plasticity: Introduction	COB-5,CO-7	Lecture Notes	Level 3
		25-12-2017	1	concepts and assumptions - yield criterions	COB-5,CO-7	Lecture Notes	Level 3
		26-12-2017	1	yield criterions	COB-5,CO-7	Lecture Notes	Level 3
		29-12-2017	1	yield criterions	COB-5,CO-7	Lecture Notes	Level 3

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SCHEDULE OF INSTRUCTIONS

UNIT PLAN

Academic Year : 2022-23
Semester : I UNIT NO.: 1
Name of the Program: M.TECH. STRUCTURAL ENGINEERING
Course/Subject: ADVANCED SILID MECHANICS Course Code: GR22D5002
Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING
Designation: PROFESSOR.

Lesson No.	Date	No. of Periods	Topics / Sub - Topics	Objective & Outcome Nos.	References (Text Book, Journal…) Page Nos.: ____to ____	Bloom's Knowledge Levels
1.	26-10-2022	1	Introduction to Elasticity	COB-1 CO-1	Lecture Notes	Level 2
2.	28-10-2022	1	Notation for forces and stresses	COB-1 CO-1	Lecture Notes	Level 2
3.	29-10-2022	1	Components of stresses	COB-1 CO-1	Lecture Notes	Level 2
4.	02-11-2022	1	Components of strain	COB-1 CO-1	Lecture Notes	Level 2
5.	04-11-2022	1	Hooke's law	COB-1 CO-1	Lecture Notes	Level 3
6.	05-11-2022	1	Strain and Stress Fields	COB-1 CO-1	Lecture Notes	Level 3
7	09-11-2022	1	Stress and strain at a Point	COB-1 CO-1	Lecture Notes	Level 3
8	11-11-2022	1	Stress Components on an Arbitrary Plane	COB-1 CO-1	Lecture Notes	Level 3
9	12-11-2022	1	Hydrostatic and Deviatoric Components	COB-1 CO-1	Lecture Notes	Level 3
10	16-11-2022	1	Saint- Venant's principle.	COB-1 CO-1	Lecture Notes	Level 3

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SCHEDULE OF INSTRUCTIONS

UNIT PLAN

Academic Year : 2022-23
Semester : I UNIT NO.: 1I
Name of the Program: M.TECH. STRUCTURAL ENGINEERING
Course/Subject: ADVANCED SILID MECHANICS Course Code: GR22D5002
Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING
Designation: PROFESSOR.

Lesson No.	Date	No. of Periods	Topics / Sub - Topics	Objective & Outcome Nos.	References (Text Book, Journal…) Page Nos.: ____to ____	Bloom's Knowledge Levels
1.	18-11-2022	1	Equations of Elasticity in Two-dimensional problems in rectangular coordinates	COB-2 CO-2	Lecture Notes	Level 2
2.	19-11-2022	1	Equations of Elasticity in Two-dimensional problems in polar coordinates	COB-2 CO-2	Lecture Notes	Level 2
3.	23-11-2022	1	Equations of Equilibrium	COB-2 CO-2	Lecture Notes	Level 2
4.	25-11-2022	1	Stress- Strain relations	COB-2 CO-2	Lecture Notes	Level 2
5.	26-11-2022	1	Strain –Displacement and Compatibility Relations	COB-2 CO-2	Lecture Notes	Level 3
6.	30-11-2022	1	Boundary conditions	COB-2 CO-2	Lecture Notes	Level 3
7	02-12-2022	1	Plane stress and plane strain analysis	COB-2 CO-2	Lecture Notes	Level 3
8	03-12-2022	1	stress function	COB-2 CO-2	Lecture Notes	Level 3
9	07-12-2022	1	Two dimensional problems in rectangular coordinates	COB-2 CO-2	Lecture Notes	Level 3
10	09-12-2022	1	solution by polynomials.	COB-2 CO-2	Lecture Notes	Level 3
11	10-12-2022	1	solution by polynomials.	COB-2 CO-2	Lecture Notes	Level 3

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SCHEDULE OF INSTRUCTIONS

UNIT PLAN

Academic Year : 2022-23
Semester : I UNIT NO.: 1II
Name of the Program: M.TECH. STRUCTURAL ENGINEERING
Course/Subject: ADVANCED SILID MECHANICS Course Code: GR22D5002
Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING
Designation: PROFESSOR.

Lesson No.	Date	No. of Periods	Topics / Sub - Topics	Objective & Outcome Nos.	References (Text Book, Journal…) Page Nos.: ____to ____	Bloom's Knowledge Levels
1.	14-12-2022	1	Analysis of stress and strain in three dimensions in rectangular coordinates	COB-3 CO-3	Lecture Notes	Level 2
2.	16-12-2022	1	Analysis of stress and strain in three dimensions in polar coordinates	COB-3 CO-3	Lecture Notes	Level 2
3.	17-12-2022	1	principal stresses	COB-3 CO-3	Lecture Notes	Level 2
4.	21-12-2022	1	Worked out example on Principal stresses	COB-3 CO-3	Lecture Notes	Level 2
5.	30-12-2022	1	stress ellipsoid	COB-3 CO-3	Lecture Notes	Level 3
6.	31-12-2022	1	determination of principal stresses	COB-3 CO-3	Lecture Notes	Level 3
7	04-01-2023	1	max shear stresses	COB-3 CO-3	Lecture Notes	Level 3
8	06-01-2023	1	equations of equilibrium in terms of displacements	COB-3 CO-3	Lecture Notes	Level 3
9	07-01-2023	1	equations of equilibrium in terms of displacements	COB-3 CO-3	Lecture Notes	Level 3

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Gokaraju Rangaraju Institute of Engineering and Technology (Autonomous)
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SCHEDULE OF INSTRUCTIONS

UNIT PLAN

Academic Year : 2022-23
Semester : I UNIT NO.: 1V
Name of the Program: M.TECH. STRUCTURAL ENGINEERING
Course/Subject: ADVANCED SILID MECHANICS Course Code: GR22D5002
Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING
Designation: PROFESSOR.

Lesson No.	Date	No. of Periods	Topics / Sub - Topics	Objective & Outcome Nos.	References (Text Book, Journal…) Page Nos.: ____to ____	Bloom's Knowledge Levels
1.	11-01-2023	1	Torsion of Prismatic Bars	COB-4 CO-4	Lecture Notes	Level 2
2.	13-01-2023	1	Saint Venant's Method	COB-4 CO-4	Lecture Notes	Level 2
3.	18-01-2023	1	Prandtl's Membrane Analogy	COB-4 CO-4	Lecture Notes	Level 2
4.	20-01-2023	1	Torsion of Rectangular Bar	COB-4 CO-4	Lecture Notes	Level 2
5.	21-01-2023	1	Use of soap films in solving torsion problems	COB-4 CO-4	Lecture Notes	Level 3
6.	25-01-2023	1	Use of soap films in solving torsion problems	COB-4 CO-4	Lecture Notes	Level 3
7	27-01-2023	1	Bending of Prismatic Bars: Stress function.	COB-4 CO-4	Lecture Notes	Level 3
8	28-01-2023	1	bending of cantilever	COB-4 CO-4	Lecture Notes	Level 3
9	01-02-2023	1	circular cross section	COB-4 CO-4	Lecture Notes	Level 3
10	03-02-2023	1	circular cross section	COB-4 CO-4	Lecture Notes	Level 3

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Bachupally, Kukatpally, Hyderabad – 500 090. (040) 6686 4440

SCHEDULE OF INSTRUCTIONS

UNIT PLAN

Academic Year : 2022-23
Semester : I UNIT NO.: V
Name of the Program: M.TECH. STRUCTURAL ENGINEERING
Course/Subject: ADVANCED SILID MECHANICS Course Code: GR22D5002
Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING
Designation: PROFESSOR.

Lesson No.	Date	No. of Periods	Topics / Sub - Topics	Objective & Outcome Nos.	References (Text Book, Journal…) Page Nos.: ____to ____	Bloom's Knowledge Levels
1.	04-02-2023	1	Concepts of plasticity	COB-5 CO-5	Lecture Notes	Level 2
2.	08-02-2023	1	Concepts of plasticity	COB-5 CO-5	Lecture Notes	Level 2
3.	10-02-2023	1	Concepts of plasticity	COB-5 CO-5	Lecture Notes	Level 2
4.	11-02-2023	1	Plastic Deformation	COB-5 CO-5	Lecture Notes	Level 2
5.	15-02-2023	1	Strain Hardening	COB-5 CO-5	Lecture Notes	Level 3
6.	17-02-2023	1	Idealized Stress- Strain curve	COB-5 CO-5	Lecture Notes	Level 3
7.	22-02-2023	1	Yield Criterions	COB-5 CO-5	Lecture Notes	Level 3
8.	24-02-2023	1	Plastic Stress-Strain Relations	COB-5 CO-5	Lecture Notes	Level 3
9.	25-02-2023	1	Failure theories	COB-5 CO-5	Lecture Notes	Level 3

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Gokaraju Rangaraju Institute of Engineering and Technology

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I M.Tech. I Semester 2022-23 I Mid-Term Examinations – Dec 2022

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Name: **Branch: Structural Engineering**

Subject: Advanced Solid Mechanics **Code: GR22D5002**

Date: 26 - 12-2022 (FN)

Objective **(10 X 1 = 10 Marks)**

(Answer All Questions) **Time: 15 min.**

Q. No.	PART-A	CO	BL*	PI
1	The relation between elastic constants E, G and K a) E=9KG/3K+G b) E=KG/3K+G c) E=9KG/K+G d) E=9KG/2K+G	CO1	1	2.1.1
2	In continuum theory, the internal forces are introduced due to a) body forces and surface forces b) contact forces and field forces c) Only body forces d) Only surface forces	CO1	1	2.1.1
3	Why is the strain the fundamental property but not the stress? a) Because it is dimensionless b) Because it is a ratio and it occurs first c) Because it's value is calculated in the laboratory and is independent d) Because stress is a derived property	CO1	2	2.1.2
4	If a material has uniform composition and uniform properties throughout, then it is called a) Homogeneous b) Isotropic c) Continuum d) Heterogeneous	CO1	1	2.1.2
5	Which one is the graphical method to analyze stresses a) Mohr's Circle b) Von Misses c) Moment-Area d) Venn Diagram	CO2	1	2.1.1
6	The body will regain it is previous shape and size only when the deformation caused by the external forces, is within a certain limit. What is that limit? a) Plastic limit b) Elastic limit c) Deformation limit d) Yield limit	CO2	1	2.1.2
7	The materials which have the same elastic properties in all directions are called ___________ a) Isotropic b) Brittle c) Homogeneous d) Hard	CO2	1	2.1.2
8	The slope of the stress-strain curve in the elastic deformation region is _____________ a) Elastic modulus b) Plastic modulus c) Poisson's ratio d) Rigidity modulud	CO2	2	2.1.2
9	The 3x3 matrix form of the stress and strain tensors are_______	CO3	2	2.1.1

10	In any loaded member, there exists a three mutually perpendicular planes on which the Shear stress vanishes (zero),the Three planes are called ___________ and the normal force acting acting on that principal plane are called ____________	CO3	2	2.1.1

Gokaraju Rangaraju Institute of Engineering and Technology

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I M.Tech. I Semester 2022-23 I Mid-Term Examinations – Dec 2022

Subject: Advanced Solid Mechanics **Code: GR22D5002**

Branch: Structural Engineering **Date:** 26 - 12-2022 (FN)

Subjective **(4 X 5 = 20 Marks)**

(Answer Any FOUR Questions) **Time: 105 min.**

Q. No.	PART B	M	CO	BL	PI
1	Identify the plane stress and plane strain problems and derive the corresponding equations	5	CO1	2	4.1.1
2	Derive the strain displacement relations and equations of compatibility	5	CO1	2	2.1.2
3	Develop the differential equations of equilibrium for 2-D problems in elasticity using Cartesian and Polar coordinate system with detailed Illustrations.	5	CO2	2	2.2.1
4	Develop the differential equations of equilibrium for 3-D problems in elasticity using Cartesian and Polar coordinate system with detailed Illustrations	5	CO2	2	2.2.1
5	The three stress components at a point are given by $\begin{pmatrix} 100 & 50 & 60 \\ 50 & 80 & 100 \\ 60 & 100 & 60 \end{pmatrix}$ kPa. Calculate the principal stresses and principal planes	5	CO3	4	3.3.2
6	Discuss the solutions for 2D problems using stress polynomials	5	CO2	3	3.3.2

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Gokaraju Rangaraju Institute of Engineering and Technology

(Autonomous)

I M.Tech. I Semester 2022-23 II Mid-Term Examinations – March 2023

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Name: **Branch: Structural Engineering**

Subject: Advanced Solid Mechanics **Code: GR22D5002**

Date: 03- 03-2023 (FN)

Objective **(10 X 1 = 10 Marks)**

(Answer All Questions) **Time: 15 min.**

Q. No.	PART-A	CO	BL*	PI
1	The state of stress at any point can be characterized by the a) one rectangular stress components b) three rectangular stress components c) six rectangular stress components d) nine rectangular stress components	CO3	1	2.1.1
2	Stresses which can cause the change in volume are called a) Deviatoric stresses b) Octahedral stresses c) shear stresses d) hydrostatic stresses	CO3	1	2.1.1
3	Prandtl introduced the membrane analogy, showing that the torsion in a section is governed by the equation: $\frac{\delta^2\phi}{\delta x^2} + \frac{\delta^2\phi}{\delta y^2} =$ a) -2Gθ b) 2Gθ c) 3Gθ d) 4Gθ	CO4	2	2.1.1
4	In the torsion equation T/J=τ/R=Cθ/l, the term J/R is called a) shear modulus b) section modulus c) polar modulus d) Young's modulus	CO4	1	2.1.1
5	In addition to shear stresses, some members carry torque by axial stresses. This is called a) warping b) shearing c) slipping d) twisting	CO4	2	2.1.1
6	The following analogy is used to describe the stress distribution on a long bar in torsion a) soap film b) plastic membrane c) Pranda film d) bauschinger effect	CO4	2	2.1.1
7	When the load is increased further beyond its elastic limit, a kind of rearrangement occurs at atom level and the mobility of the dislocation decreases makes the metal harder and stronger through the resulting plastic deformation. This process is called a) strain hardening b) strain softening c) necking d) yielding	CO5	1	2.1.1
8	For brittle materials failure is occurred by a) fracture b) yielding c) shearing d) twisting	CO5	2	2.1.1
9	In which of the failure theory, the hydrostatic stresses are significant a) Coulomb-Mohr's theory b) Rankine theory c) Tresca theroy d) von Mises theory	CO5	2	2.1.1
10	Theory which states that the yielding occurs when the maximum shear stress is equal to the shear stress at yielding in a uniaxial tensile test. a) Maximum principal stress theory b) Maximum shear stress theory c) Maximum distortion energy theory d) Maximum strain energy theory	CO5	2	2.1.1

Gokaraju Rangaraju Institute of Engineering and Technology

(Autonomous)

I M.Tech. I Semester 2022-23 II Mid-Term Examinations – March 2023

Subject: Advanced Solid Mechanics **Code: GR22D5002**

Branch: Structural Engineering **Date:** 03- 03-2023 (FN)

Subjective **(4 X 5 = 20 Marks)**

(Answer Any FOUR Questions) **Time: 105 min.**

Q. No.	PART B	M	CO	BL	PI
1	The state of stress at a point is given by $\sigma_{XX} = 10$, $\tau_{XY} = 8$ $\sigma_{YY} = -6$, $\tau_{YZ} = 0$ $\sigma_{ZZ} = 4$, $\tau_{ZX} = 0$ Consider another set of co-ordinate axis X^1, Y^1, Z^1 in which Z^1 coincides with Z-axis and X^1 is rotated by 30° anti-clock wise from the X axis. Determine the stress components in the new system?	5	CO3	4	4.2.2
2	Develop the differential equation for bending of a cantilever by terminal loads with (i) circular section and (ii) with elliptical section	5	CO4	3	3.2.1
3	Discuss about Saint Venant's Semi Inverse Method for prismatic bars under torsion and arrive at shear stress and torque values in terms of stress function Ø. Apply the method to a bar of elliptic c/s to obtain distribution of shear stress and warping displacement in c/s.	5	CO4	3	3.2.1
4	Derive the Saint-Venant's solution for the problem of torsion in straight bars with elliptical cross-section in obtaining shear stress distribution	5	CO4	3	3.3.1
5	Discuss the idealized stress-strain curve and plastic stress strain relations	5	CO5	2	3.1.1
6	Explain the von Mises and Tresca yield criterion	5	CO5	2	3.1.1

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CODE: GR22D5002	GR 22	SET - 2

I M.Tech I Semester Regular Examinations, March/April 2023

ADVANCED SOLID MECHANICS

(Structural Engineering)

Time: 3 hours **Max Marks: 60**

Instructions:

1. Question paper comprises of **Part-A** and **Part-B**
2. **Part-A** (for 10 marks) must be answered at one place in the answer book.
3. **Part-B** (for 50 marks) consists of **five questions with internal choice,** answer all questions.
4. **CO** means Course Outcomes. **BL** means Blooms Taxonomy Levels.

PART – A

(Answer ALL questions. All questions carry equal marks)

10 * 1 = 10 Marks

1.	**a)**	Define stress vector.	**1M**	CO1	BL1
	b)	What are the components of strain tensor?	**1M**	CO1	BL1
	c)	Explain Saint-Venant's Principle	**1M**	CO2	BL2
	d)	What is Biharmonic equation in terms of stress function.?	**1M**	CO2	BL1
	e)	What are the stress invariants?	**1M**	CO3	BL1
	f)	Write a note on stress transformation.	**1M**	CO3	BL2
	g)	Give the expressions for strain energy due to torsion.	**1M**	CO4	BL1
	h)	Explain membrane analogy.	**1M**	CO4	BL2
	i)	What is strain hardening?	**1M**	CO5	BL2
	j)	What is strain energy of deformation?	**1M**	CO5	BL2

PART – B

(Answer ALL questions. All questions carry equal marks)

5 * 10 = 50 Marks

2.	**a)**	State Hooke's law and explain about pure shear.	**5M**	CO1	BL2
	b)	Obtain the Cauchy's Stress Formulae.	**5M**	CO1	BL2
		OR			
3.	**a)**	What is a plane strain? Explain it.	**5M**	CO1	BL2
	b)	Derive the strain displacement relations and equations of compatibility.	**5M**	CO1	BL2

CODE: GR22D5002 | GR 22 | SET - 2

4.	a)	Derive the compatibility conditions for the two-dimensional Cartesian coordinates.	5M	CO2	BL2
	b)	Derive the differential equilibrium equation in polar coordinates for two dimensional elastic bodies.	5M	CO2	BL2
		OR			
5.	a)	Write down the differential equation of equilibrium in a polar coordinate system in 2 dimensions.	5M	CO2	BL2
	b)	Assume the fifth-order polynomial degree for the rectangular beam strip and find Airy's stress function with the different stress components. Analyze the behavior of the beam and draw the stress distribution diagram.	5M	CO2	BL4
6.	a)	Derive the compatibility relation of strain in a 3D elastic body. What it is its significance?	5M	CO3	BL3
	b)	Determine the principal stress tensor at a point in a material if the strain tensor at a point is given below And Poisson's ratio is 0.3. Define stress invariants also.	5M	CO3	BL4

$$\begin{bmatrix} +600 & -200 & +300 \\ -200 & +200 & +450 \\ +300 & +450 & -400 \end{bmatrix} \times 10^{-6}. \quad E = 2 \times 10^5\ N/mm^2$$

OR

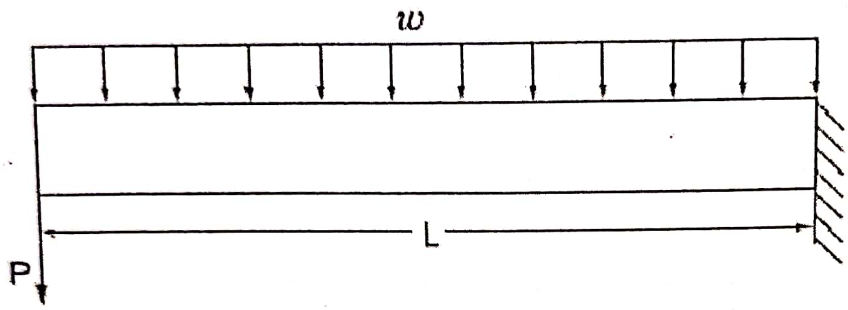
7.	a)	Derive the equation of equilibrium for the 3-D stress state.	5M	CO3	BL2
	b)	What is meant by Homogenous deformation? Explain with examples.	5M	CO3	BL3
8.	a)	Derive an expression for torsion of a bar of narrow rectangular cross-section.	5M	CO4	BL2
	b)	Explain and derive the equation for Prandtl's membrane analogy.	5M	CO4	BL2

OR

CODE: GR22D5002 | GR 22 | SET - 2

9. a) A rectangular beam of width '2a' and '2b' is subjected to torsion. Derive the equation for obtaining maximum shear stress — 5M CO4 BL3

b) The cantilever beam supports a uniformly distributed load w and a concentrated load P as shown in the figure. Also, it is given that L=2m, w=4kN/m, P=6kN, and EI=5 MN.m^2. Determine the deflection at the free end using Castigliano's theorem. — 5M CO4 BL4



10. a) State and explain the assumptions of Plasticity. — 5M CO5 BL2

b) Explain various failure theories. — 5M CO5 BL2

OR

11. a) Explain the Von Mises and Tresca yield criterion. — 5M CO5 BL2

b) What is yield criteria in theory of plastic deformation? — 5M CO5 BL2

*****

1 Derive Equilibrium Equations for a 2 Dimensional State of Stress? [15]

2 What is Plane strain & Plane stress problems? Explain with an example and derive appropriate equations for the above problems? [15]

3 For the stress function $\Phi = -(F/d^3) \times XZ^2(3d-2Z)$, determine the stress components and sketch their variations in a region included in Z = 0, Z = d, X = 0 ,on the side X-positive [15]

4 Derive Equilibrium & Compatibility equations for a body in Polar co-ordinate system? [15]

5 Derive the equation of motion for an damped free vibration of motion for Single degree of Freedom (SDOF) from first principles. Write the equations for maximum displacement amplitude [15]

6 a) A mass of 7 kg is attached to a spring with a stiffness of 4 N/mm. Determine the critical damping coefficient. [8]

b) Derive the expression for the dynamic displacement of an SDOF system for the un damped free vibrations. Sketch the response. [7]

7 A SDOF system is subjected to a harmonic loading defined by $P(t) = P_0 \sin \omega t$. Derive the expression for the dynamic displacement for the under damped vibrations. Sketch the response [15]

8 Discuss about the response of a system under general loading using Duhamel integral [15]

1 Derive the compatibility equations for a 2 Dimensional state of strain? [15]

2 Derive [**C**] which relates stress & strain, for plane stress & plane strain problems? [15]

3 For the following stress function $\Phi = -(H/\Pi)\ Z \tan^{-1}(X/Z)$

Determine the stress components σ_{XX}, σ_{YY} and τ_{XZ} [15]

4 Derive Compatibility & boundary condition for a body in Polar Co-ordinate system? [15]

5 a) Discuss the differences between the Free vibration and the forced vibration [7]

b) Derive the equations of motion for an undamped free vibration using

i) Simple harmonic motion ii) Newton's second law [8]

6 The successive amplitude from afree vibration test for a structure are 0.69, 0.632, 0.1&0.099 units respectively. Determine the damping ratios of the system, considering each cycle separately and considering them all together [15]

7 A SDOF system is subjected to a harmonic loading defined by $P(t) = P_0 \text{ Sin } \omega t$.

Derive the expression for the dynamic displacement for the under damped vibrations. Sketch the response. [15]

8 Discuss about the response of a damped system for an impulsive load by using Duhamel integral. [15]

*****

1 The state of stress is at a point is given by

$\sigma_{XX} = 10$, $\tau_{XY} = -20$

$\sigma_{YY} = 20$, $\tau_{YZ} = 10$

$\sigma_{ZZ} = -10$, $\tau_{ZX} = -30$ [15]

Determine direction cosines of the Principal stresses?

2 Derive Equilibrium Equations for a 3 Dimensional State of Stress [15]

3 For the stress function $\Phi = -(F/d^3)\ x\ X Z^2(3d-2Z)$, Determine the stress components and sketch their variations in a region included in Z = 0, Z = d, X = 0 ,on the side X-positive. [15]

4 Derive Equilibrium & Compatibility equations for a body in Polar co-ordinate system? [15]

5 a) Discuss the differences between the Free vibration and the forced vibration [7]

b) Derive the equations of motion for an undamped free vibration using
i) D'Alembert's principle ii) Newton's second law [8]

6 a) A water tank of weight 150 KN is supported by 4 columns built in at ends, each column has $EI=2x10^6\ KN/m^2$ .calculate the period of vibration of the tank in its horizontal direction. Neglecting the distributed mass of the columns. [7]

b) Find the frequency of oscillation for the floating pole of the cross section area A having a mass M at one end and the density of the pole is ρ. [8]

7 A SDOF system is subjected to a harmonic loading defined by $P(t) = P_0 \sin \omega t$. Derive the expression for the dynamic displacement for the under damped vibrations. Sketch the response. [15]

8 Discuss about the response of a system under general loading using Duhamel integral. [15]

.. ..

1 The state of stress at a point is given by

$\sigma_{xx} = 10$, $\tau_{XY} = 8$

$\sigma_{YY} = -6$, $\tau_{YZ} = 0$

$\sigma_{ZZ} = 4$, $\tau_{ZX} = 0$

Consider another set of Co-ordinate axis X^1, Y^1, Z^1 in which Z^1 coincides with Z-axis and X^1 is rotated by 30^0 anti clock wise from the X axis. Determine the stress components in the new system? [15]

2 Derive Equilibrium Equations for a 3 Dimensional State of Stress [15]

3 Check whether the system of strains is possible for the following strains?

$\epsilon_{XX} = 5 + X^2 + Y^2 + X^4 + Y^4$

$\epsilon_{YY} = 6 + 3X^2 + 3Y^2 + X^4 + Y^4$

$\nu_{XY} = 10 + 4XY\,(X^2 + Y^2 + 2)$ [15]

4 Derive Compatibility & boundary condition for a body in Polar Co-ordinate system? [15]

5 a) Discuss the differences between the Free vibration and the forced vibration [7]

b) Derive the equations of motion for an undamped free vibration using
i) D'Alembert's principle ii) Newton's second law [8]

6 a) A mass of 6 kg is attached to a spring with a stiffness of 3.5 N/mm. Determine the critical damping coefficient. [7]

b) Derive the expression for the dynamic displacement of an SDOF system for the undamped free vibrations. Sketch the response. [8]

7 A SDOF system is subjected to a harmonic loading defined by $P(t) = P_0 \sin \omega t$. Derive the expression for the dynamic displacement for the under damped vibrations. Sketch the response [15]

8 Discuss about the response of a damped system for an impulsive load by using Duhamel integral. [15]

ADVANCED SOLID MECHANICS
ASSIGNMENT 1

1. The three stress components at a point are given by

$$\begin{pmatrix} 100 & 50 & 60 \\ 50 & 80 & 100 \\ 60 & 100 & 60 \end{pmatrix}$$ kPa. Calculate the principal stresses and principal planes

2. Derive the Saint-Venant's equations of compatibility
3. Develop the differential equations of equilibrium for 2-D and 3-D problems in elasticity using Cartesian coordinate system with detailed Illustrations.
4. Develop the differential equations of equilibrium for 2-D and 3-D problems in elasticity using polar coordinate system with detailed Illustrations.
5. Discuss the solutions for 2D problems using stress polynomials.
6. Derive the strain displacement relations
7. Explain Plane stress and Plane strain case

Assignment -2

1. Young's modulus is defined as the ratio of
   a) Volumetric stress and volumetric strain
   b) Lateral stress and lateral strain
   c) Longitudinal stress and longitudinal strain
   d) Shear stress to shear strain
2. When a body is subjected to a direct tensile stress (σx) in one plane accompanied by a simple shear stress (τxy), the minimum normal stress is
   a) $(\sigma x/2) + (1/2) \times \sqrt{(\sigma x^2 + 4\, \tau^2 xy)}$
   b) $(\sigma x/2) - (1/2) \times \sqrt{(\sigma x^2 + 4\, \tau^2 xy)}$
   c) $(\sigma x/2) + (1/2) \times \sqrt{(\sigma x^2 - 4\, \tau^2 xy)}$
   d) $(1/2) \times \sqrt{(\sigma x^2 + 4\, \tau^2 xy)}$
3. The materials which have the same elastic properties in all directions are called ___________
   a) Isotropic b) Brittle c) Homogenous d) Hard
4. As the elastic limit reaches, tensile strain ___________
   a) Increases more rapidly b) Decreases more rapidly
   c) Increases in proportion to the stress d) Decreases in proportion to the stress
5. What the number that measures an object's resistance to being deformed elastically when stress is applied to it?
   a) Elastic modulus b) Plastic modulus c) Poisson's ratio d) Stress modulus
6. Find the strain of a brass rod of length 100mm which is subjected to a tensile load of 50kN when the extension of rod is equal to 0.1mm?
   a) 0.01 b) 0.001 c) 0.05 d) 0.005

7. The law which states that within elastic limits strain produced is proportional to the stress producing it is known as _____________
   a) Bernoulli's law   b) Hooke's law   c) Stress law   d) Poisson's law
8. For an isotropic, homogeneous and elastic material obeying Hooke's law, the number of independent elastic constants is
   a) 2   b) 3   c) 9   d) 1
9. What is Hooke's law for the 1-D system?
   a) The relation between normal stress and the corresponding strain
   b) The relation between shear stress and the corresponding strain
   c) The relation between lateral strain and the corresponding stress
   d) None of the mentioned
10. The slope of the stress-strain curve in the elastic deformation region is ____________
    a) Elastic modulus   b) Plastic modulus   c) Poisson's ratio   d) None of the mentioned

Short answer questions:

1. What are body forces and surface forces?
2. Give the relation between elastic constants.
3. What is stress and strain tensor?
4. Explain the Plane Stress and Plane Strain.
5. What are Lame's constants?
6. What is stress tensor?
7. What are direction cosines?
8. What is strain tensor?
9. What you understand if the material is homogeneous and isotropic?
10. What are Stress Invariants ?
11. What are the Components of Stresses and Strains?

M.Tech I Year I Semester Regular Examinations, March 2023

ADVANCED SOLID MECHANICS

Structural Engineering (Civil Engineering)

Time: 3 hours **Max Marks: 60**

Note:

1. Please verify the regulation of question paper and subject name
2. Question Paper Consists of Part - A and Part - B
3. Assume required data, if not given in the question

Bloom's (Taxonomy) Levels	Percentage of weight age	Marks allotted
BL1 (Knowledge: Remember) BL2 (Comprehension: Understand)	30 to 40	18 to 24
BL3 (Application: Apply) BL4 (Analysis: Analyze)	60 to 70	36 to 42
Total	100	60

PART – A (BL1 to BL4)

(Answer ALL Questions) **(10x1 = 10 Marks)**

1	What are plane strain & plane stress problems?	BL2 CO1 1 M
2	Define Stress and Strain tensors?	BL2 CO1 1 M
3	Explain Saint-Venant's Principle	BL2 CO2 1 M
4	What is Biharmonic equation in terms of stress function.?	BL2 CO2 1 M
5	What is strain rosette?	BL2 CO3 1 M
6	Give expressions for stress and strain invariants?	BL1 CO3 1 M
7	Define warping in torsion	BL2 CO4 1 M
8	Explain Soap film analogy method	BL2 CO4 1 M
9	What is strain hardening?	BL2 CO5 1 M
10	Draw idealized stress-strain curve for mild steel.	BL1 CO5 1 M

PART – B (BL1 to BL4)

(Answer ALL Questions) **(5X10 = 50 Marks)**

Each Question Carries 10 marks and may have a, b. as sub Questions

11	a)	Identify the plane stress and plane strain problems and derive the corresponding equations	BL4 CO1 Marks-5
	b)	Derive the strain displacement relations and equations of compatibility	BL3 CO1 Marks-5
		[OR]	
12	a)	Obtain the Cauchy's Stress Formulae	BL4 CO1 Marks-5
	b)	Derive the expressions for the hydrostatic and deviatoric components	BL3 CO1 Marks-5
13	a)	Develop the differential equations of equilibrium for 2-D problems in elasticity using Cartesian and Polar coordinate system with detailed Illustrations.	BL3 CO2 Marks-5
	b)	Discuss the solutions for 2D problems using stress polynomials	BL4 CO2 Marks-5
		[OR]	
14	a)	Derive Bi-harmonic equation in terms of Airy's stress function	BL3 CO2 Marks-5

	b)	Derive the compatibility equations for a 3-D system	BL3 CO2 Marks-5
15	a)	Develop the differential equations of equilibrium for 3-D problems in elasticity using Cartesian and Polar coordinate system with detailed Illustrations	BL3 CO3 Marks-5
	b)	Write the equation of equilibrium for a 3-D problem in elasticity in terms of displacements	BL3 CO3 Marks-5
[OR]			
16	a)	The three stress components at a point are given by $\begin{pmatrix} 100 & 50 & 60 \\ 50 & 80 & 100 \\ 60 & 100 & 60 \end{pmatrix}$ kPa. Calculate the principal stresses and principal planes	BL4 CO3 Marks-5
	b)	The state of stress at a point is given by $\sigma_{XX} = 10$, $\tau_{XY} = 8$ $\sigma_{YY} = -6$, $\tau_{YZ} = 0$ $\sigma_{ZZ} = 4$, $\tau_{ZX} = 0$ Consider another set of co-ordinate axis X^1, Y^1, Z^1 in which Z^1 coincides with Z-axis and X^1 is rotated by 30° anti-clock wise from the X axis. Determine the stress components in the new system?	BL4 CO3 Marks-5
17	a)	Discuss about Saint Venant's Semi Inverse Method for prismatic bars under torsion and arrive at shear stress and torque values in terms of stress function Ø. Apply the method to a bar of elliptic c/s to obtain distribution of shear stress and warping displacement in c/s.(	BL3 CO4 Marks-5
	b)	Derive the Saint-Venant's solution for the problem of torsion in straight bars with elliptical cross-section in obtaining shear stress distribution	BL4 CO4 Marks-5
[OR]			
18	a)	Explain Soap film analogy method and its applications	BL3 CO4 Marks-5
	b)	Develop the differential equation for bending of a cantilever by terminal loads with (i) circular section and (ii) with elliptical section	BL3CO4 Marks-5
19	a)	State and explain the assumptions of Plasticity	BL2 CO5 Marks-5
	b)	Explain various failure theories	BL3 CO5 Marks-5
[OR]			
20	a)	What is yield criteria in theory of plastic deformation?	BL2 CO5 Marks-5
	b)	Explain the Von Mises and Tresca yield criterion	BL3 CO5 Marks-5

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Assignment 5

1. Define Stress and Strain
2. Explain stress ellipsoid
3. What are stress invariants?
4. Define principal stress and the principal planes
5. Explain the stress concentration factor
6. Explain the Strain components in polar coordinates.
7. Explain plane stress and plane strain cases
8. What is a strain rosette?
9. Explain Saint-Venant’s Principe.
10. Give the basic equations of equilibrium
11. Explain the phenomenon “Strain Hardening”.
12. State “Maximum principal stress theory”.
13. Explain the equations of compatibility.
14. State the stress and strain transformation laws.
15. Establish the relationship between various constants of elasticity.
16. Define bending stress and shear stress
17. What are 2 dimensional rectangular coordinates.
18. Deifne torsion
19. Establish the torsional
20. Name the Theories of Failure and their limitations.
21. Explain Hooke’s law
22. What are stress strain relations.
23. explain stress & strain components.
24. Define boundary conditions
25. Stress invariants
26. Explain membrane analogy
27. Explain soap film method
28. Explain the principle of superpositions
29. write the assumptions of plasticity
30. Evaluate shear stresses in a rectangular section of a cantilever beam loaded at the free end.
31. Explain the different theories failure and write yield criterion for each.
32. Explain Homogeneous deformation
33. Define warping.
34. Define state of plasticity
35. Obtain the strain displacement relations.
36. Explain airy’s stress function
37. Explain stress tensor and strain tensor.
38. What do you understand about stress function.
39. Explain Stress- Strain diagram of mild steel
40. What is principle of virtual work
41. What is principle of superposition
42. Uniqueness theorem
43. Recipocal theorem
44. Octohedral stresses and plane
45. Types of stresses and strains
46. Body forces and surface forces
47. Define strain energy
48. What is Shear centre
49. Stress – strain relations using Lame’s constants
50. Cauchy’s strain displacement relations
51. Relation between elastic constants
52. Elasticity and Plasticity difference
53. Shear flow means
54. Maximum shear stress equation
55. Bending equation
56. Torsion equation
57. Isotropics means
58. Assumptions of elasticity
59. State of pure shear

Assignment 3

The state of stress at a point with respect to x,y,z system is $\begin{pmatrix} 10 & 5 & -15 \\ 5 & 10 & 20 \\ -15 & 20 & 25 \end{pmatrix}$ kN/sq.m Determine the stress relative to x^1, y^1, z^1 coordinate systems obtained by a rotation through 45^0 about Z axis.
Obtain equilibrium equations and boundary conditions and hence arrive at compatibility condition in term of stress components for a plane stress condition.
A thick cylinder is subjected to internal and external pressures define equations for radial and circumferential stresses at the boundaries.

Assignment 4

1. State and explain saint venants semi inverse method for prismatic bars under torsion. Hence arrive at shear stress and torque values in terms of stress function Ø. Applying the same to a bar of elliptic c/s obtain distribution of shear stress in the c/s and warping displacement in c/s.
2. Explain any three Theories of Failure and give the governing equations. Also explain the limitations of those theories.
3. At a point in a stressed body, the Cartesian components of stresses are □x =80 MPa, □y = 50 MPa, □z = 30 MPa, □xy =30 MPa, □yz =20 MPa, □zx =40 MPa. Determine a) the normal and shear stresses on a plane whose normal has the direction cosines of cos(n,x)= 1/3 , cos(n,y)=2/3, cos(n,z)=2/3 ; b) angle between resultant stress and outward normal n.
4. Explain membrane analogy.Apply this analogy to the problem of bar of narrow rectangular section subjected to torsion and calculate the stresses.

ADVANCED SOLID MECHANICS

Assignment 1

Subjective

1. State and explain Saint-Venant's semi-inverse method for prismatic bars under torsion. Hence determine torsional moment and shear stresses in terms of Prandtl's stress function Ø.
2. Derive the Saint-Venant's solution for the problem of torsion in straight bars with circular and elliptical cross-section in obtaining shear stress distribution.
3. Determine the stresses due to bending of a prismatic cantilever subjected to terminal load and having circular cross-section.
4. a) Explain Soap film analogy method or Membrane Analogy approach for torsional problems

   b) Explain in detail the different theories of failure and write yield criterion for each.

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GOKARAJU RANGARAJU INSTITUTE OF ENGINEERING AND TECHNOLOGY

Department of Civil Engineering

ADVANCED SOLID MECHANICS

Assignment 1

Objective

1. According to this theory, the maximum principal stress in the material determines failure regardless of the other two principal stresses which are algebraically smaller.
   a) Maximum Principal Stress Theory b) Maximum shearing stress theory
   c) Maximum elastic energy theory d) Energy of distortion theory
2. Permanent deformations involve the dissipation of energy; such processes are termed
   a) Reversible b) Irreversible c) Does not change d) Statement is wrong
3. The ____________ are independent of the rate of deformation (or rate of loading)
   a) Plastic deformation b) elastic deformation c) viscoplastic d) viscoelastic
4. _______theory deals with yielding of materials under complex stress states
   a) Plasticity b) Elasticity c) Elasto-plasticity d) Rankine's
5. One aspect of plasticity in the viewpoint of structural design is that it is concerned with predicting the_________, which can be applied to a body without causing excessive yielding.
   a) Maximum load b) Maximum moment c) Maximum shear d) Service load
6. If specimen is deformed plastically beyond the yield stress, it is found that the yield stress on reloading in compression is less than the original yield stress. The dependence of the yield stress on loading path and direction is called the
   a) Bauschinger effect b) Tresca effect c) Von mises effect d) St.venant's effect
7. A true stress – strain curve provides the stress required to cause the material to flow plastically at any strain. This is often called as
   a) flow curve b) force-displacement curve c) stress-strain curve d) true curve
8. In formulating a basic plasticity theory the following assumption is not correct
   a) No Bauschinger effect b) the response is independent of rate of loading or deforming
   c) The material is isotropic d) the material is compressible even in the plastic range
9. Which of the following matches are correct
   1. Maximum Principal Stress Theory - Rankine
   2. Maximum shearing stress theory - Tresca
   3. Maximum elastic energy theory- Beltrami
   4. Energy of distortion theory - Von Mises
   5. Maximum Principal strain theory – St,Venant

   a) 1, 2,3,4,5 b) 1,2,4,5 c) 1,2,3,4 d) 1,3,4,5

10. At ________region in stress- strain curve, with the increasing stresses, stacking up of atoms happens. This provides resistance to the dislocation movement and thereby decreasing the the deformation and increasing the strength of material.
    a) Strain hardening b) Strain softening c) Necking d) Yielding

****

GOKARAJU RANGARAJU INSTITUTE OF ENGINEERING AND TECHNOLOGY

Department of Civil Engineering

ADVANCED SOLID MECHANICS

Assignment 2

1. Develop the differential equations of equilibrium for 2-D and 3-D problems in elasticity using Cartesian coordinate system with detailed Illustrations.
2. Develop the differential equations of equilibrium for 2-D and 3-D problems in elasticity using polar coordinate system with detailed Illustrations.
3. The state of stress at a point with respect to x,y,z system is

$$\begin{pmatrix} 10 & 5 & -15 \\ 5 & 10 & 20 \\ -15 & 20 & 25 \end{pmatrix} \text{kN/sq.m.}$$

   Determine the stresses relative to x^1, y^1, z^1 coordinate systems obtained by a rotation through 45^0 about Z axis.

4. The three stress components at a point are given by

$$\begin{pmatrix} 100 & 50 & 60 \\ 50 & 80 & 100 \\ 60 & 100 & 60 \end{pmatrix}$$ kPa. Calculate the principal stresses and principal planes

5. Derive the Saint-Venant’s equations of compatibility.

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Department of Civil Engineering

ADVANCED SOLID MECHANICS

Assignment 2

1. Young's modulus is defined as the ratio of
   a) Volumetric stress and volumetric strain
   b) Lateral stress and lateral strain
   c) Longitudinal stress and longitudinal strain
   d) Shear stress to shear strain
2. When a body is subjected to a direct tensile stress (σx) in one plane accompanied by a simple shear stress (τxy), the minimum normal stress is
   a) $(\sigma x/2) + (1/2) \times \sqrt{(\sigma x^2 + 4\, \tau^2 xy)}$
   b) $(\sigma x/2) - (1/2) \times \sqrt{(\sigma x^2 + 4\, \tau^2 xy)}$
   c) $(\sigma x/2) + (1/2) \times \sqrt{(\sigma x^2 - 4\, \tau^2 xy)}$
   d) $(1/2) \times \sqrt{(\sigma x^2 + 4\, \tau^2 xy)}$
3. The materials which have the same elastic properties in all directions are called __________
   a) Isotropic b) Brittle c) Homogenous d) Hard
4. As the elastic limit reaches, tensile strain __________
   a) Increases more rapidly b) Decreases more rapidly
   c) Increases in proportion to the stress d) Decreases in proportion to the stress
5. What the number that measures an object's resistance to being deformed elastically when stress is applied to it?
   a) Elastic modulus b) Plastic modulus c) Poisson's ratio d) Stress modulus
6. Find the strain of a brass rod of length 100mm which is subjected to a tensile load of 50kN when the extension of rod is equal to 0.1mm?
   a) 0.01 b) 0.001 c) 0.05 d) 0.005
7. The law which states that within elastic limits strain produced is proportional to the stress producing it is known as _____________
   a) Bernoulli's law b) Hooke's law c) Stress law d) Poisson's law
8. For an isotropic, homogeneous and elastic material obeying Hooke's law, the number of independent elastic constants is
   a) 2 b) 3 c) 9 d) 1
9. What is Hooke's law for the 1-D system?
   a) The relation between normal stress and the corresponding strain
   b) The relation between shear stress and the corresponding strain
   c) The relation between lateral strain and the corresponding stress
   d) None of the mentioned
10. The slope of the stress-strain curve in the elastic deformation region is ____________
    a) Elastic modulus b) Plastic modulus c) Poisson's ratio d) None of the mentioned

SEMESTER I

CORE 2 - ADVANCED SOLID MECHANICS (CREDITS - 3)

Course objectives:

1. To explain the theory, concepts and principles of Elasticity
2. To generalize the equations of elasticity and their correlations
3. To demonstrate the Two-Dimensional Problems of Elasticity in terms of Cartesian and polar coordinates
4. To apply principles of elasticity to analyze the torsion in prismatic bars
5. To extend the principles of stress/strain for plastic deformation to study the modes of failure

Course Outcomes:

At the completion of this course, the student is expected to be able to –

1. Develop equations of equilibrium and draw relations among stress, strain and displacements
2. Utilize equations of elasticity such as equilibrium equations, compatibility equations and various boundary conditions to analyze elastic problems.
3. Gain the understating of Two-Dimensional Problems of Elasticity in Cartesian and polar coordinates system
4. Apply the principles of elasticity to solve torsional problems in prismatic bars and tubes.
5. Use the concepts of stresses and strains for plastic deformation to comprehend the yield criteria of materials.

Syllabus Contents:

UNIT 1: Introduction to Elasticity: Displacement, Strain and Stress Fields, Constitutive Relations, Cartesian Tensors and Equations of Elasticity. Strain and Stress Field: Elementary Concept of Strain, Stain at a Point, Principal Strains and Principal Axes, Compatibility Conditions, Stress at a Point, Stress Components on an Arbitrary Plane, Differential Equations of Equilibrium, Hydrostatic and Deviatoric Components.

UNIT 2: Equations of Elasticity: Equations of Equilibrium, Stress- Strain relations, Strain Displacement and Compatibility Relations, Boundary Value Problems, Co-axiality of the Principal Directions.

UNIT 3: Two-Dimensional Problems of Elasticity: Plane Stress and Plane Strain Problems, Airy's stress Function, Two-Dimensional Problems in Polar Coordinates.

UNIT 4: Torsion of Prismatic Bars: Saint Venant's Method, Prandtl's Membrane Analogy, Torsion of Rectangular Bar, Torsion of Thin Tubes.

UNIT 5: Plastic Deformation: Strain Hardening, Idealized Stress- Strain curve, Yield Criteria, Von Mises Yield Criterion, Tresca Yield Criterion, Plastic Stress-Strain Relations, Principle of Normality and Plastic Potential, Isotropic Hardening.

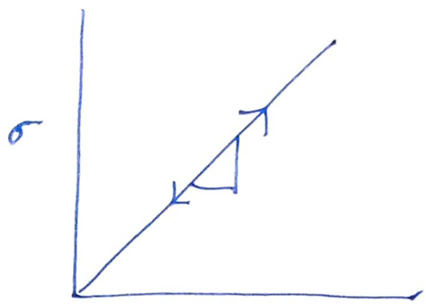
References:

1. Theory of Elasticity, Timoshenko S. and GoodierJ. N., McGraw Hill, 1961.
2. Elasticity, Sadd M.H., Elsevier, 2005.
3. Engineering Solid Mechanics, Ragab A.R., Bayoumi S.E., CRC Press, 1999.
4. Computational Elasticity, Ameen M., Narosa, 2005.
5. Solid Mechanics, Kazimi S. M. A., Tata McGraw Hill, 1994.
6. Advanced Mechanics of Solids, Srinathl .S., Tata McGraw Hill, 2000.

Tensor - stress, strain

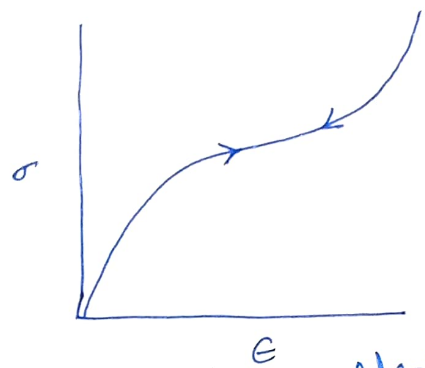
What is elasticity?

tendency of material to return to its original shape and size when forces causing deformation are removed.



Linear elasticity

load and unload
straight line $y = mx + c$ — m is slope
slope of the line is 'E' young's modulus.



Non-linear elasticity.

curve can be any function but not straight line

Elasticity < linear / non-linear.

stress need not be proportional to strain to be elastic. So elasticity can be linear or non-linear.

Perfectly elastic - if same loading and unloading paths are same - no dissipation of energy

Area under stress-strain curve is strain energy.

* If the area under loading curve and unloading curve are different of the area under loading and unloading are different then strain energies for loading and unloading are different then cannot be elasticity possible case.

In case of elasticity, energy losing or adding to the system cannot happen to elastic systems.

linear elastic (or) non-linear elastic materials
or
Hyper elastic materials

Perfect Elasticity — { State of stress at any time is independent of previous history of stresses.
— stress is unique function of strain
↳ derivative of strain will be stress.
$\sigma = E\epsilon$ linear function.

↓

time independent

If how I load and unload the material specimen it depends on previous history — then it is time dependent

Non-linear elastic material

↳ stress is non linear function of strain

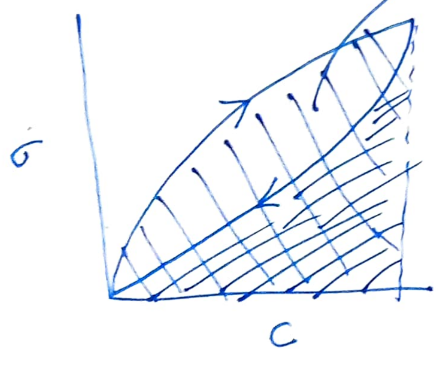
↳ No Hooke's Law is valid.

$\sigma = E_1 \epsilon^2 + E_2 \epsilon$ etc.

non-linear function

Partial derivative of S.E function w.r.t strain gives stress for both linear and non-linear elastic cases.

Visco-elastic Material → Loss of energy or dissipated



S.E during loading and S.E during unloading are different.
energy is lost — out of the system
no residual strain means it reaches to the point where it starts
↓
permanent deformation

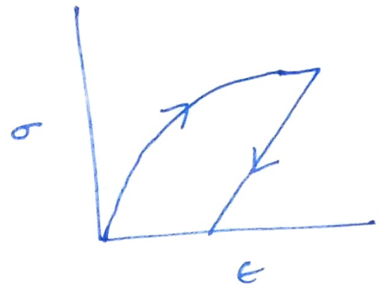
* This material is not elastic because it loss energy.

This happens because stress not only depends on the strain but also on the rate of strain

$\sigma = \sigma(\epsilon, \dot{\epsilon})$

$\dot{\epsilon} = \frac{\partial \epsilon}{\partial t}$

Inelastic material



- permanent or residual strains when body is unloaded. It does not come to original position.
- Dissipation of energy happens converted to heat.
- called as plastic material
- work with increments of strain than total accumulated strain as stress is not a unique function of strains

$$\epsilon = \int_{i=1}^{n} \epsilon_i$$

Why TOE?

SOM :- assumptions are made for closed form solution → valid for slender member

example :- flexural formula

$$\boxed{\frac{M}{I} = \frac{f}{y} = \frac{E}{R}}$$

assumptions:

- plane section remain plane before and after bending only for slender members and for pure bending case.
- plane section remain plane before and after bending is not valid for cantilever beam where sections do not remain plane

* Every member is a 3D members so effect of c/s should be considered not only longitudinal dimensions should be taken into account. c/s stress strain is also important like longitudinal stress-strain.

SOM will not account c/s stress-strain

SOM is for slender member only.

Torsion using SOM assumptions are correct for circular sections for rectangular sections SOM is not valid because rectangular ... [illegible] ... warping

To solve such problem TOP is used.

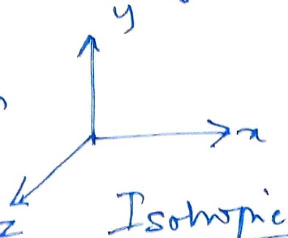
↳ forming governing d.E and apply boundary conditions to solve the problem without any simplified assumptions.

Some assumptions are true for certain classes of problems

Many problems need more knowledge than some assumptions

Isotropic - in any direction material properties are same
↓
E, ν, G, K etc

ex: steel material remains same in all directions.



Isotropic material can be characterized by simple relations in x, y and z directions
or
using very less independent constants material can be characterized.

only 2 constants:

$$\begin{cases} E_x = E_y = E \\ \nu_{xy} = \nu_{yx} = \nu \\ G_{xy} = G = \dfrac{E}{2(1+\nu)} \end{cases}$$

Anisotropic material (composites)

Cannot be characterized by simple relations or requires many independent constants.

$$\left.\begin{aligned} E_x &\neq E_y \\ \nu_{xy} &\neq \nu_{yx} \\ G &= \frac{E}{2(1+\nu)} \end{aligned}\right\}$$ total 5 constants are required.

← this can be one in 2D case

Concrete - heterogeneous, Isotropic material
↓
material Isotropy may not be necessarily true.

Tensor Algebra

Scalar – has only one value

vector – magnitude or value and direction.

Tensor – One magnitude and two directions.

↓

Second order tensor

4th order tensor – four direction which is unphysical from the 3D systems of coordinates.

σ_{xy} or σ_{12} – { normal is along 1, acting along the direction of 2

↓

normal direction is along x and is acting along y direction.

<u>Identity matrix</u> is second order tensor.

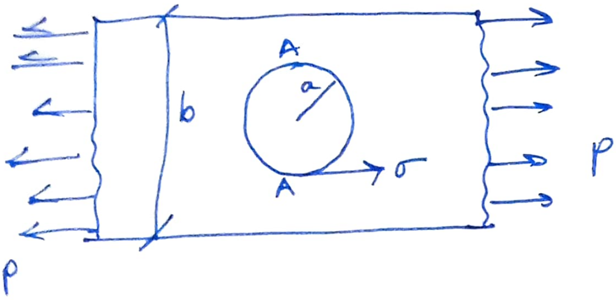
* <u>4th order tensor</u> is a transformation which transfers a second order tensor (higher order tensor) to ~~any~~ another second order tensor.

* <u>2nd order tensor</u> is transformation which transfers a vector to another vector.

<u>e_i is unit vector</u>

Theory of elasticity deals with the rigorous mathematical modelling techniques, numerically computing methods and experimental findings of the stresses, strains and displacements within an elastic body knowing the boundary conditions on it explicitly. Thus these are essentially the <u>boundary value problems</u>.

At <u>sharp corners</u> and <u>discontinuities</u>, the stress concentration occurs where the fracture originates. These increased stresses <u>cannot be predicted by SOM</u> so <u>TOE principles</u> are used to find the solution.



Circular hole of radius 'a' if ($b \approx 20a$) is large, the SOM solution gives stress A in the horizontal direction $\sigma_x = p$ but TEP the stress point A is $\sigma_x = 3p$.

If the width of this plate is small, ie $b \simeq 4a$, then SOM gives stress at a point A is $\sigma_x = 2p$, where as TEP gives solution $\sigma_x = 4.3p$.

SOM cannot predict the stress directly under a load or at supports in a simple beam, whereas photoelasticity can give the exact experimental value of these stresses.

In SOM, factor of safety is more (more factor of ignorance) where as in TEP factory of safety is very less.

Where as TEP solution is accurate where as SOM solution is approximate.

∇ - Del / Nabla

α - alpha
β - beta
Γ / γ - gamma
δ / Δ - delta
ϵ - epsilon
ζ = zeta
η - eta
θ - theta
κ - Kappa
λ - lambda
μ - Mu
ν - Nu
ξ - Xi
π - pi
ρ - Rho
σ - sigma
τ - tau
υ - upsilon
ϕ - phi
χ - chi
ψ - psi
ω / Ω - omega

· TEP gives solutions to many engineering problems.

SOM - elementary theory inadequate to give complete picture of stress distribution in engineering structures.

SOM limitations:-

→ insufficient to give information regarding local stresses near the loads and near the supports of beams.

→ stresses in the case of rollers and in balls of bearings can be found only by using the methods of TEP.

→ does not give the distribution of stresses in regions of sharp variations in c/s of beams or shafts.

In TEP, we will be evaluating stress (force experienced at every point in space) and strain (the displacement caused).

→ In elementary theory approach, assumed deformation is used and hence the average stress at a section is obtained under a given loading. SOM (elementary theory approach) treats separately each simple type of complex loading for example axial centric, bending or torsion.

→ Formulae of elementary theory is best suited for slender members and are derived based on very restrictive conditions.

→ In TEP, no assumption is made regarding stress distribution. Hence, Hooke's law cannot be applied directly.

→ TEP does not depend on prescribed deformation mode and deals with general equations to be satisfied by a body in equilibrium under any external force system.

→ TEP is preferred when critical design constraints such as minimum weight, minimum cost or high reliability are to be obtained exactly with no errors.

or when no prior experience in solving the problems. or not simple assumptions are not adequate to find the solutions to the problems of engineering.

of applied properly TEP will yield results or solutions more closely approximating the actual distributions of strain, stress and displacement.

→ In elementary theory (SOM), distribution of strain and corresponding stresses are assumed. It is assumed that strain due to bending varies linearly over the c/s and the stresses are obtained by Hooke's law and the stress distributions in the body is simplified. (same throughout - assumption in SOM).

→ In TEP, Hooke's law is valid but no assumption is made regarding the stress distribution. Hence Hooke's law cannot be used directly. Where as in SOM (elementary theory) distributions of stresses and strains can be obtained by Hooke's law - is the assumption.

→ optical instrument (→ physics) are used to measure displacements, slope and curvature. For each parameter measurement we have different optical measurements.

* Stress analysis { 6 stress components
↓ 6 strain components
stress resultants 3 displacement components.

at every point of body (away from points of grip)

* closed form solutions → solution at every point can be obtained using all stress components. Always we don't require 15 components.

$\sigma = \frac{P}{A}$

$\sigma_y = \frac{P}{A}$

Stress is a tensor of rank 2 — $3^2 = 9$ components

Strain is a tensor of rank 2

Analysis – at all points

Design point of view – only at a particular point is considered.

* Stress Analysis can be done Analytical method, Experimental method and Numerical methods.

Each method has its own advantages

In complex problem situations, you may have to use combinations called hybrid analysis..

* Analytical Methods (Based on concepts)
  - Conceptual approach
  - Two methods — SOM (Elementary approach)
    - MOS < TOE, TOP

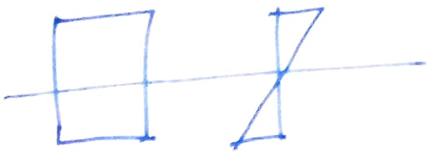
In both the above methods, the range of problems that can be solved is limited.

* SOM assumptions :-

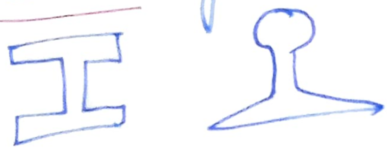
Plane sections remain plane before and after loading. This assumption is applicable only to slender sections. If you have hole in an object, this assumption is not valid.

Assumptions are not applicable at supports only at the centre.

* Stress is varying linearly at centre but not at support



So inner core is not contributing to load sharing



Cantilever - donot remain plane - shear exists

Deep beams - have shear

→ SOM can be applicable to axial members, bending members and circular sections subjected to torsion.

→* Circular shaft (hollow shaft) - shear varies linearly.

SOM solves circular shaft subjected to torsion.

- ↳ avoids use of differential equations
- ↳ applies to slender members and simple problems
- ↳ cannot be applied to rectangular shaft subjected to torsion.

Nature laws understand the solid mechanics
to
* humans understand these principles and embed it in the form of mathematics and develop into engineering tools

SOM (one dimensional) — assumptions on displacements
— No differential equations, No need for B.C's

TEP of
MOS (2D) - No assumptions on displacements, displacements are part of solution in D.Es

↳ applies to problems where boundaries are easily identified by coordinate system. So BC's are to be defined.

→ In TEP (2D & Partly 3D)

problems with small holes can be solved. If the problems are with big holes then FEM is applied.

Stress fields near big holes are different from the small hole.

→ TEP gives closed form solutions such as small hole in object, circular disc subjected to compressive loading and for TEP or MOS is applied only if BC's are amenable.

2D problems

All 3D problems cannot be solved using TEP because 3D shapes are arbitrarily shaped.

*Numerical Methods.

↳ arbitrary shaped (complex geometry)

↳ 3D problems

Other methods are
- Finite difference Method
- Finite element Method
- Boundary elements

| software packages available

In numerical approach to problems - numerical model is to be validated for BC's - to be handled carefully - Once validated ideal choice for parametric analysis.

*Experimental Methods - on prototype or on Model
- for complex geometry

Model -
- not functional
- can be of any scale
- for display
- cheap

Prototype -
- fully functional
- expensive
- actual version of main product
- close to reality

* Experimental methods are best to solve problems on prototypes.

In analytical and numerical methods, all the stress components are evaluated first and then get strain and displacements but this is not the case in experimental methods, not all components are found.

* TOE procedure consists of assuming a stress distribution, which is then checked to see whether the conditions are satisfied or not. If it is satisfied, then the assumed stress distribution is valid for the given problem else search for new stress distribution.

* Conditions of TOE :-

The distribution of stress, strain and displacement (assumed) within the body which is subjected to prescribed system of forces requires to satisfy various conditions-

1. equations of equilibrium
2. stress-strain relation (Hooke's law)
3. Conditions of compatibility
4. Boundary conditions

1) Equations of equilibrium -

equations of statics to be satisfied throughout the body.
By considering the static equilibrium of a solid subjected to the body force and applying newton's laws of motion will lead to set of differential equations which govern the stress distributions within the solid.

2) Stress-strain relation (Hooke's law)

Constitutive relations connects stress and strain fields in terms of material properties.

3) Compatibility conditions

The geometry of deformation and the distribution of strain must be consistent with the preservation of body's continuity.

4) Boundary Conditions

This is the conditions of loading imposed at the boundaries.
If the problem is ~~static~~ dynamic then the equations of equilibrium becomes the more general conservation of momentum, and may be further requirements to be conservation of energy satisfied.

Assumptions -

SOM (Elementary theory) and Theory of linear elasticity

To evaluate the stresses, strains and displacements in an elasticity problems, we need to derive set of basic equations and boundary conditions. During this process of deriving equations consider all the factors influencing the equations so the result obtained is very complicated and practically no solution can be found. So the basic assumptions are to be made about the properties of the body (influential factors) to arrive at possible solutions. Some influential factors are neglected based on their importance and effect on result approximation.

The Assumptions are -

1. Linearity is assumed
2. Body is continuous
3. Body is perfectly elastic
4. Body is homogeneous
5. Body is Isotropic
6. Displacements and strains are small

① Linearity :- Two types of linearity are normally assumed.

Material linearity (known as Hookean stress-strain behaviour or linear relation between stress and strain)

Geometric linearity (strains and deformations are very small).

② Continuous :-

Whole volume of the body is considered to be filled with continuous matter, without any voids. So we assume that physical quantities in the body such as stresses, strains and displacements will be continuously distributed and are expressed

by continuous functions of the coordinates in space.
This assumption holds true or valid as long as the dimensions of the body is very large in comparision with those of the particles and with the distances between neighbouring particles.

③ Perfectly elastic -

Body is considered to obey Hooke's law of elasticity so linear relations between the stress components and strain components are independent. Elastic perfectly means elastic constants are independent of the magnitudes of stress and strain components.

Deformations due to external loads are completely and instantaneously reversible upon load removal. Almost all materials possess to a certain degree the property of elasticity. If the external forces producing deformations do not exceed a certain limit, the deformation disappears with the removal of forces.

Depending on the nature of loads, problems of TOE are classified into elastostatics and elastodynamics.

elastostatics - loading is assumed to be independent of time or loading is static

All the loads acting on solids and structure are always dependent on time but this dependence can be neglected without any appreciable error. In elastodynamics, dynamic loading will be considered.

④ Homogenous -

If elastic properties are same through out the body. Elastic constants are independent of the location in the body. Hence we can analyze an elementary volume isolated from the body and then apply the same results of analysis to the entire body.

(5) Isotropic :-

Elastic properties in a body are the same in all directions.

Elastic constants will be independent of the orientation of coordinate axes.

In reality, none of the structural materials are purely homogenous and isotropic. Materials like steel satisfy the requirements of homogenous and Isotropy to a certain extent so the theory of elasticity principles can be applied to the analysis of steel structures with great degree of accuracy.

* Main advantage of assuming a material to be homogenous and Isotropic is that elastic properties such as E, ν can be assumed to be constant. else analysis will be complicated

→ Displacement components of all points of the body during deformation are very small in comparision with its original dimensions. The strain components and the rotations of all the line elements are much smaller than unity.

Hence when formulating the equilibrium equations relevant to the deformed state, the lengths and angles of the body before deformations are used.

In equations involving strains and displacements are formulated, the squares and products of the small quantities are usually neglected.

So linear algebraic and differential equations in elasticity are used to obtain solutions for the problems assuming that displacements and strains are small.

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* Science - finding facts through observations and experiments
  - scientia means knowledge

Systematic Comprehensive Investigation and Exploration of Nature's Causes and effect.

* Important properties of materials are elasticity, plasticity, brittleness, malleability and ductility.
- materials cannot exhibit simultaneously all the above properties.

Cast Iron - brittle
lead - plastic
Wrought Iron - malleable
Copper - ductility
mild steel - elasticity

* Elastic material - undergoes deformation when subjected to an external loading and deformation disappears on the removal of the loading. ex: Rubber

* Plastic material - undergoes continuous deformation during the period of loading and the deformation is permanent and the material does not retain its original dimensions on the removal of the loading. - ex: Aluminium

* Rigid material - does not undergo any deformation when subjected to an external loading. - glass / cast Iron

* Malleability - ability to form sheets or thin sheets - lead

* Brittle - relatively small extensions to fracture
no necking at failure for brittle materials

* The purpose of application of elasticity is analyse for stresses and displacements of elements within the elastic range and to check the sufficiency of their strength, stiffness and stability.

* Mechanics of materials deals with the stresses and displacements of structural or machine element of any shape subjected to tension, compression, shear, bending or torsion.

* Structural mechanics on the basis of mechanics of materials deals with the stresses and displacements of a structure as a whole such as truss or rigid frame.
If the structural elements are not in the form of bars such as blocks, plates, shells, domes and foundations are analysed using TOE. If you want to analyse an element at each and every point then principles of TOE are used.
Methods of analysis in Mechanics of materials and TOE are not same.
* For analysis assumptions are made on the strain condition or stress distribution, to simplify the mathematical derivation depends on degree of accuracy of the results.
* TOE gives more accurate results than mechanics of materials.
* Assumptions of SOM leads to the uniform or linear distribution of bending stresses. But TOE principles won't assume the assumptions of SOM and also proves that stress distribution will be far from linear variation. In tension members with hole with SOM principles it is assumed that the tensile stresses are uniformly distributed across the net section of the member where as in exact analysis of TOE shows that stresses are not uniform but are concentrated near the hole. Max. stress at the end of the hole is far greater than the average stress across the net section.

Components of deformations (x, y, z direction) are ϵ_x ϵ_y ϵ_z γ_{xy} γ_{yz} γ_{zx}

Components of displacements (x, y, z direction) are u, v, w resp.

* Theory of elasticity uses -

1. Equilibrium equations relating to stresses
2. Kinematic equations relating the strains and displacements (Compatibility equations)
3. Constitutive equations relating the stresses and strains
4. B.C's relating to the physical domains
5. Uniqueness constraints relating to the applicability of the solution.

→ Elastic behaviour implies the absence of any permanent deformation. Deformation disappears with the removal of the forces.

→ Engineering materials possess a certain extent of elasticity. Construction materials however remain elastic only for very small strains before exhibiting either plastic straining or brittle failure.

* Types of forces :-

In continuum theory, the internal forces are introduced due the body forces or field forces and surface forces or contact forces.

Surface forces or contact forces - acting on the common surfaces of contact between two bodies or fluids and transferred indirectly to inside the body through media.

- Also known as surface tractions and are expressed in terms of per unit area of the surface influenced by the action.

example :- air pressure, water pressure, earth pressure and any force applied on the surface of a body (N/mm²)

* Body forces or field forces

- any solid body is formed of molecules which are made up of atomic particles or atoms.
- the internal forces within the continuous body are those due to the interaction between the molecules.

In the absence of any external forces, these internal forces keep the body in equilibrium and produce no deformation. But due action of external forces the body will deform and involves changes in position of molecules and distance between them so that internal forces will change between the molecules. These forces are distributed throughout the mass of the body and are exerted by agencies not in contact with the body.

- expressed in terms of force per unit volume or per unit mass of the body

example :- gravitational force, centrifugal force, electromagnetic forces, inertia forces (in motion) etc. (N/mm^3)

Components of body force per unit volume in Cartesian coordinate directions are F_x, F_y and F_z.

Body force is a vector.

Body force vector (matrix notation)

$$B = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \text{ or } \{F_x \; F_y \; F_z\}$$

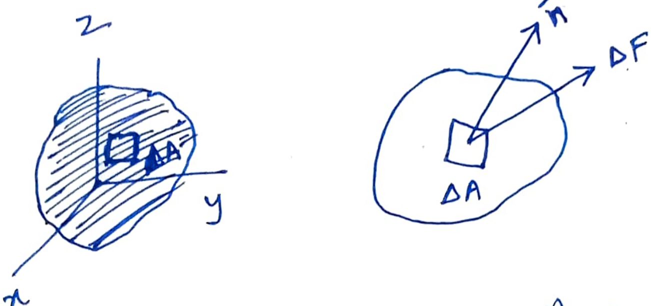
Matrix with only one element is called scalar.

* Stress field - is the distribution of internal 'tractions' that balance a given set of external tractions and body forces.

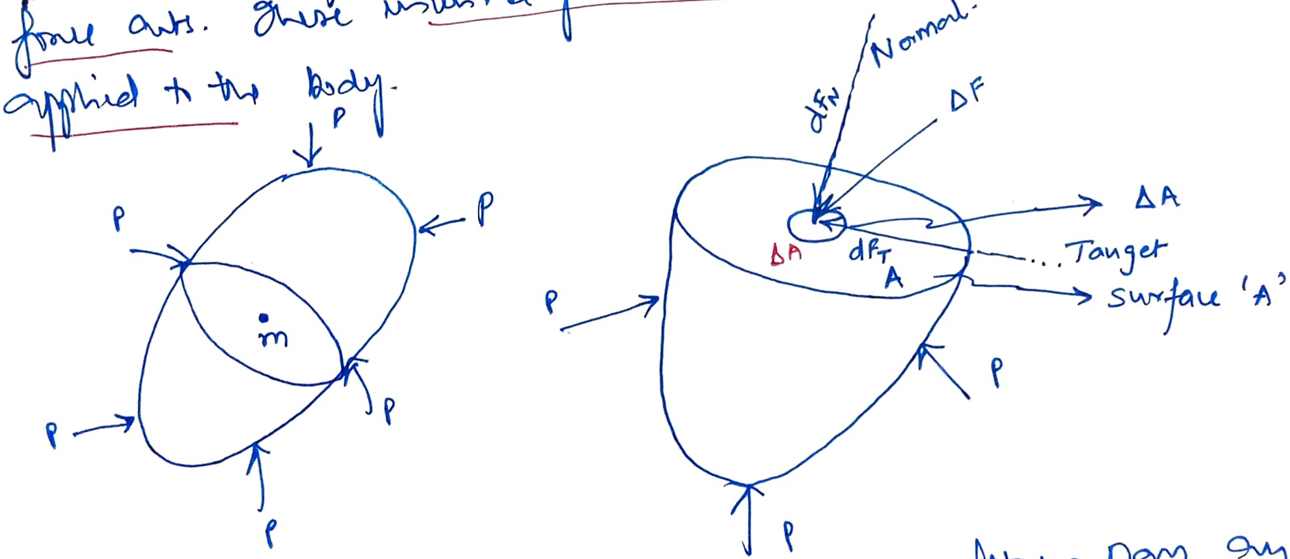
Surface traction -

External traction 'T' represents the force per unit area acting at a given location on the body's surface. T is a bound vector means T cannot slide along its line of action or translate to another location with the same meaning. So Traction vector cannot be fully described unless both force and the surface area on which force acts are known.

$$T = \lim_{dA \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$



In continuum mechanics, stress is the measure of the internal forces acting within the deformable body. It is the average force per unit area of a surface within the body on which internal force acts. These internal forces are reaction to external forces applied to the body.



Consider the state of stress at a point 'm', let us pass any arbitrary plane passing through 'm'. Plane divides the body into two parts. If the upper part is removed, then the equilibrium of forces on the plane will not be satisfied even though the remaining bottom portion is in equilibrium. So the system of internal forces in the upper part is converted into external forces on the bottom portion. Resultant of internal forces exerted by the upper part on bottom part is 'F'. On elemental area 'ΔA' around point 'm', the resultant of the internal forces on the elemental area is 'ΔF'.

Average stress of internal forces on this area $= \frac{\Delta F}{\Delta A}$

To define the stress at point 'm', around around 'm' should be diminished indefinitely so that ΔA and ΔF becomes small. The limiting ratio is known as the traction or stress vector or stress at a point

$$S = \sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

n - denotes that the stress defined is applicable only to the particular plane whose outward normal is 'n'.

→ Stresses are interpreted as internal tractions that act on a defined internal datum plane. So cannot measure stress without specifying the datum plane.

dF_x, dF_y, dF_z are components of force dF on the small area dA of the free body surface in x, y, z directions.

Vectorially total force on the small surface dA can be represented as $dF = dF_x\bar{i} + dF_y\bar{j} + dF_z\bar{k}$

The traction components along x, y, z directions are

$$\bar{X} = S_x = \lim_{dA \to 0} \frac{dF_x}{dA} \; ; \; \bar{Y} = S_y = \lim_{dA \to 0} \frac{dF_y}{dA} \; ; \; \bar{Z} = S_z = \lim_{dA \to 0} \frac{dF_z}{dA}$$

S_x, S_y, S_z are called Traction components.

Stresses acting on an internal element plane are resolved into three mutually orthogonal components. One component is normal to the surface (Direct stress). The other two components are tangential to the surface (shear stresses).

→ Direct stresses tends to change the volume of the material (hydrostatic pressure) and are resisted by the body's bulk modulus (depends on young's modulus and poisson's ratio)

→ Shear stresses tends to deform the material without changing its volume and are resisted by the body's shear modulus.

dF_N → normal component of force 'dF' on the small area 'dA'

dF_{T1} and dF_{T2} → Tangential components.

Then stress or traction components are defined as

Direct or normal stress (along N direction)

$$\sigma = \lim_{dA \to 0} \frac{dF_N}{dA}$$

Shear stress (perpendicular to N direction)

$$\tau_1 = \lim_{dA \to 0} \frac{dF_{T1}}{dA}$$

$$\tau_2 = \lim_{dA \to 0} \frac{dF_{T2}}{dA}$$

→ Direct and shear stresses vary from point to point

~~Shear stresses are~~ Normal stresses may be tensile or compressive in nature.

Shear stresses are not sub-classified but their directions are important to specify.

Schematically show the stress acting on a body in 2D and 3D cases

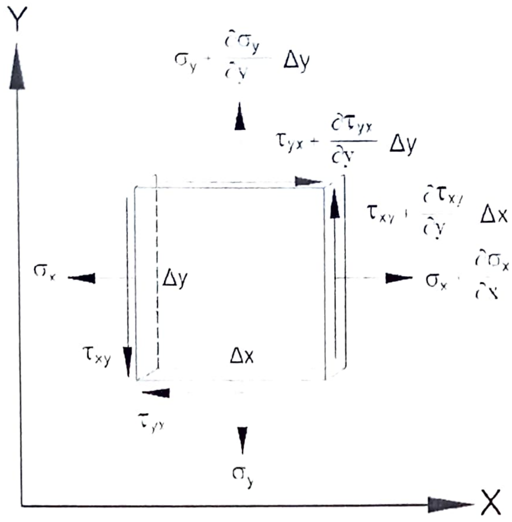


Figure 2.11(a) Stress components acting on a plane element (2D)

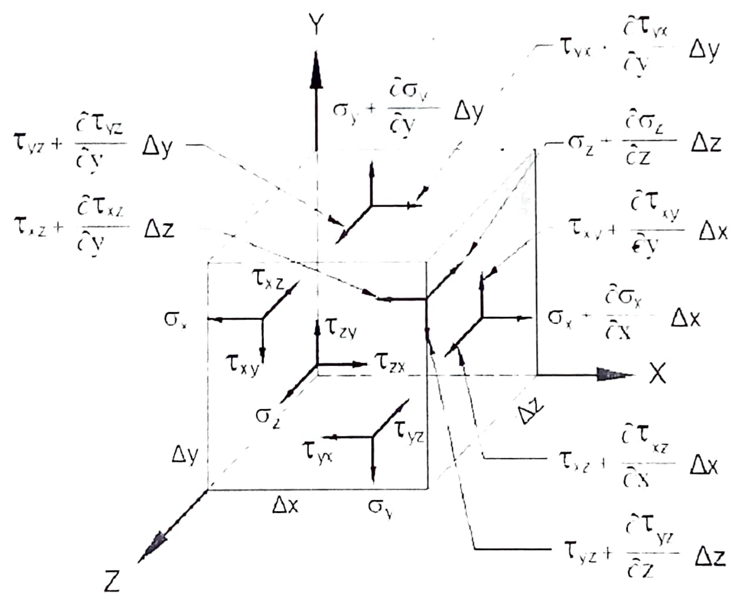


Figure 2.11(b) Stress components acting on a three dimensional element (3D)

TOE determines the state of stress in a body subjected to the action of given forces. In 2D problems, it is necessary to solve the differential equations of equilibrium and solution must satisfy the B.C's. These equilibrium equations derived by the application of equations of statics and contains three stress components σ_x, σ_y and τ_{xy} are not sufficient for determining the elastic deformation of a body. Since the problem is statically indeterminate one, we need to obtain the solution where elastic deformation of a body must also be considered.

State of stress at a point (general – Cartesian coordinate)

→ Stress is not uniformly distributed over the c/s of the body

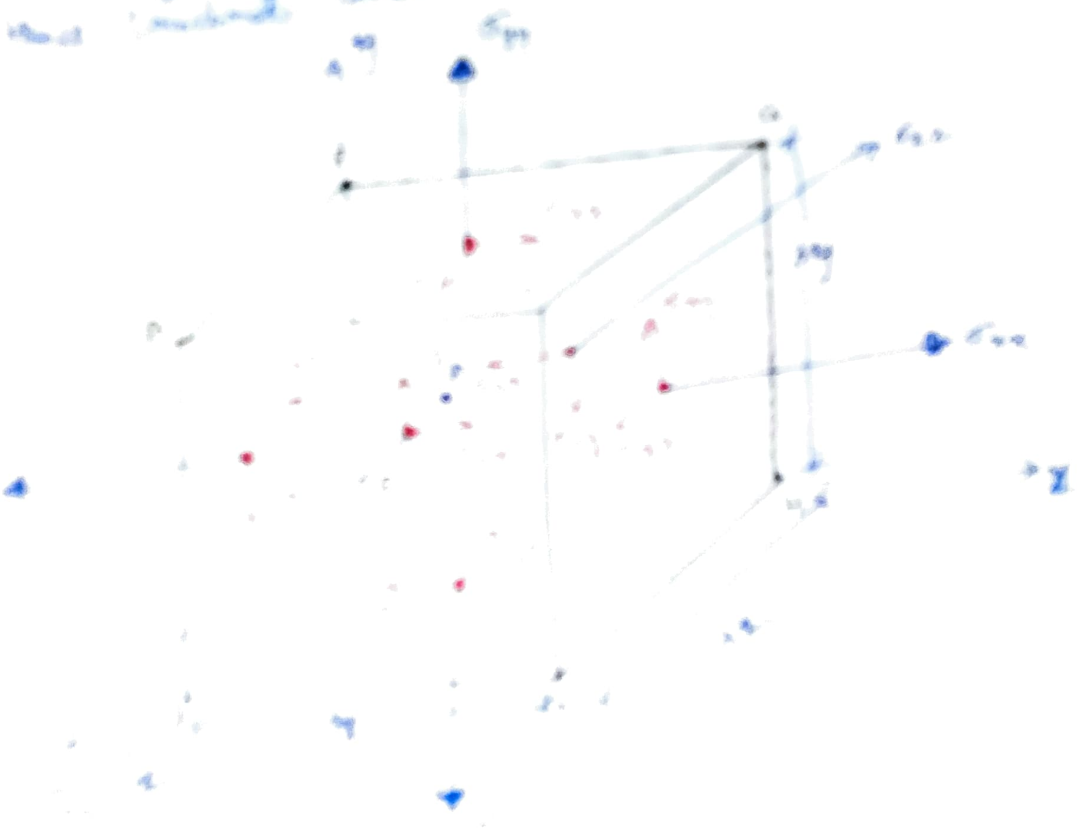
→ Stress at a given point differs from the average stress over the entire area

→ Define the stress at a given point in the body

→ There is [illegible] stress at any point [illegible] assumed [illegible] depend by nine [illegible] in [illegible] and is completely depend by nine [illegible] stress and [illegible] components (three orthogonal normal stresses and six orthogonal shear stresses)

→ Stress components are functions of both the position of the point in a body and the orientation of the plane passing through that point

→ Stress having [illegible] can be resolved into three components [illegible] [illegible] [illegible]



State of Stress on an Element :- (Cartesian Coordinates)

→ Stress is not uniformly distributed over the c/s of the body

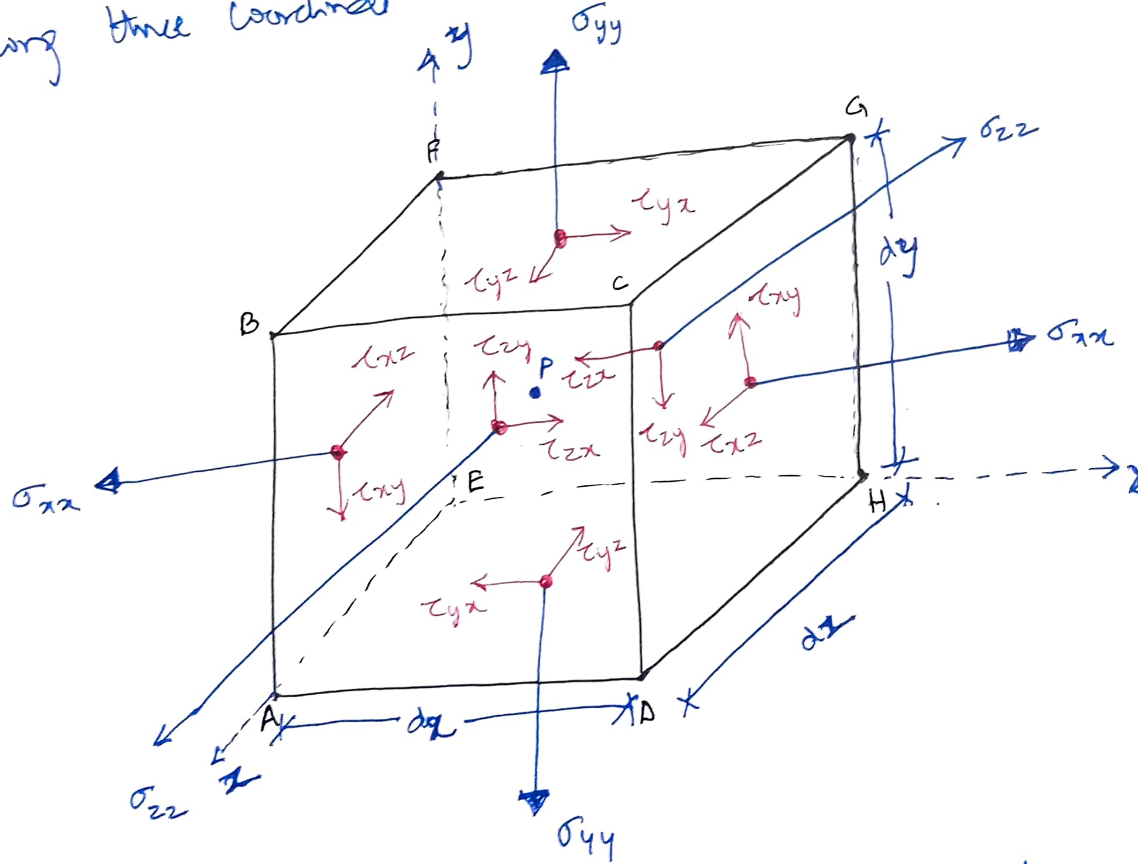
→ Stress at a given point differs from the average stress over the entire area.

→ Define the stress at a given point in the body.

→ Acc. to Cauchy, stress at any point in an object assumed to behave as a continuum and is completely defined by nine component stresses. (three orthogonal normal stresses and ~~three~~ six orthogonal shear stresses).

* → Stress components are functions of both the position of the point in a body and the orientation of the plane passed through that point.

→ Stress being vector can be resolved into three components along three coordinate axes



On any plane or face there is one normal and two shear components.

* τ_{xy} = stress on x plane along y-direction

x → gives the direction of the normal of the face

y → direction of stress itself.

→ Stress is second order tensor. (has one magnitude and two directions)

→ Surface traction is first order tensor (vector)

Zero order tensors or scalars → example - mass

→ 3-D stress tensor - cartesian coordinate system

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$\tau_{xy} = \tau_{yx}$
$\tau_{xz} = \tau_{zx}$
$\tau_{yz} = \tau_{zy}$
} result of static equilibrium (no net moment)

symmetric

Stress matrix.

$x \to 1 \quad y \to 2 \quad z \to 3$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

or

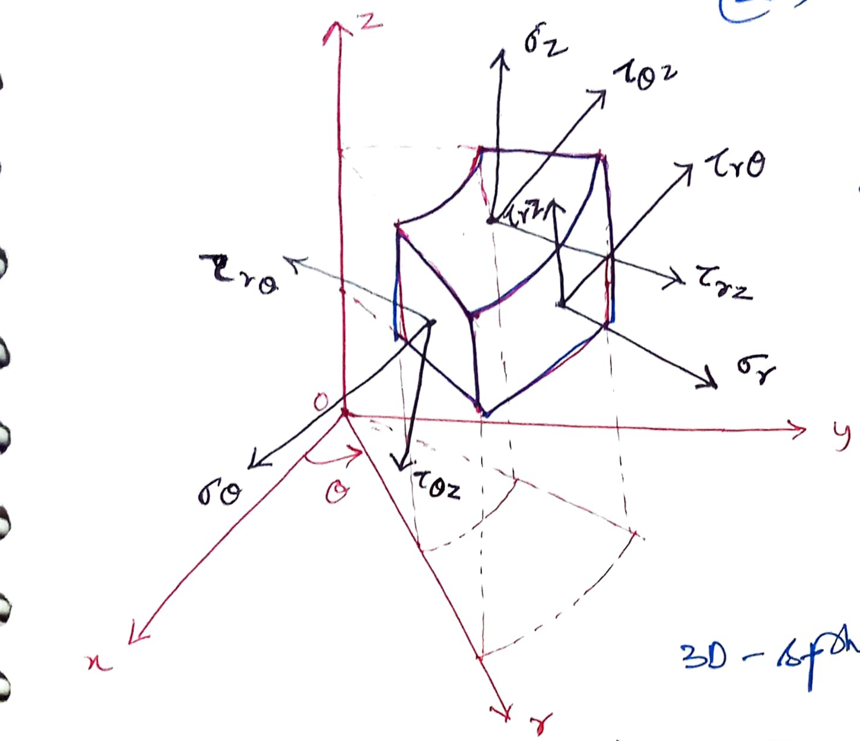
$$\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

The above 3D stress tensor is known as Cauchy stress tensor.

Note:- For perfectly elastic material the state of stress is independent of history of stresses and stress is unique function of strains.

Stress at a point (Cylindrical Coordinates) (Polar Coordinates) (2D)



each point in the plane is identified by the distance from the reference point and angle from a reference direction.

radial coordinate, distance (r or e)

angular coordinate, polar angle or azimuth

(θ, ϕ, t)

r - radial distance

θ - polar angle (zenith angle).

z - azimuthal angle

3D - spherical coordinate system

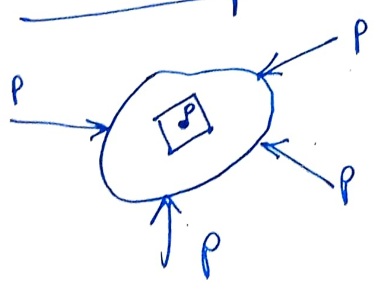
Stress tensor in the cylindrical system

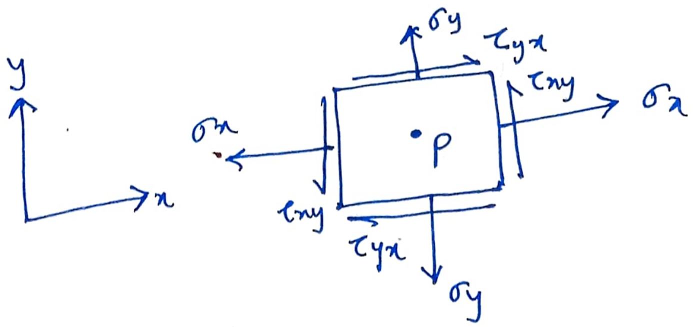
$$\begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_z \end{bmatrix}$$

stress tensor is symmetric

$\tau_{r\theta} = \tau_{\theta r}$
$\tau_{rz} = \tau_{zr}$
$\tau_{z\theta} = \tau_{\theta z}$

2D state of stress





Stress components in cartesian coordinate system

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$ 2D stress tensor.

For unit dimensions, and thickness unit then the net force = 0, $\Sigma M = 0$

$$\Sigma M_p = \text{moment about } P = \left(\tau_{xy} \times \frac{1}{2}\right) \times 2 - \left(\tau_{yx} \times \frac{1}{2}\right)^2 = 0$$

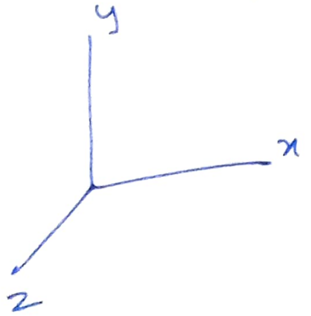
$$\boxed{\tau_{xy} = \tau_{yx}}$$

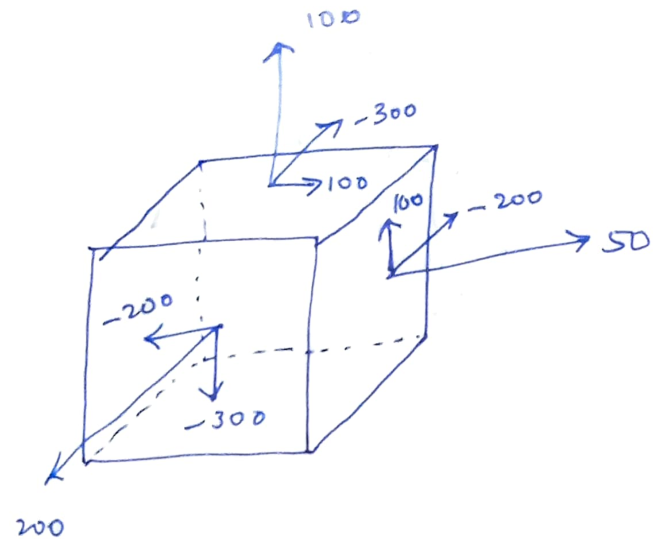
For shear stress component on one plane at point "p", there will be complementary shear stress on the other plane in order to maintain equilibrium due to rotation.

Example:- The state of stress at a point is given by

$$\begin{bmatrix} 50 & 100 & -200 \\ 100 & 100 & -300 \\ -200 & -300 & 200 \end{bmatrix} N/mm^2$$

Show the stresses on the element around the given point





$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

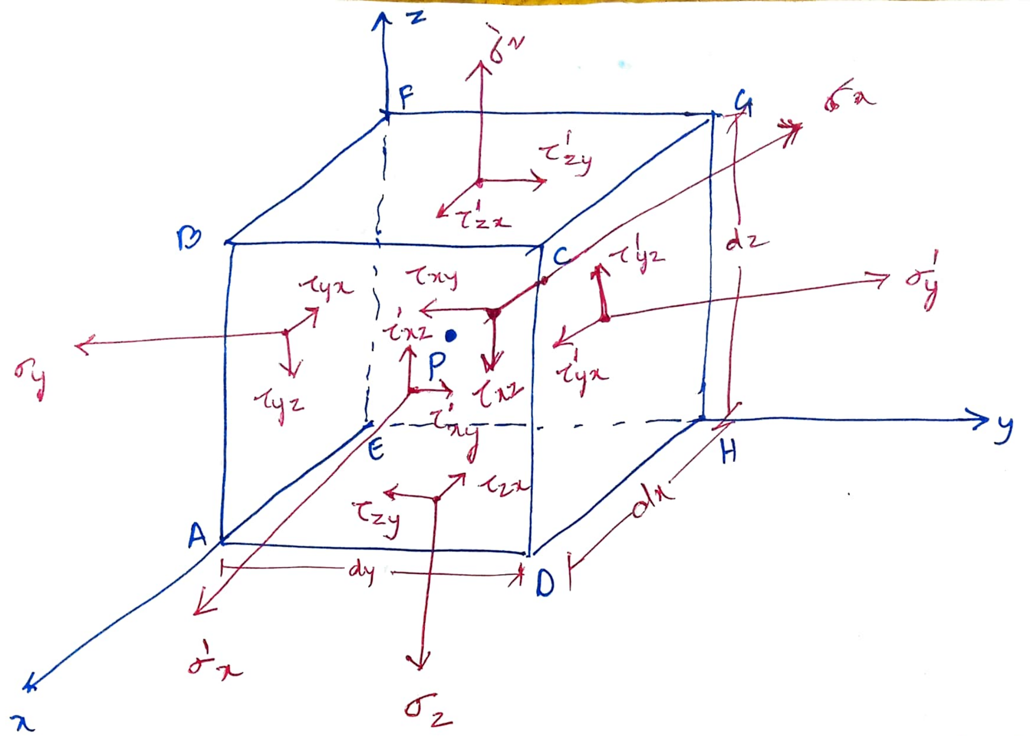
(#) Differential Equation of equilibrium in 3D stress systems

Stress intensities vary from point to point within a continuum. So it is necessary to establish the conditions to control the way in which the stress components vary.

Stresses are functions of coordinates, small change in the motion within the body will result in variation of stress by a value.

Consider an infinitesimal element in the form of parallelepiped with its faces parallel to planes referred as Cartesian coordinate systems. Components of stress near the point P are shown in fig. Thus the whole parallelepiped is in equilibrium and at rest thus the body forces are assumed to be absent.

(σ_x) normal stress on face EFGH, opposite face ABCD stress will be ($\sigma_x + \Delta \sigma_x$) of opposite sign.



The intensity of variation of a function with that of a variable is the derivative of the function w.r.t argument.

Increment of stress σ_x by a unit length $\frac{\partial \sigma_x}{\partial x}$

For the length dx, the increment of stress is given by

$$\Delta \sigma_x = \frac{\partial \sigma_x}{\partial x} . dx$$

$\partial x \rightarrow \partial \sigma_x$

$1 \rightarrow ?$

$\frac{\partial \sigma_x}{\partial x}$

$1 \rightarrow \frac{\partial \sigma_x}{\partial x}$

$dx \rightarrow \frac{\partial \sigma_x}{\partial x} . dx$

$$\sigma'_x = \sigma_x + \frac{\partial \sigma_x}{\partial x} . dx$$

Similarly

$$\sigma'_y = \sigma_y + \frac{\partial \sigma_y}{\partial y} . dy$$

$$\sigma'_z = \sigma_z + \frac{\partial \sigma_z}{\partial z} . dz$$

$$\tau'_{xy} = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} . dx$$

$$\tau'_{yz} = \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} . dy$$

$$\tau'_{zx} = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} . dz$$

In addition there may exist body forces whose components are X, Y, Z in x, y, z directions.

Since the parallelepiped is in equilibrium, the two conditions are to be satisfied are -
(i) The sum of the forces in each direction must be zero
(ii) The sum of the moments of the forces about the reference axis should be zero.

$\Sigma F_x = 0$

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} . dx\right) dy \times dz - \sigma_x dy \times dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\right) dz \times dx$$

$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} . dz\right) dx \times dy - \tau_{zx} \, dx \times dy$$

$$- \tau_{yx} \, dz \times dx$$

$$+ X \underbrace{dx \times dy \times dz}_{dv} = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z = 0$$

} equations of equilibriums

These equations represent the balance between the externally applied ~~load~~ body force field and the internally developed stress field.

Case 1:- In case of point loads, the stress near the point load becomes infinity so above equations cannot be used as such.

Case 2:- If weight is the only body force, | $X = Y = 0$, $Z = -\rho g$

$\rho = \frac{M}{V} = \frac{kg}{m^3} = \frac{W}{g.V} = \frac{W}{g.m^3}$ | W = body force acting downwards)

body force = force/unit volume

force = Body force . m^3

$mg = \rho g.m^3$

$-mg = -\rho g.m^3$ (negative because downwards)

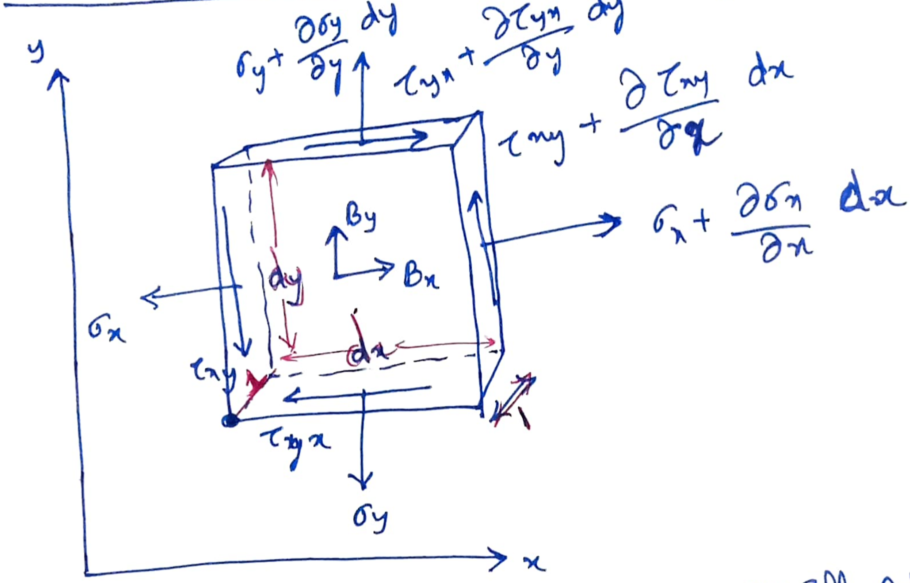
force = Body force . m^3

$Z = -\rho g$

Case 3:- If body forces are absent, then the equations of equilibrium

$X = Y = Z = 0$

Derive equations for equilibrium of a differential element (2D)



If the body is in equilibrium, thus small representative part is also in equilibrium.

In a stressed body components of stress will vary.

$\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx}$ are functions of x and y (independent of z) and other stress components do not vary through thickness are zero. B_x, B_y are components of body forces per unit volume in x and y directions. $(B_z = 0)$

σ_x	τ_{xy}
$\sigma_x + d\sigma_x$	$\tau_{xy} + d\tau_{xy}$

$$d\sigma_x = \frac{\partial \sigma_x}{\partial x} \cdot dx$$

(see back for reasons pg 15)

For equilibrium in x-direction, $\Sigma F = 0$

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy - \sigma_x dy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot dy\right) dx - \tau_{yx} dx + f_x \, dx \, dy = 0$$

$$\boxed{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + B_x = 0}$$

2D

$\Sigma F_y = 0$

Equilibrium along y-direction (y=0)

$$\Rightarrow \left(\sigma_y + \frac{\partial \sigma_y}{\partial y}\cdot dy\right)(dx\times 1) - \sigma_y \cdot (dx\times 1) + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\cdot dx\right)(dy \times 1)$$

$$- \tau_{xy}(dy\times 1) \quad \tau_{zx}\, dy\times 1$$

$$+ f_x\, dx \cdot dy \cdot 1 = 0$$

$$\boxed{\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + B_y = 0}$$ 2D

$\Sigma M = 0$

Taking moments of force about lower left corner and equating to zero.

(annotations: $(dy\times1)$, HOT, too small $dy\times1$, → unit thickness, HOT, $(dx\times1)$)

$$\Rightarrow -(\sigma_x \cdot dy)\left(\frac{dy}{2}\right) + (\tau_{xy}\cdot dy)\frac{1}{2} - \left(\sigma_y + \frac{\partial \sigma_y}{\partial y}\cdot dy\right) dx \cdot \frac{dx}{2}$$

(annotations: $dx\times1$, $dy\times1$)

$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\cdot dy\right) dx\cdot dy - \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\cdot dx\right) dx\, dy$$

(annotations: $dy\times1$, HOT, $dx\times1$, HOT, $dx\times1$)

$$+ \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\cdot dx\right) dy \cdot \frac{dy}{2} + \sigma_y \cdot dx \cdot \frac{dx}{2} - \tau_{yx}\cdot dx \cdot \frac{1}{2}$$

$$+ \left(B_x\, dy\cdot dx\right)\frac{dy}{2} - \left(B_y \cdot dx \cdot dy\right)\frac{dx}{2} = 0$$

(HOT) (HOT)

∵ dz is very too small negligible

HOT: $(dx)^2$ or $(dy)^2$ terms are neglected, only dx, dy terms are considered

$$\tau_{xy}\, dx \cdot dy = \tau_{yx}\, dx\, dy$$

$$\boxed{\tau_{xy} = \tau_{yx}}$$ equality of shears

Similarly $$\boxed{\begin{aligned}\tau_{yz} &= \tau_{zy} \\ \tau_{zx} &= \tau_{xz}\end{aligned}}$$

Note τ_{xy} = is the direction of y and parallel to Normal to x

(or)

y direction and plane cutting x axis.

Science → finding facts through observation and experiments.

α - alpha	ζ - zeta	μ = Mu	σ Sigma
β - beta	η - eta	ν Nu	τ tau
Γ / γ - gamma	θ - theta	ξ Xi	υ upsilon
δ / Δ - delta	κ - kappa	π pi	ϕ phi
ϵ - epsilon.	λ - lambda	ρ Rho	χ chi
			ψ psi
			ω / Ω omega

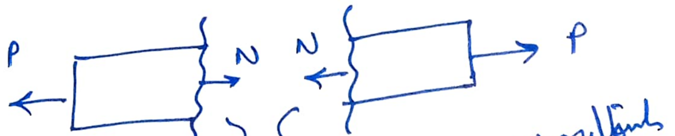
Science - Scientia means knowledge (Systematic Comprehensive Investigation and Exploration of nature's causes and effects).

* **Components of deformation** are (in x, y, z direction) are ϵ_x ϵ_y ϵ_z γ_{xy} γ_{yz} γ_{zx}

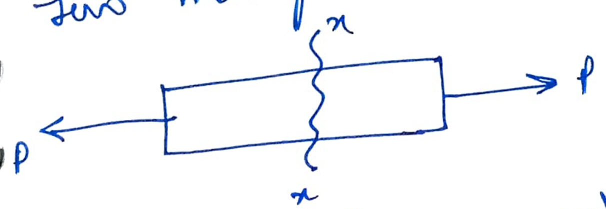
* **Components of displacement** are x, y, z directions are u, v, w resp.

* In practice, a structural element or machine component is subjected to complex loading systems requiring stresses to be developed in all principal directions X, Y and Z (σ_{xx}, σ_{yy}, σ_{zz}). Similarly along the sides there are tangential stresses in the three directions (τ_{xy}, τ_{yz}, τ_{zx}).

* If the stress in one of the direction (say - z direction) is zero then you have 2D case

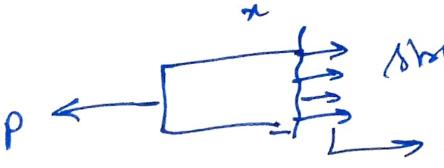


Internal stress resultants



$N = P$

$\sigma = \frac{P}{A} = \frac{N}{A}$



τ_{xy} – shear stress on a plane is y-direction.

Normal strains – means change in length along a specific direction.

↓

also called extensional strains / dimensional strains.

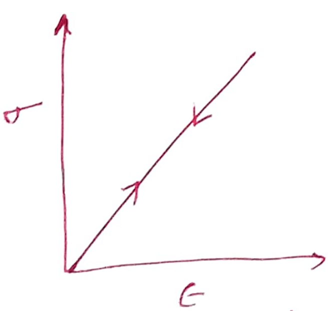
Shear strains – means changes in angles w.r.t two specific directions.

$\sigma_{x,y,z}$
$\epsilon_{x,y,z}$ } functions of coordinates of space.
u, v, w

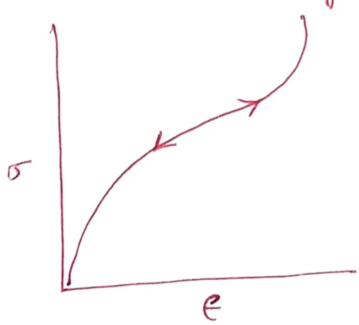
Isotropic – Elastic constants are independent of coordinate axis.

Concept of stress :- When a certain system of external forces act on a body then the body offers resistance to these forces. The internal resistance offered by the body per unit area is called the stress induced in the body.

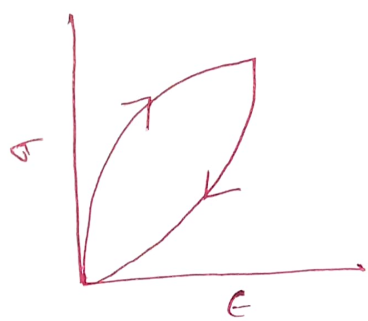
Note:- What kind of material is it? (stress-strain diagram of a material is given)



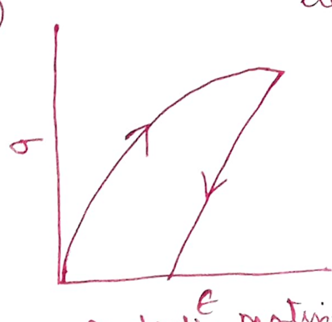
elastic material (linear)



non linear elastic material



visco-elastic material



inelastic material.

Homogeneous And Isotropic

People find it difficult to differentiate between the words homogeneous and isotropic, but they are two different words, which have no relationship. Uniformity is discussed in both words, yet both are defined with no connection. Depending on the subject, properties and the classification, these terms can be distinguished.
Homogeneous means that something is uniform throughout. Homogeneity depends on the context which it is based on. A homogeneous material means a material which has uniform composition and uniform properties throughout. Metals, alloys, ceramics are examples of homogeneous materials. The opposite term of homogeneous is heterogeneous.
In an isotropic material, physical and mechanical properties are equal in all orientations or directions. The isotropic nature of the material depends on its crystal structure. If the grains of the material are not oriented uniformly in all directions, it is not an isotropic material.
Properties like Young's modulus, thermal expansion coefficient, magnetic behavior can vary with directions in such anisotropic (not isotropic) materials.
Materials that do not have such "directionality" are called "isotropic".
homogenuis means the same properties at every point. it is independent of translation .
Isotropy means same properties an all directions for a specific point. the modulus of elasticity is same in x,y and z direction i.e $E_x=E_y=E_z$ it is independent of rotation you may rotate in any direction this point having same value
homogeneous : the property is not a function of position, i.e. it does not depend on x, y or z.
isotropic: the property does not depend on a particular direction.
isotropic is always homogeneous but the reverse is not true. And another way to say it all is that an isotropic property is invariant under translation and rotation
most materials are homogeneous at a large enough scale, but they can reveal inhomogeneities if we look close enough
Isotropic Material is defined as if its mechanical and thermal properties are the same in all directions. Isotropic materials can have a homogeneous or non-homogeneous microscopic structures. For example, steel demonstrates isotropic behavior, although its microscopic structure is non-homogeneous.
Physical properties are things that are measureable. Those are things like density, melting point, conductivity, coefficient of expansion, etc. Mechanical properties are how the metal performs when different forces are applied to them.
Physical properties can be observed or measured without changing the composition of matter. Physical properties are used to observe and describe matter. Chemical properties are only observed during a chemical reaction and thus changing the substance's chemical composition.

Generalised Hooke's laws (or) Lamé's Constants

Stress - strain relationships :- (Isotropic materials)

Acc. to Hooke's law, stress is proportional to strain for uniaxial stress action.

Modulus of elasticity $E = \frac{\sigma}{\epsilon}$ or $\frac{\sigma_x}{\epsilon_x}$

Poisson's ratio (μ) or ν $= \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{\epsilon_y}{\epsilon_x}$ or $\frac{\epsilon_z}{\epsilon_x}$

$\epsilon_y = \epsilon_z = \mu \epsilon_x$

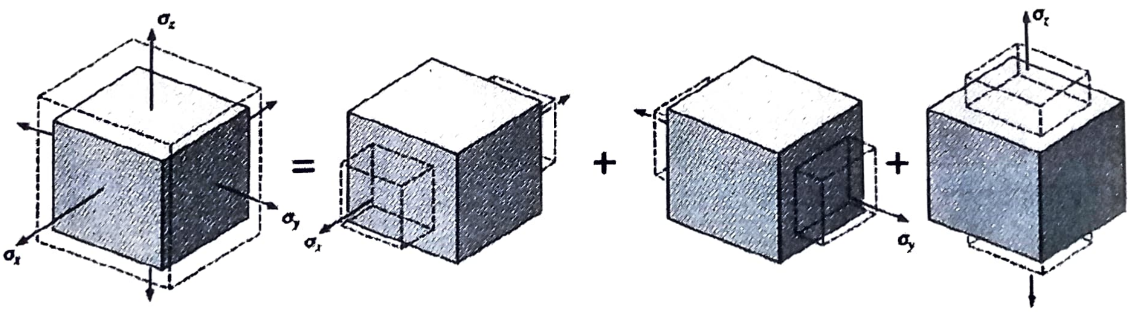
Modulus of rigidity $G = \frac{\text{Shear stress}}{\text{shear strain}}$

$G = \frac{E}{2(1+\mu)}$

Bulk modulus $K = \frac{\text{Normal stress}}{\text{Volumetric strain}}$

$K = \frac{E}{3(1-2\mu)}$

Consider a cubic volume element subjected to a state of triaxial normal stress $\sigma_x, \sigma_y, \sigma_z$ and associated normal strains ϵ_x, ϵ_y and ϵ_z are developed in the material. Since the material is isotropic, the cubic volume element will deform to a rectangular element, no shear strains are produced in the material. By using the principle of superposition the deformation of the cubic volume element subjected to each normal stress can be drawn.



(a) (b) (c) (d)

Under σ_x, element is elongated in x- direction and the associated strain in the direction is $\epsilon_x^1 = \frac{\sigma_x}{E}$

when σ_y is applied the element contracts in x direction due to poisson's effect and the associated strain in x direction is

$$\epsilon_x^2 = -\mu \frac{\sigma_y}{E}$$

Similarly when σ_z is applied, element contracts in x direction due to poisson's effect and the associated strain in x direction is

$$\epsilon_x^3 = -\mu \frac{\sigma_z}{E}$$

Superimposing these three normal strains, the total normal strain $\epsilon_x = \epsilon_x^1 + \epsilon_x^2 + \epsilon_x^3$ (when body subjected to triaxial state of stress)

$$\epsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} = \frac{1}{E}\left(\sigma_x - \mu(\sigma_y + \sigma_z)\right) \quad \text{— (1)}$$

Similarly the normal strains in y and z direction can be determined as

$$\epsilon_y = \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_z}{E} = \frac{1}{E}\left(\sigma_y - \mu(\sigma_x + \sigma_z)\right) \quad \text{— (2)}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} = \frac{1}{E}\left(\sigma_z - \mu(\sigma_x + \sigma_y)\right) \quad \text{— (3)}$$

Add (1), (2) & (3)

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{1}{E}\left(\sigma_x - \mu(\sigma_y + \sigma_z)\right) + \frac{1}{E}\left(\sigma_y - \mu(\sigma_x + \sigma_z)\right) + \frac{1}{E}\left(\sigma_z - \mu(\sigma_x + \sigma_y)\right)$$

let $\epsilon_x + \epsilon_y + \epsilon_z = e$

$$e = \frac{1}{E}\left((\sigma_x + \sigma_y + \sigma_z) - 2\mu(\sigma_x + \sigma_y + \sigma_z)\right)$$

$$e = \frac{1}{E}\left((\sigma_x + \sigma_y + \sigma_z)(1 - 2\mu)\right)$$

$$eE = (\sigma_x + \sigma_y + \sigma_z)(1-2\mu)$$

$$\sigma_y + \sigma_z = \frac{eE}{(1-2\mu)} - \sigma_x \quad \text{——— (4)}$$

Substitute (4) in (1)

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \mu\left(\frac{eE}{(1-2\mu)} - \sigma_x\right)\right]$$

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \mu\frac{eE}{(1-2\mu)} + \mu\sigma_x\right]$$

$$\sigma_x(1+\mu) = E\epsilon_x + \frac{\mu eE}{(1-2\mu)}$$

$$\sigma_x = \frac{E\epsilon_x}{(1+\mu)} + \frac{\mu}{(1+\mu)(1-2\mu)} eE$$

Substitute Lami's constants
$$\lambda = \frac{\mu}{(1+\mu)(1-2\mu)}$$
$$2G = \frac{E}{(1+\mu)}$$

$$\sigma_x = 2G\epsilon_x + \lambda eE$$

Similarly
$$\sigma_y = 2G\epsilon_y + \lambda eE$$
$$\sigma_z = 2G\epsilon_z + \lambda eE$$

Application of shear stress τ_{xy} to the cubic volume element of isotropic material only produces the shear strain γ_{xy} in the element. Likewise the shear stress τ_{yz} and τ_{zx} only produce the shear strain γ_{yz} and γ_{xz} on the element.

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx}$$

Generalized Hooke's law statement " When more than one strains component exists within the elastic limit, then at every point of the body each of the six stress components may be expressed as a function of the six components of strains and vice versa."

Example :- The following Cartesian stresses act at a point in a body subjected to a complex loading system. If $E = 210\,GPa$ and $\mu = 0.28$. Determine the equivalent strains present.

$\sigma_x = 150\,MPa$ $\quad \tau_{xy} = 90\,MPa$

$\sigma_y = 100\,MPa$ $\quad \tau_{yz} = 120\,MPa$

$\sigma_z = 75\,MPa$ $\quad \tau_{zx} = 50\,MPa$

Sol:-

$$e_x = \frac{1}{E}\left(\sigma_x - \mu(\sigma_y + \sigma_z)\right) = 4.81 \times 10^{-4}$$

$$e_y = \frac{1}{E}\left(\sigma_y - \mu(\sigma_x + \sigma_z)\right) = 1.76 \times 10^{-4}$$

$$e_z = \frac{1}{E}\left(\sigma_z - \mu(\sigma_x + \sigma_y)\right) = 2.38 \times 10^{-4}$$

$$G = \frac{E}{2(1+\mu)} = 82.03\,GPa$$ pg (33)

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 1.10 \times 10^{-3}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = 1.46 \times 10^{-3}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = 6.10 \times 10^{-3}$$

Example :- Given that the following strains exist at a point in a 3D system, determine the equivalent stresses which act at the point. Take $E = 200\,GPa$ and $\mu = 0.3$. Also find the Lame's constants.

$e_x = 0.003$ $\quad \gamma_{xy} = 0.0001$

$e_y = 0.0008$ $\quad \gamma_{yz} = 0.0005$

$e_z = 0.0007$ $\quad \gamma_{zx} = 0.0002$

Soln- $e = \epsilon_x + \epsilon_y + \epsilon_z = 0.0045$

Lami's constant $\lambda = \dfrac{\mu}{(1+\mu)(1-2\mu)} = 0.577$

$$G = \frac{E}{2(1+\mu)} = 76.92 \text{ GPa}$$

$\sigma_x = 2G\epsilon_x + \lambda e E = 980.82$ MPa

$\sigma_y = 2G\epsilon_y + \lambda e E = 642.37$ MPa

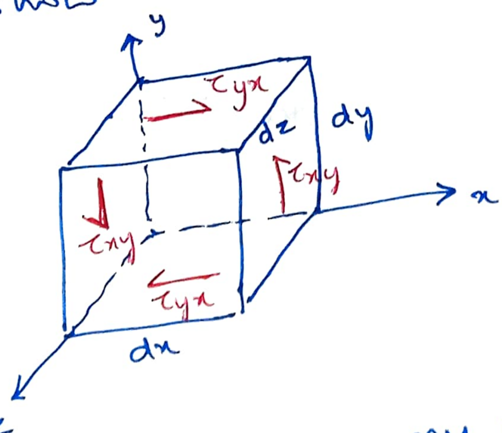
$\sigma_z = 2G\epsilon_z + \lambda e E = 626.08$ MPa

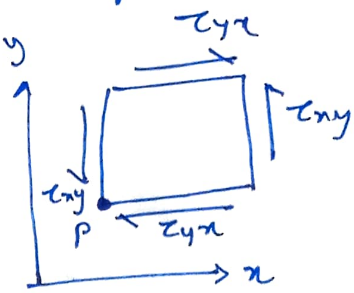
$\tau_{xy} = G\gamma_{xy} = 7.69$ MPa

$\tau_{yz} = G\gamma_{yz} = 38.47$ MPa

$\tau_{zx} = G\gamma_{zx} = 15.39$ MPa

Example :- Show that the stress tensor is symmetrical





Consider the stresses acting on xy plane

By condition of equilibrium, moment about z axis $\Sigma M_z = 0$

$$(-\tau_{xy}\, dy\, dz)\, dx + (\tau_{yx}\, dx\, dz)\, dy = 0$$

$$\tau_{xy} = \tau_{yx}$$

Similarly $\tau_{xz} = \tau_{zx}$ (consider xz plane)

$\tau_{yz} = \tau_{zy}$ (consider yz plane)

Hence stress tensor is symmetrical

Strain tensor is also symmetrical

$(\gamma_{xy} = \gamma_{yx},\ \gamma_{yz} = \gamma_{zy},\ \gamma_{zx} = \gamma_{xz})$

Example - The stress components at a point are given by

$\sigma_x = x + y^3$ $\qquad$ $\tau_{xy} = x^2 y$

$\sigma_y = y^2 + zx$ $\qquad$ $\tau_{yz} = y^2 z$

$\sigma_z = x^3 + z^2$ $\qquad$ $\tau_{xz} = x z^2$

What must be the body force stress in order to satisfy the conditions of equilibrium?

Sol:-

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0$$

$$1 + x^2 + 2xz + f_x = 0$$

Body force stress in the x-direction $f_x = -(x^2 + 2xz + 1)$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0$$

$$2xy + 2y + y^2 + f_y = 0$$

Body force stress in the y-direction $f_y = -(y^2 + 2y + 2xy)$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + f_z = 0$$

$$z^2 + 2yz + 2z + f_z = 0$$

Body force stress in the z-direction } $f_z = -(z^2 + 2yz + 2z)$

Example:- Given the following stress fields

$\sigma_x = 40x^2 + 6xy$ $\qquad$ $\tau_{xy} = 0$

$\sigma_y = 60y + 20$ $\qquad$ $\tau_{yz} = xy^3 + yz^3 + zx^3$

$\sigma_z = 50z^2$ $\qquad$ $\tau_{xz} = 9xy + 6y^2 z$

Find the body force distribution required for equilibrium and the body force stress components at point (3,4,2)

Sol:-

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

$x = 3$, $y = 4$, $z = 2$

$$80x + 6y + 0 + 6y^2 + f_x = 0$$

$$f_x = -360$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$$

$$0 + 60 + 6y^2 + f_y = 0 \qquad f_y = -156$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + f_z = 0$$

$$9y + 3y^2x + 100z + f_z = 0 \qquad f_z = -380$$

Example: for previous problem assume $E = 200$ GPa and $\mu = 0.25$. Calculate the strains at the point $(1, 3, 2)$

$x = 1$
$y = 3$
$z = 2$

$\sigma_x = 40x^2 + 6xy = 58$ MPa

$\sigma_y = 200$ MPa

$\sigma_z = 50z^3 = 200$ MPa

$\tau_{xy} = 0$

$\tau_{yz} = xy^3 + yz^3 + zx^3 = 53$ MPa

$\tau_{xz} = 9xy + 6y^2z = 135$ MPa

$$\epsilon_x = \frac{1}{E}(\sigma_x - \mu(\sigma_y + \sigma_z)) = -2.1 \times 10^{-4}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \mu(\sigma_x + \sigma_z)) = 6.78 \times 10^{-4}$$

$$\epsilon_z = \frac{1}{E}(\sigma_z - \mu(\sigma_x + \sigma_y)) = 6.78 \times 10^{-4}$$

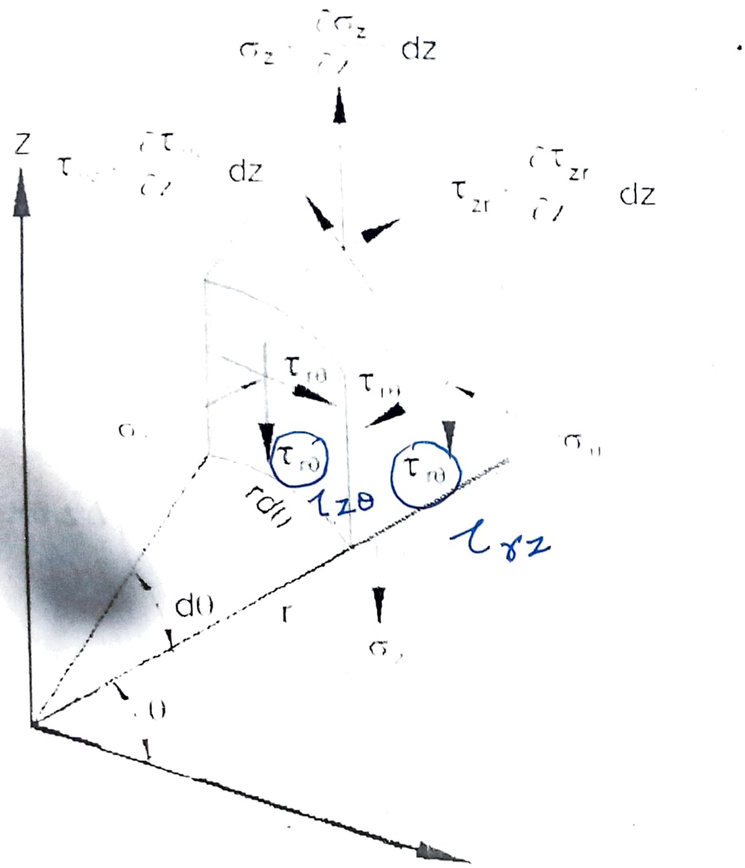
$$G = \frac{E}{2(1+\mu)} = 80 \text{ GPa}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = 6.625 \times 10^{-4}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = 1.687 \times 10^{-3}$$

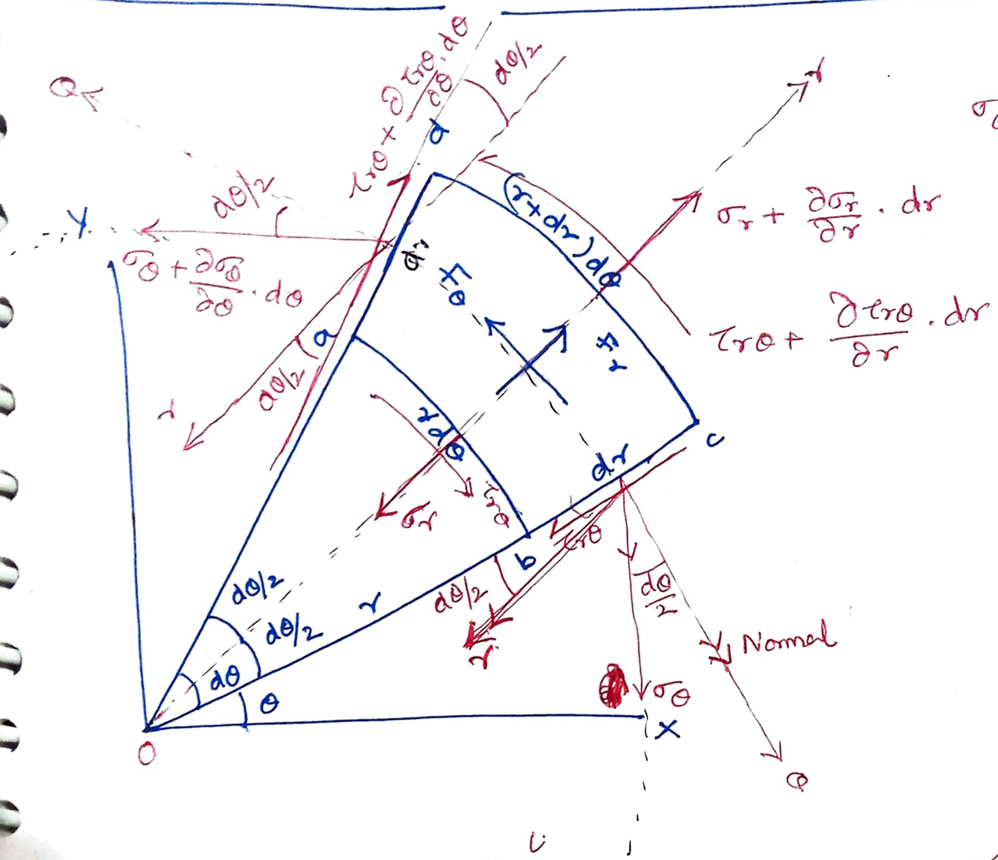
Equilibrium equations in cylindrical co-ordinate system and show schematically the stress acting on a body

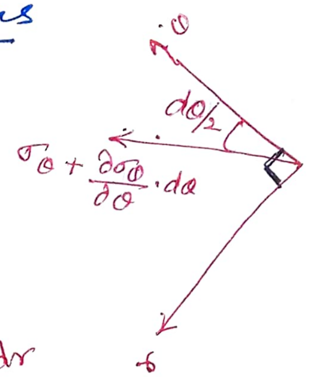


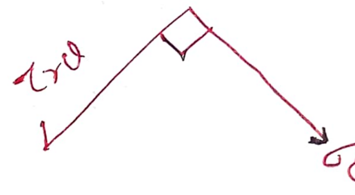
Two-dimensional problems in Polar Coordinates

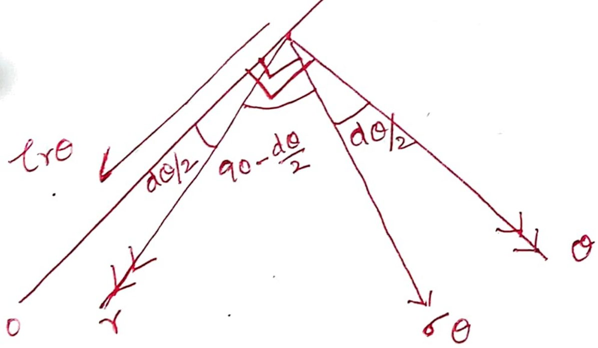
The polar coordinates of a point describe its position in terms of distance from a fixed point (the origin) and an angle measured from a fixed direction which is normally the horizontal axes. Many engineering components have a degree of axial symmetry that is they are either rotationally symmetric about a central axis as in a case of circular ring or contain circular holes or made up of parts of hollow discs like a curved bar. In such cases, it is advantageous to use cylindrical coordinate system or polar coordinate system instead of rectangular or cartesian coordinate system.

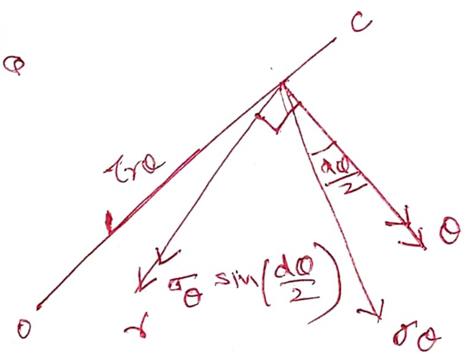
Equilibrium equations in Polar Coordinates











$\tau_{r\theta} \perp r$ to σ_θ

Normal stress component in radial direction - σ_r

" " " in circumferential direction - σ_θ

Shearing stress component $\tau_{r\theta}$.

On account of the variation of stress the values at the sides are not same.

F_r = force × surface area × (unit width)

The polar coordinate (r, θ) system and the cartesian coordinate (x, y) systems are related as

$x = r\cos\theta$ $\qquad r^2 = x^2 + y^2$

$y = r\sin\theta$ $\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

State of stress on element abcd of unit thickness are expressed in polar coordinates as shown in fig. $dz = 1$

F_r and F_θ are body forces in r and θ directions resp.

$\Sigma F_r = 0$ condition of equilibrium, forces in r direction.

$$\Rightarrow \begin{cases} -\sigma_r (r\,d\theta \times 1) + \left(\sigma_r + \frac{\partial \sigma_r}{\partial r}.dr\right)(r+dr)\,d\theta \times 1 - \sigma_\theta \sin\left(\frac{d\theta}{2}\right)(dr\times 1) \\ + F_r (r\,d\theta \times dr) - \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}.d\theta\right)\sin\left(\frac{d\theta}{2}\right)(dr \times 1) - \tau_{r\theta}\cos\left(\frac{d\theta}{2}\right)(dr\times 1) \\ + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}.d\theta\right)\cos\left(\frac{d\theta}{2}\right)(dr.\times 1) = 0 \end{cases}$$

$d\theta$ is very small so $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$; $\cos\frac{d\theta}{2} = 1$

neglecting HOT (higher order terms)

$$\begin{cases} -\sigma_r r\,d\theta + \sigma_r r\,d\theta + \sigma_r\,dr\,d\theta + \frac{\partial \sigma_r}{\partial r}.r\,dr\,d\theta + \frac{\partial \sigma_r}{\partial r}.dr.dr\,d\theta \\ -\sigma_\theta\left(\frac{d\theta}{2}\right)dr + F_r\,r\,d\theta\,dr - \sigma_\theta.\frac{d\theta}{2}.dr - \frac{\partial \sigma_\theta}{\partial \theta}.d\theta.\frac{d\theta}{2}.dr \\ -\tau_{r\theta}\,dr + \tau_{r\theta}\,dr + \frac{\partial \tau_{r\theta}}{\partial \theta}.d\theta.dr = 0 \end{cases}$$

(got cancelled; got cancelled; HOT; HOT; cancelled; cancelled)

$$\Rightarrow r.\frac{\partial \sigma_r}{\partial r}.dr\,d\theta + \sigma_r\,dr\,d\theta - \sigma_\theta\,dr.d\theta + \frac{\partial \tau_{r\theta}}{\partial \theta}.d\theta.dr + F_r\,r\,d\theta\,dr = 0$$

divide throughout by $r\,d\theta\,dr$

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} - \frac{\sigma_\theta}{r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + F_r = 0$$

$$\boxed{\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \left(\frac{\sigma_r - \sigma_\theta}{r}\right) + F_r = 0} \quad \text{—— (1)}$$

Similarly resolving all forces in θ-direction right angle to r-direction

$\Sigma F_\theta = 0$

$$\left(-\sigma_\theta \cos\frac{d\theta}{2}\right)(dr\times 1) + \left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}.d\theta\right)\cos\frac{d\theta}{2}(dr\times 1) + \left(\tau_{r\theta}\sin\frac{d\theta}{2}\right)$$

$$(dr\times 1) + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}.d\theta\right)(dr\times 1)\sin\frac{d\theta}{2} - \tau_{r\theta}(r\,d\theta\times 1)$$

$$+ \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r}.dr\right)(r+dr)\,d\theta + f_\theta(r\,d\theta.dr) = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + f_\theta = 0 \quad \text{—— (2)}$$

In the absence of body forces, the equilibrium equations can be represented as

2D

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{\sigma_r} = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0$$

3D

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \left(\frac{\sigma_r - \sigma_\theta}{r}\right) = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0$$

Pg (107)

Stress tensor in cylindrical coordinates (r, θ, z)

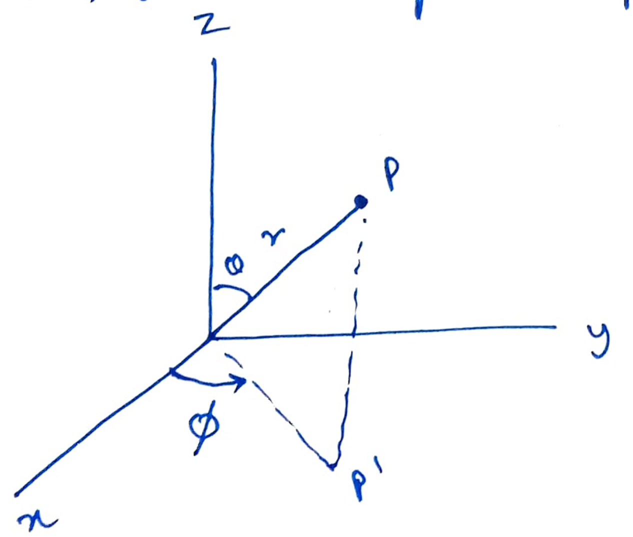
r - radial

θ - circumferential

z - axial (polar) direction

$$[\sigma] = \begin{bmatrix} \sigma_r & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_\theta & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_z \end{bmatrix} \simeq \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$$

This coordinate system define the location of a point in 3D space



z zenith direction

r radial coordinate

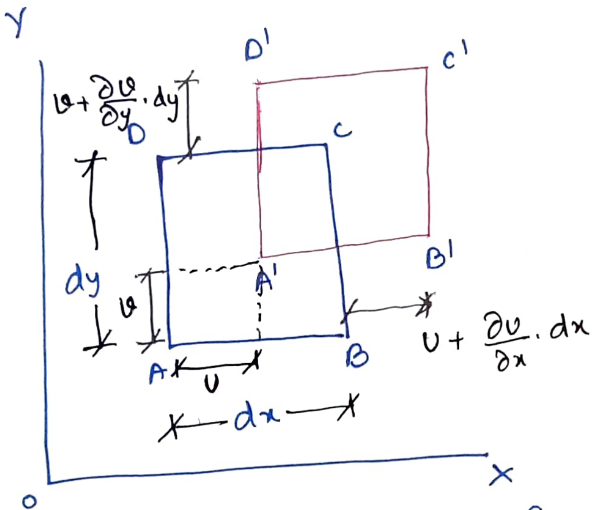
θ polar coordinate or angle, colatitude, zenith angle normal angle.

ϕ azimuthal angle

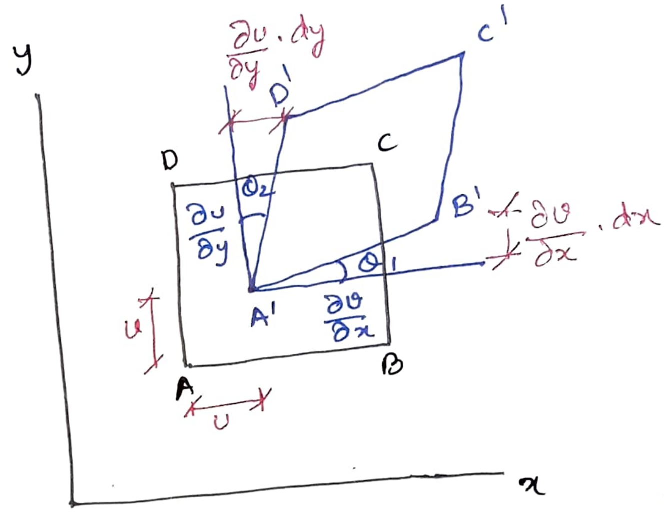
Components of Strain or Cauchy's Strain - displacement relations

Strain is defined as measure of deformation in the body.

- direct strain or extensional strain (in x or y direction)
- shear strain (in x-y plane)

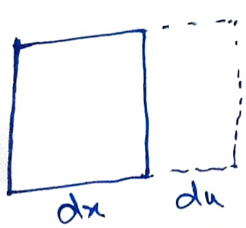


direct strain

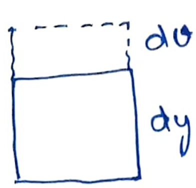


($\theta_1 = \frac{\partial v}{\partial x}$ $\theta_2 = \frac{\partial u}{\partial y}$) Shear strain

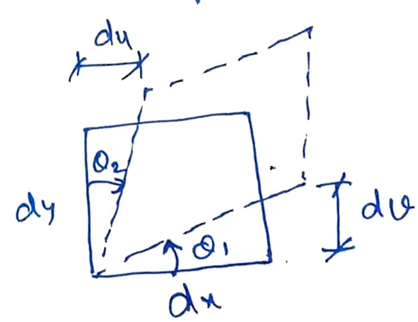
When an elastic body is deformed all the particles of the body are displaced. The small displacements of particles of a deformed body can be resolved into components u, v, w parallel to coordinate axes x, y, z resp. It is assumed that these quantities are very small and varying continuously over the volume of the body.



linear strain in x-direction



linear strain in y direction



linear strain in x direction }

$$e_x = \frac{\text{change in length in x-direction}}{\text{original length}} = \frac{du}{dx}$$

$$e_y = \frac{dv}{dy}$$

Shear strain γ_{xy} is defined as the change in the initial right angle between the two line elements originally parallel to x and y axes

The total change in angle is $\theta_1 + \theta_2$

$$\theta_1 = \tan\theta_1 = \frac{dv}{dx}$$

$$\theta_2 = \tan\theta_2 = \frac{du}{dy}$$

Shear strain $\gamma_{xy} = \theta_1 + \theta_2 = \frac{du}{dy} + \frac{dv}{dx}$

$$\boxed{\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx}}$$

Reduction in the initial right angle is considered to be a positive shear strain, since positive shear stress components τ_{xy} and τ_{yx} cause a decrease in right angle.

Components of Strain or Strain - displacement relations

Normal or longitudinal strains $\left\{ \epsilon_x = \frac{\partial u}{\partial x} \quad \epsilon_y = \frac{\partial v}{\partial y} \quad \epsilon_z = \frac{\partial w}{\partial z} \right.$ — (1)

+ve sign elongation

-ve sign contraction

$$\left.\begin{aligned} \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial w} \end{aligned}\right\} \text{shearing strains} \quad \text{— (2)}$$

We know that

$$\left.\begin{aligned} \gamma_{xy} &= \gamma_{yx} \\ \gamma_{yz} &= \gamma_{zy} \\ \gamma_{zx} &= \gamma_{xz} \end{aligned}\right\} \text{strain tensor is symmetrical}$$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$; $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}$; $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$ in x, y, z directions.

are gradients of the displacement components

Displacement - gradient matrix

$$\left[\frac{\partial u_i}{\partial x_j}\right] = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial w/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{bmatrix}$$

Equations (1) and (2) are called strain - displacement relations

$$\gamma_{xy} = \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}$$

$\epsilon_x = \epsilon_{xx}$
$\gamma_{xy} = \epsilon_{xy}$

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$\gamma_{xy} = \gamma_{yx}$

$\gamma_{xy} = \epsilon_{xy}$

Similarly

$$\epsilon_{xy} = \frac{\partial u_x}{\partial y}$$

$\epsilon_{xy} = \epsilon_{yx}$

$$\epsilon_{yx} = \frac{\partial v_y}{\partial x}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial v_y}{\partial x}$$

$\gamma_{xy} = \epsilon_{xy} + \epsilon_{yx}$

$\gamma_{xy} = 2\,\epsilon_{xy}$

$\epsilon_{xy} = \frac{1}{2}\gamma_{xy}$

$$\text{strain tensor } [\epsilon] = \begin{bmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{bmatrix}$$

The deformation or straining of an elastic body results in the relative displacement between the points.

$\epsilon_x, \epsilon_y, \epsilon_z$ (direct strains)

$\epsilon_{xy} = \epsilon_{yx}$, $\epsilon_{yz} = \epsilon_{zy}$, $\epsilon_{zx} = \epsilon_{xz}$ (shear strains)

Average shear strains are reported as "Engineering shear strains".

Engineering shear strain is the average change in angle between two perpendicular components.

shear strain is differt from engineering shear strain
↓

Strain - displacement relation

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \partial/\partial u & 0 & 0 \\ 0 & \partial/\partial v & 0 \\ 0 & 0 & \partial/\partial w \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$[\epsilon] = \{D\}\{u\}$$

strain matrix; $\{D\}$ ↓ operator matrix; $\{u\}$ displacement matrix

Equations of Compatibility for strain
or
Compatibility Conditions

u, v, w are displacement components.
Deformation at a point is specified by six (6) strain components.

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

These 6 components of strain will be related to three components of displacement only. Determining 6 strain components using three displacement functions is straight forward solution (first differentiation) but determining three displacement functions using 6 strain components is complicated because of integrations involved. and these 6 strain components cannot exist independently and there must be certain relation among these. These relations are called compatibility equations.

In 2D strain, differentiation of ϵ_x twice w.r.t y, ϵ_y twice w.r.t x ~~and γ_{xy} w.r.t x and y~~ results in

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial y \partial x^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial v}{\partial x}\right)$$

Adding these two

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\boxed{\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}} \quad \text{—} \; (1)$$

Similarly considering ϵ_y, ϵ_z and γ_{yz} ; ϵ_z, ϵ_x and γ_{zx} we get two more conditions

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad \text{—} \; (2)$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

equation (1) & (2) are one set of first three compatibility conditions which shows dependency between strain components.

To establish conditions among shear strains

$$\left.\begin{aligned} \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned}\right\}$$

Differentiating w.r.t x, y, z

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 v}{\partial z \partial x}$$

$$\frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y}$$

$$\frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z}$$

Adding last two equations and subtracting first

$$\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} = 2 \frac{\partial^2 w}{\partial x \partial y}$$

Differentiating the above equation w.r.t z and observing that

$$\frac{\partial^3 w}{\partial x \partial y \partial z} = \frac{\partial^2 \epsilon_z}{\partial x \partial y}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^3 w}{\partial x \partial y \partial z} = \frac{2\,\partial^2 \epsilon_z}{\partial x \partial y}$$

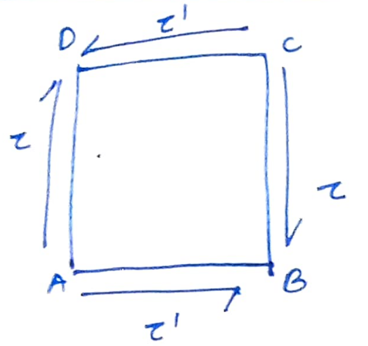
$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \epsilon_z}{\partial x \partial y}$$

Similarly
$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) = 2\frac{\partial^2 \epsilon_x}{\partial y \partial z}$$

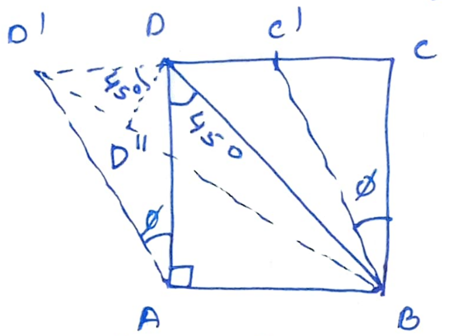
$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \epsilon_y}{\partial x \partial z} \quad \text{--- (3)}$$

The above six equations are called Saint-Venant's equations of compatibility. If the component of strains are not related it is impossible to define continuous deformed solid. These equations are also known as "~~Continu~~ Continuity equations". or Compatibility equations

Relation between Modulus of elasticity and Modulus of Rigidity



(Before distortion)



(After Distortion)

shear stress τ causes shear strain ϕ

$$\text{Strain of } BD = \frac{BD' - BD}{BD} = \frac{D'D''}{BD} = \frac{DD' \cos 45^\circ}{AD\sqrt{2}} = \frac{DD'}{2\,AD} = \frac{\phi}{2}$$

linear strain of diagonal is half of the shear strain and is tensile in nature

linear strain of the diagonal $BD = \frac{\phi}{2} = \frac{\tau}{2G}$ — (1) $\because G = \frac{\tau}{\phi}$

Tensile strain on diagonal BD due to tensile stress on diagonal BD is $\frac{\tau}{E}$ — (2)

Tensile strain on diagonal BD due to compressive strain on diagonal AC is $\nu \frac{\tau}{E}$ — (3) $\nu = \frac{\text{lateral strain}}{\text{linear strain}}$

Combining the effect of two stresses on diagonal BD strain is

$$= \frac{\tau}{E} + \nu \frac{\tau}{E} = \frac{\tau}{E}(1+\nu) \quad \text{— (4)}$$

equate (1) and (4)

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu) \Rightarrow \boxed{G = \frac{E}{2(1+\nu)}}$$

Relationship between Young's Modulus 'E' and Bulk Modulus 'K'

$$\text{Bulk modulus 'K'} = \frac{\text{stress (direct)}}{\text{volumetric strain}} = \frac{\sigma}{\epsilon_v}$$

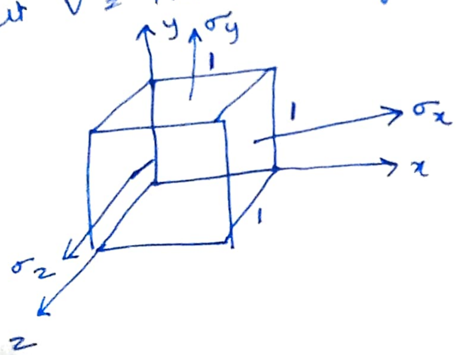
$$\epsilon_v = \frac{\delta V}{V} = \frac{\text{change in volume}}{\text{original volume}}$$

ϵ_v (dilatation) change in volume per unit volume.

In multi axial stress case, volumetric strain $\left(\frac{\delta V}{V}\right)$ is sum of all strains in all three mutually perpendicular directions

$$\epsilon_v = \left(\frac{\delta V}{V}\right) = \epsilon_x + \epsilon_y + \epsilon_z$$

Let $V = 1 \times 1 \times 1$ original volume



Due to $\sigma_x, \sigma_y, \sigma_z$ body is deformed and its deformed volume

$$V' = (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z)$$

$$V' = 1 + \epsilon_x + \epsilon_y + \epsilon_z + \epsilon_x\epsilon_y + \epsilon_x\epsilon_z + \epsilon_y\epsilon_z + \epsilon_x\epsilon_y\epsilon_z$$

Since ϵ_x, ϵ_y and ϵ_z are small, terms $\epsilon_x\epsilon_y$, $\epsilon_y\epsilon_z$ and $\epsilon_z\epsilon_x$, $\epsilon_x\epsilon_y\epsilon_z$ are neglected as they are ~~too~~ small.

$$V' = 1 + \epsilon_x + \epsilon_y + \epsilon_z$$

$$\text{change in volume } \delta V = V' - V = \epsilon_x + \epsilon_y + \epsilon_z$$

$$V = 1$$

$$\text{Volumetric strain } \epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$$

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\epsilon_v = \frac{(\sigma_x + \sigma_y + \sigma_z)(1-2\nu)}{E}$$

if $\sigma_x = \sigma_y = \sigma_z = \sigma$ (same stress) (direct stress)

then $\epsilon_v = \frac{3\sigma}{E}(1-2\nu) \Rightarrow E = \frac{3\sigma}{\epsilon_v}(1-2\nu)$

K (Bulk Modulus) $= \frac{\sigma}{\epsilon_v}$

$$E = 3K(1-2\nu)$$

$$K = \frac{E}{3(1-2\nu)}$$

Since $G = \frac{E}{2(1+\nu)}$

Relation between G and K is

$$G = \frac{3K(1-2\nu)}{2(1+\nu)}$$

$$E = \frac{9GK}{3K+G}$$

Plane stress and Plane strain :-

A state of plane stress is said to exist when the elastic body is very thin and there are no loads applied in the coordinate direction parallel to the thickness.

In 2D objects, stresses can be produced in only two directions and not possible in the third direction.

In case of plane stress problems, the out of plane stress components are considered to be zero.

(non-zero stress components)

$\sigma_x, \sigma_y, \tau_{xy}, \epsilon_z$ are present

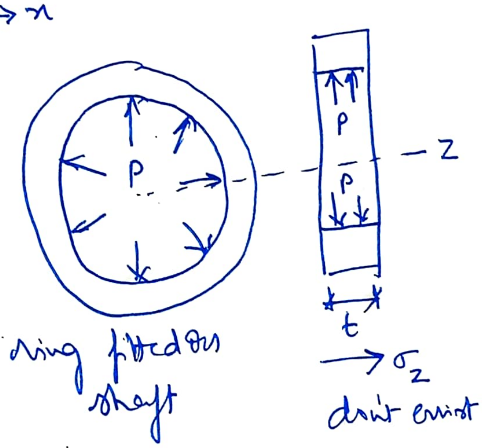
$\sigma_z = \tau_{yz} = \tau_{zx} = 0$ (out of plane)

$\epsilon_{yz} = \epsilon_{zx} = 0$

ϵ_z is produced by the stress σ_x and σ_y

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\sigma_x \nu}{E} - \frac{\sigma_y \nu}{E}$$

$\epsilon_x, \epsilon_y, \epsilon_z, \epsilon_{xy} \neq 0$



In the direction parallel to the axis i.e. stress cannot be distributed in the direction parallel to the thickness.

Example: Domains bounded by two parallel planes separated by a distance that is small in comparison to other dimensions. i.e, thin plate.

This formulation theory of 2D problems assume that these planes are stress free i.e $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ on each face. In z direction, the region is thin so stress variation is very small i.e zero so $\sigma_z = 0$.

Plane stress problems are in-plane deformations of thin elastic plates.

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y)$$

$$\epsilon_x = \frac{1}{E}(\sigma_y - \nu \sigma_x)$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = \frac{-\nu}{(1-\nu)}(\epsilon_x + \epsilon_y)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$G = \frac{E}{2(1+\nu)}$$

ϵ_z represents out of plane strain component instead of inplane components so ϵ_z will exist for plane stress.

Plane strain

The state of plane strain occurs in members that are not free to expand in the direction ⊥r to the plane of applied loads.

$\epsilon_z = 0 \quad \gamma_{yz} = 0 \quad \gamma_{zx} = 0$ but σ_z may not be zero

Example infinitely long cylinder (prismatic body)

u, v are present but $w = 0$

$\epsilon_x = \frac{\partial u}{\partial x} \quad \epsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

$\sigma_z = \frac{-\nu}{E}(\sigma_x + \sigma_y)$

though $\epsilon_z = 0$ corresponding σ_z will not vanish

NT: In plane stress:- non zero strain components are 4 $\epsilon_x, \epsilon_y, \epsilon_z$ and ϵ_{xy} are present because in plane stress case ($\sigma_x, \sigma_y, \tau_{xy}$ and also present so corresponding stress only present) though $\sigma_z = 0$ because ϵ_z is also present but ϵ_z is present even dependent on $\sigma_z, \sigma_x, \sigma_y$ $\left(\epsilon_z = \frac{\sigma_z}{E} - \nu\left(\frac{\sigma_x + \sigma_y}{E}\right)\right)$ $\left(\epsilon_z = -\nu\left(\frac{\sigma_x + \sigma_y}{E}\right)\right)$

So non-zero stress components are 3 $(\sigma_x, \sigma_y, \tau_{xy})$.

plane strain - non-zero strain components for plane strain problems are 3. $(\epsilon_x, \epsilon_y, \epsilon_{xy})$ and non-zero stress components are 4 $(\sigma_x, \sigma_y, \tau_{xy}, \sigma_z)$

$\epsilon_x, \epsilon_y, \epsilon_{xy} \neq 0 \; ; \; \epsilon_z = \epsilon_{xz} = \epsilon_{yz} = 0$ (out of plane strain components)

$\epsilon_z = \frac{\sigma_z}{E} - \nu\frac{(\sigma_x + \sigma_y)}{E}$

$0 = \frac{\sigma_z}{E} - \nu\left(\frac{\sigma_x + \sigma_y}{E}\right) \Rightarrow \sigma_z = \nu(\sigma_x + \sigma_y)$

$\sigma_z \neq 0$

Prob:- In a plane stress problem, $\sigma_x = 5\,MPa$, $\sigma_y = -10\,MPa$, σ_{xy} or $\tau_{xy} = 7.5\,MPa$. Calculate ϵ_z if the E value $= 2\,GPa$ ($2\times10^9\,Pa$) and ν is 0.15.

$\sigma_z = 0$ plane stress problems

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu(\sigma_x + \sigma_y)}{E} = \frac{-0.15\,(5-10)\times10^6}{2\times10^9}$$

$$\epsilon_z = 3.75\times10^{-4}$$

Prob:- In plane strain problem, $\epsilon_x = 0.005$, $\epsilon_y = -0.001$, $\epsilon_{xy} = 0.006$. Calculate σ_{xz} if $E = 2\,GPa$ and $\nu = 0.25$

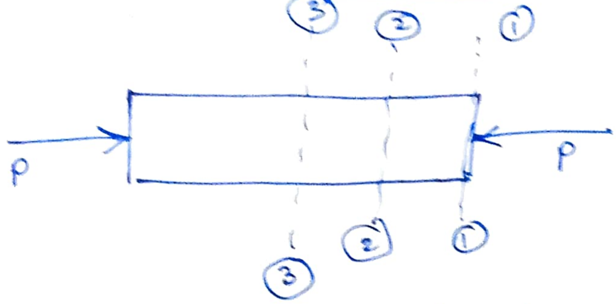
$\epsilon_{xz} = \epsilon_{yz} = \epsilon_z = 0$ plane strain problems

$\sigma_{xz} = 0 \leftarrow \sigma_{xz} = G.\epsilon_{xz}$

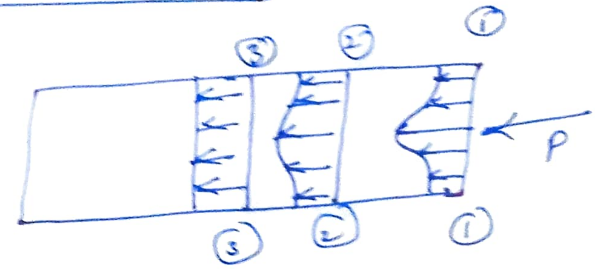
Note: In plane stress problem, the average out of plane displacements are zero.

See pg (87)

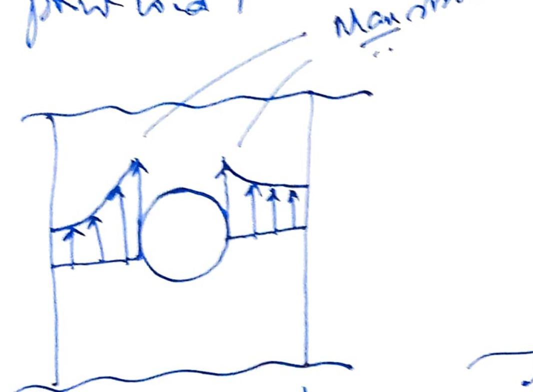
Stress Concentration :- (St. Venant's Principle)



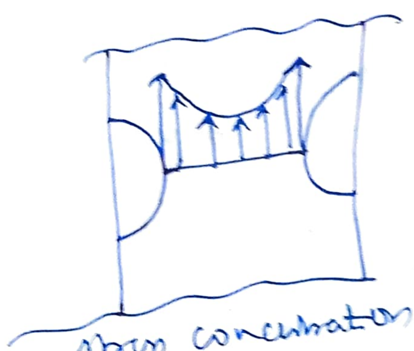
Stress concentration due to point load P



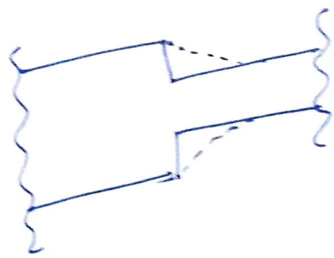
Stress diagrams



Stress concentration at a hole



Stress concentration at the notches



Stresses at a section sufficiently away from the load is uniform whereas stress at the section where the load is applied or near that end is not uniform. This principle is known as <u>St. Venant's principle</u>. The local effect of the concentrated load is to increase the stress around the load point. This effect is called <u>stress concentration</u> or <u>stress localization</u>. Any dis-continuity in the material like a hole or notches in the section causes stress-concentration as shown above. The stress near the hole or notch is much higher than the average stress normally calculated as load / area.

<u>Stress concentration factor</u> is the ratio of max stress to the average stress.

<u>St. Venant's principle</u> states that the stress at a distance away from the load point is equal to the avg. stress.

The stress is much higher at or near the edge of the hole than average stress (stress concentration factor can be as high as 3). In case of sudden change in section, the stress concentration near the change of section is more so to reduce this effect, the change of section is made gradual by providing fillets.

Stress concentration factors are worked out using the theory of elasticity or experimentally by photo elasticity. Using finite element analysis, stress concentration can be determined accurately at any form of discontinuity.

So as per St. Venant's principle, stress-distribution is uniform over the c/s at a section at a distance away from the application of external forces. Near ends, stress-distribution is uncertain and non-uniform (stress concentration).

State of stress at a point

Cartesian (x, y, z) co-ordinate system

Nine stress components must be known at each point to define completely state of stress at a point

But it is proved that shear stresses are complementary

i.e. $\tau_{xy} = \tau_{yx}$; $\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$

Therefore there are only six components of stress at a point, three normal stresses and three shear stresses. Therefore stress at a point is specified as

$$[\sigma] = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix}$$

Similarly, six stress components in the cylindrical (r,Θ , z) co-ordinate system

$$[\sigma] = \begin{pmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{pmatrix}$$

The state of strain at a point of a body in the Cartesian (x, y, z) co-ordinate system can be expressed in the matrix form as

$$[\varepsilon] = \begin{pmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{pmatrix}$$

Similarly, six strain components in the cylindrical (r,Θ , z) co-ordinate system

$$[\varepsilon] = \begin{pmatrix} \varepsilon_r & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_\theta & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_z \end{pmatrix}$$

Strain Displacement relationship

The six strain components, three linear strain and three shear strains, at a point of the body are related to the three displacements u, v, and w by the following expressions in the Cartesian (x, y, z) co-ordinate system

Normal strain: $\varepsilon_x = \dfrac{\partial u}{\partial x}$, $\varepsilon_y = \dfrac{\partial v}{\partial y}$, $\varepsilon_z = \dfrac{\partial w}{\partial z}$

Shear strain: $\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$, $\gamma_{yz} = \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}$, $\gamma_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}$.

Strain displacement relationship for cylindrical (r, θ, z) co-ordinate system

Normal strain: $\varepsilon_r = \dfrac{\partial u}{\partial r}$, $\varepsilon_\theta = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{u}{r}$, $\varepsilon_z = \dfrac{\partial w}{\partial z}$

Shear strain: $\gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} + \dfrac{\partial v}{\partial r} - \dfrac{u}{r}$, $\gamma_{\theta z} = \dfrac{\partial v}{\partial z} + \dfrac{1}{r}\dfrac{\partial w}{\partial \theta}$, $\gamma_{rz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r}$.

Equilibrium Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \; ; \; \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \text{ and } \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

Where X, Y and Z are the components of body force such as gravitational, centrifugal, or other inertia forces.

The equilibrium equations for a body referred in cylindrical co-ordinates (r,Θ , z) system.

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \left(\frac{\sigma_r - \sigma_\theta}{r}\right) + P_r = 0 \; ; \; \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + P_\theta = 0$$

and

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + P_z = 0$$

Where P_r, P_θ and P_z are the components of body force such as gravitational, centrifugal, or other inertia forces.

Strain compatibility equations

It is clear from the strain displacement relationship that if the three displacement components are given, then the strain components can be uniquely determined. If, on the other hand, the six strain components are arbitrarily specified at a point, then the displacement components cannot be uniquely determined. This is because the six strain components are related to only three displacement components viz u,v andw. Hence if displacement components are to be single valued and continuous, then there must exist certain interrelationship among the strain components. These relations are called the strain compatibility equations. For three dimensional bodies there exist six strain compatibility equations. In the Cartesian (x, y, z) co-ordinate system.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}; \quad \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad \text{and} \quad \frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}$$

And

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right]; \quad 2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x}\right]$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z}\right]$$

Similarly strain compatibility equations, for the case of small displacements, in terms of cylindrical coordinates (r,Θ , z) can be obtained as

$$\frac{\partial^2 \varepsilon_r}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial r^2} = \frac{\partial^2 \gamma_{rz}}{\partial r \partial z}; \quad -r\frac{\partial \varepsilon_r}{\partial r} + \frac{\partial^2 \varepsilon_r}{\partial \theta^2} + r\frac{\partial^2 (r\varepsilon_\theta)}{\partial r^2} = \frac{\partial^2 (r\gamma_{r\theta})}{\partial r \partial \theta}$$

$$r^2\frac{\partial^2 \varepsilon_\theta}{\partial z^2} + r\frac{\partial \varepsilon_z}{\partial r} + \frac{\partial^2 \varepsilon_z}{\partial \theta^2} - r\frac{\partial \gamma_{rz}}{\partial z} = r\frac{\partial^2 \gamma_{\theta z}}{\partial \theta \partial z}$$

And

$$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \theta}(r\gamma_{r\theta})\right] + \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}(r\gamma_{\theta z}) - \frac{\partial \gamma_{rz}}{\partial \theta}\right] = 2r\frac{\partial}{\partial z}\left[\frac{\partial}{\partial r}(r\varepsilon_\theta) - \varepsilon_r\right]$$

$$\frac{r^2 \partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial}{\partial r}(r\gamma_{\theta z}) - \frac{\partial \gamma_{rz}}{\partial \theta}\right)\right] - \frac{\partial^2 (r\gamma_{r\theta})}{\partial r \partial z} = 2\frac{\partial^2 (r\varepsilon_r)}{\partial \theta \partial z}$$

$$\frac{\partial}{\partial z}\left[\frac{\partial \gamma_{r\theta}}{\partial z} - r\frac{\partial}{\partial r}\left(\frac{\gamma_{\theta z}}{r}\right) - \frac{1}{r}\frac{\partial \gamma_{rz}}{\partial \theta}\right] = -2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varepsilon_z}{\partial \theta}\right)$$

Stress strain relationships

The stresses and strains cannot be independent when we consider physical problem of the theory of elasticity which is concerned with the determination of stress components and deformation due to external loads acting on an elastic body. Hence the stresses need to be related to strain through a physical law. For isotropic material, generalized Hook"s law gives the following stress strain relations.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon(\sigma_y + \sigma_z)\right]; \quad \varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon(\sigma_x + \sigma_z)\right];$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon(\sigma_y + \sigma_x)\right] \quad \text{and} \quad \gamma_{xy} = \frac{\tau_{xy}}{G}, \; \gamma_{yz} = \frac{\tau_{yz}}{G}, \; \gamma_{xz} = \frac{\tau_{xz}}{G}$$

Where v, E and G are the elastic properties of the material.

Similarly in terms of cylindrical coordinates (r,Θ , z) can be obtained as

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \upsilon(\sigma_\theta + \sigma_z)\right]; \quad \varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \upsilon(\sigma_r + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon(\sigma_\theta + \sigma_r)\right] \quad \text{and} \quad \gamma_{r\theta} = \frac{\tau_{r\theta}}{G} \cdot \gamma_{\theta z} = \frac{\tau_{\theta z}}{G} \cdot \gamma_{rz} = \frac{\tau_{rz}}{G}$$

Hooke's Law

Alternately stress-strain relation for isotropic material can be written as,

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{zy}\\ \tau_{zx}\end{Bmatrix} = \frac{E}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix}(1-\upsilon) & \upsilon & \upsilon & 0 & 0 & 0\\ \upsilon & (1-\upsilon) & \upsilon & 0 & 0 & 0\\ \upsilon & \upsilon & (1-\upsilon) & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\upsilon)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\upsilon)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\upsilon)}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}$$

$$\therefore \quad [\sigma] = [D]\{\varepsilon\}$$

OR

$$\sigma_x = \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_x$$
$$\sigma_y = \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_y$$
$$\sigma_z = \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_z$$

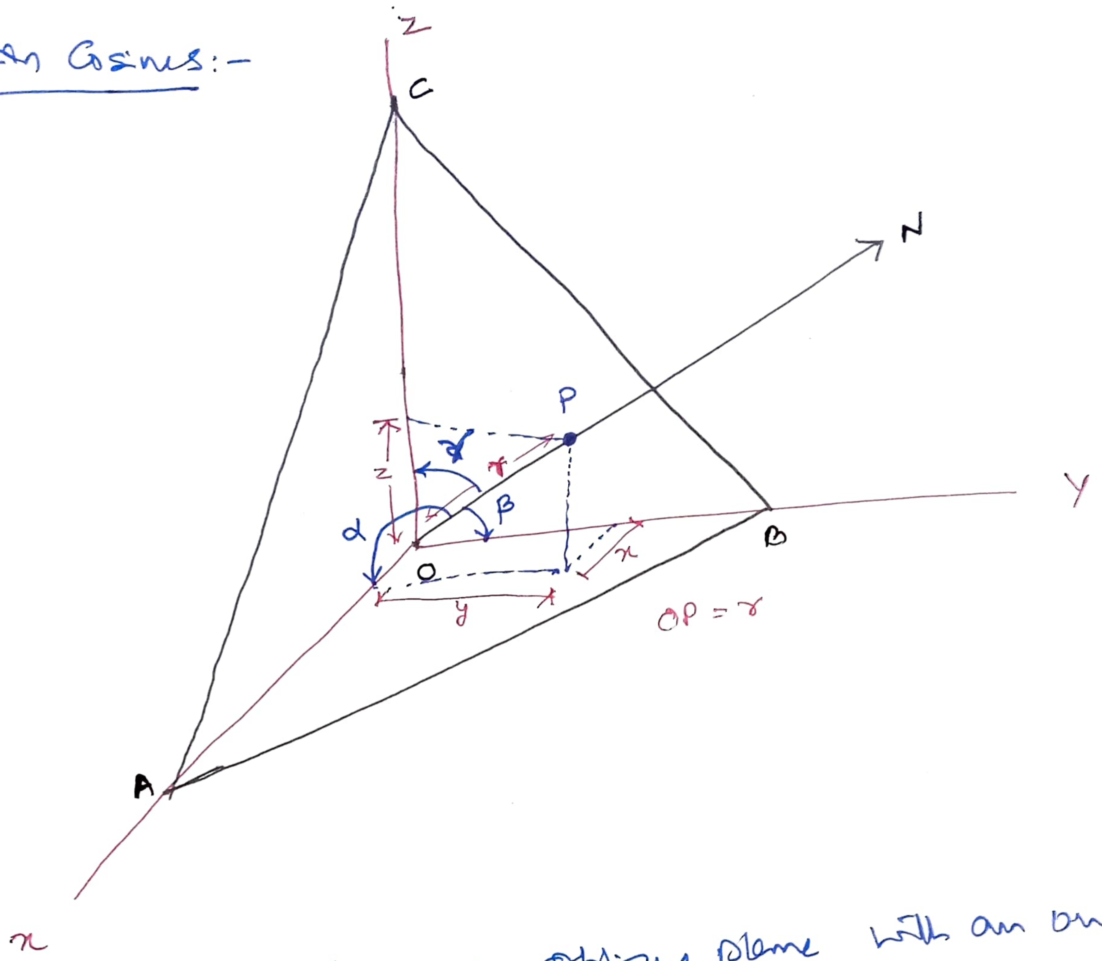
Where $\lambda = \text{Lame's constant} = \frac{\upsilon E}{(1-\upsilon)(1-2\upsilon)}$ and $G = \frac{E}{2(1+\upsilon)}$

Similarly in terms of cylindrical coordinates (r,θ,z) can be obtained as;

$$\sigma_r = \lambda\,(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_r$$
$$\sigma_\theta = \lambda\,(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_\theta$$
$$\sigma_z = \lambda\,(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_z$$

State of stress on an Oblique Plane :- (General plane)

Direction Cosines:-



ABC is a general plane or Oblique plane with an outward normal 'n'. The directions of this normal can be expressed in terms of direction cosines. Let the angle of inclination of the normal 'n' to the axes x, y and z be α, β and γ resp. and on the normal Point 'P' (x, y, z) is on the plane ABC at a distance of r from origin 'O' (OP = r)

Coordinates of P (x, y, z) can be written as

$\cos\gamma = n$ or n_z

$\cos\beta = m$ or n_y

$\cos\alpha = l$ or n_x

$z = r\cos\gamma = rn$

where $n = \cos\gamma$

$y = r\cos\beta = rm$

$x = r\cos\alpha = rl$

here l, m, n are known as direction cosines of the line 'OP'.

r is the polar coordinate of Point 'P'

We know that $r^2 = x^2 + y^2 + z^2$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1 \qquad l = \frac{x}{r} = \cos\alpha$$

$$m = \frac{y}{r} = \cos\beta$$

$$l^2 + m^2 + n^2 = 1 \qquad n = \frac{z}{r} = \cos\gamma$$

So only two direction cosines are neessary to define a point, since the third is dependent on the other two.

This above concept can be extended to planes and areas

$$l = \cos\alpha = \frac{\text{Area } OBC}{\text{Area } ABC}$$

Area OBC = projection of area ABC on yz plane

$$m = \cos\beta = \frac{\text{Area } OAC}{\text{Area } ABC}$$

$$n = \cos\gamma = \frac{\text{Area } OBA}{\text{Area } ABC}$$

If the area of plane ABC is dA, then

Area OBC = $dA \cdot l$

Area OAC = $m \cdot dA$

Area OBA = $n \cdot dA$

In addition to the nine cartesian stress components, the component of the resultant stress σ_R along x, y, z axes are σ_{Rx}, σ_{Ry}, σ_{Rz}. They act on the same plane ABC

Equilibrium of forces in x-direction

$$\sigma_x \, l dA + \tau_{xy} m dA + \tau_{xz} n dA = \sigma_{Rx} dA$$

Equilibrium of forces in y-direction

$$\tau_{yx} \, l dA + \sigma_y m dA + \tau_{yz} n dA = \sigma_{Ry} dA \qquad \text{—(1)}$$

Illy in z direction

$$\tau_{zx} \, l dA + \tau_{zy} m dA + \sigma_z n dA = \sigma_{Rz} dA$$

NOTE:- In practice, the state of stress at a point w.r.t cartesian systems is not very significant because the failure of a structure or body due to fracture may occurs due to a state of stress on a different plane which is inclined to the three coordinate axes. Therefore finding the stresses on an oblique plane due to stresses at a point given in a cartesian coordinate systems are important.

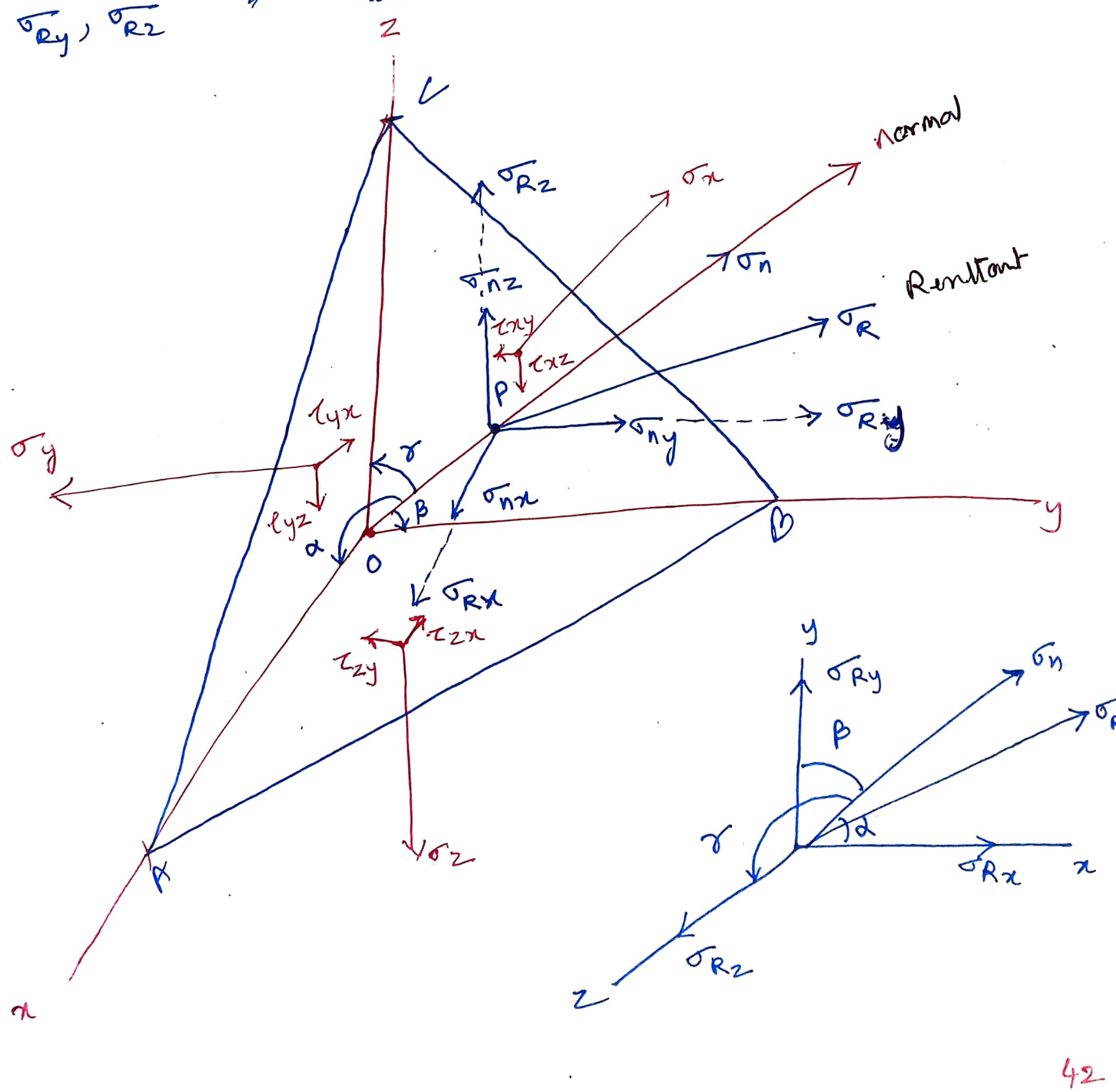
σ_x, τ_{xy}, τ_{xz} are the stresses on the plane OBC

σ_y, τ_{yx}, τ_{yz} " " " " OAC

σ_z, τ_{zx}, τ_{zy} " " " " OAB

σ_{nx}, σ_{ny}, σ_{nz} " " " " ABC

σ_{Rx}, σ_{Ry}, σ_{Rz} " resultant stress on the " ABC



from eqn (1)

$$\sigma_{Rx} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$\sigma_{Ry} = \tau_{yx} l + \sigma_y m + \tau_{yz} n$$

$$\sigma_{Rz} = \tau_{zx} l + \tau_{zy} m + \sigma_z n$$

$$\begin{bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

Cauchy's law

Resultant stress $\sigma_R = \sqrt{(\sigma_{Rx})^2 + (\sigma_{Ry})^2 + (\sigma_{Rz})^2}$

$$\sigma_n = \sigma_{Rx} l + \sigma_{Ry} m + \sigma_{Rz} n$$

$$= (\sigma_x l + \tau_{xy} m + \tau_{xz} n) l + (\tau_{yx} l + \sigma_y m + \tau_{yz} n) m$$

$$+ (\tau_{zx} l + \tau_{zy} m + \sigma_z n) n$$

normal component of stress $\sigma_n = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2(\tau_{xy} lm + \tau_{yz} mn + \tau_{xz} ln)$

$$\tau_n = \sqrt{\sigma_R^2 - \sigma_n^2} \qquad \sigma_R = \sqrt{\sigma_n^2 + \tau_n^2}$$

direct stress component } of
shear stress component }

l_R, m_R, n_R are the direction cosines of the resultant stress σ_R w.r.t the coordinate axes then

$$\sigma_{Rx} = l_R \sigma_R$$

$$\sigma_{Ry} = m_R \sigma_R$$

$$\sigma_{Rz} = n_R \sigma_R$$

$$l_R^2 + m_R^2 + n_R^2 = 1$$

Example: The cartesian stress components at a point Q are given below. Find the stress resultant at Q on a plane passing through Q whose normal is coincident with the x-axis. Also find σ_n and τ_n.

$\sigma_x = 150$ MPa $\sigma_y = -100$ MPa $\sigma_z = 200$ MPa

$\tau_{xy} = \tau_{yx} = 75$ MPa $\tau_{yz} = \tau_{zy} = 30$ MPa $\tau_{xz} = \tau_{zx} = -50$ MPa

The normal is coincident with x-axis so $\alpha = 0$, $\beta = 90^\circ$, $\gamma = 90^\circ$

$l = \cos\alpha = 1$

$m = \cos\beta = 0$

$n = \cos\gamma = 0$

$$\begin{bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

$$= \begin{bmatrix} 150 & 75 & -50 \\ 75 & -100 & 30 \\ -50 & 30 & 200 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 150 \\ 75 \\ -50 \end{bmatrix}$$

$$\sigma_R = \text{resultant stress} = \sqrt{\sigma_{Rx}^2 + \sigma_{Ry}^2 + \sigma_{Rz}^2}$$

$$= \sqrt{150^2 + 75^2 + (-50)^2} = 175 \text{ MPa}$$

$$\sigma_n \text{ normal component} = \sigma_{Rx} l + \sigma_{Ry} m + \sigma_{Rz} n$$

$$= 1 \times 150 + 0 \times 75 + (0 \times -50) = 150 \text{ MPa}$$

$$\tau_n = \sqrt{\sigma_R^2 - \sigma_n^2} = \sqrt{175^2 - 150^2} = 90.14 \text{ MPa}$$

Example:- At a point in a stressed material, the cartesian stress components are

$\sigma_x = -50$ MPa $\sigma_y = 40$ MPa $\sigma_z = 20$ MPa $\tau_{xy} = \tau_{yx} = -25$ MPa

$\tau_{xz} = \tau_{zx} = -5$ MPa. $\tau_{yz} = \tau_{zy} = 10$ MPa

Calculate the normal, shear and resultant stresses on a plane whose normal makes an angle of 62° with the x-axis and 35° with the y-axis.

$\alpha = 62^\circ$ $\quad l = \cos\alpha = 0.47$

$\beta = 35^\circ$ $\quad m = \cos\beta = 0.82$

$l^2 + m^2 + n^2 = 1$

$n = 0.33$

Applying Cauchy's law

$$\begin{Bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{Bmatrix} = \begin{bmatrix} -50 & -25 & -5 \\ -24 & 40 & 10 \\ -5 & 10 & 20 \end{bmatrix} \begin{Bmatrix} 0.47 \\ 0.82 \\ 0.33 \end{Bmatrix} = \begin{bmatrix} -45.65 \\ 24.37 \\ 12.45 \end{bmatrix}$$

σ_R = Resultant stress $= \sqrt{\sigma_{Rx}^2 + \sigma_{Ry}^2 + \sigma_{Rz}^2} = 53.22$ MPa

normal component $\sigma_n = \sigma_{Rx} l + \sigma_{Ry} m + \sigma_{Rz} n$

$= 0.47 \times -45.65 + 0.82 \times 24.37 + 0.33 \times 12.45$

$= 2.64$ MPa

shear stress $\tau_n = \sqrt{\sigma_R^2 - \sigma_n^2} = \sqrt{53.22^2 - 2.64^2} = 53.15$ MPa.

Example :- At a point in a material, a resultant stress of 25MPa acts in a direction making angle of 50° and 80° with the x and y axes resp.

(a) find the normal and shear stresses on an oblique plane whose normal makes angles of 60° and 75° resp. with the x and y axes

(b) For the same resultant stress, if $\tau_{xy} = 20$ MPa, $\tau_{yz} = 30$ MPa $\tau_{xz} = -15$ MPa. Determine σ_x, σ_y and σ_z.

$\sigma_R = 25 MPa$

$\alpha_R = 50^\circ$ $\quad l_R = \cos\alpha_R = 0.64$

$\beta_R = 80^\circ$ $\quad m_R = \cos\beta_R = 0.17$

$\sqrt{l_R^2 + m_R^2 + n_R^2} = 1$

$n_R = 0.75$

$\sigma_{Rx} = l_R \times \sigma_R = 0.64 \times 25 = 16 MPa$

$\sigma_{Ry} = m_R \times \sigma_R = 0.17 \times 25 = 4.25 MPa$

$\sigma_{Rz} = n_R \times \sigma_R = 0.75 \times 25 = 18.75 MPa$

(a) $\alpha = 60^\circ$ $\quad l = \cos\alpha = 0.5$

$\beta = 75^\circ$ $\quad m = \cos\beta = 0.26$

$\sqrt{l^2 + m^2 + n^2} = 1$

$n = 0.83$

$\sigma_n = \sigma_{Rx} l + \sigma_{Ry} m + \sigma_{Rz} n$

$= 0.5 \times 16 + 0.26 \times 4.25 + 0.83 \times 18.75 = 24.67 MPa$

$\tau_n = \sqrt{\sigma_R^2 - \sigma_n^2} = \sqrt{25^2 - 24.67^2} = 4.05 MPa$

(b) $\tau_{xy} = 20 MPa$ $\quad \tau_{yz} = 30 MPa$ $\quad \tau_{xz} = -15 MPa$

Cauchy's formula

$$\begin{Bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

$$= \begin{bmatrix} \sigma_x & 20 & -15 \\ 20 & \sigma_y & 30 \\ -15 & 30 & \sigma_z \end{bmatrix} \begin{Bmatrix} 0.50 \\ 0.26 \\ 0.83 \end{Bmatrix} = \begin{Bmatrix} 16 \\ 4.25 \\ 18.75 \end{Bmatrix}$$

$\sigma_x = 46.50 MPa$ $\quad \sigma_z = 22.25 MPa.$

$\sigma_y = 117.88 MPa$

Example: The following state of strain exists at a point 'P'

$$\begin{bmatrix} 0.04 & -0.02 & 0.01 \\ -0.02 & 0.08 & 0.03 \\ 0.01 & 0.03 & 0.06 \end{bmatrix}$$

in the direction of PQ having direction cosines $l = 0.8$ and $m = 0.6$. Find σ_{PQ} and τ_{PQ}

ϵ_{PQ} $\quad$ γ_{PQ}

$l = 0.8 \quad m = 0.6$

$\sqrt{l^2 + m^2 + n^2} = 1 \qquad n = 0$

Cauchy's law

$$\begin{Bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{Bmatrix} \begin{matrix} \epsilon_{Rx} \\ \epsilon_{Ry} \\ \epsilon_{Rz} \end{matrix} = \begin{bmatrix} 0.04 & -0.02 & 0.01 \\ -0.02 & 0.08 & 0.03 \\ 0.01 & 0.03 & 0.06 \end{bmatrix} \begin{Bmatrix} 0.8 \\ 0.6 \\ 0 \end{Bmatrix}$$

$$\begin{matrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{matrix} \begin{Bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{Bmatrix} = \begin{bmatrix} 0.020 \\ 0.032 \\ 0.026 \end{bmatrix}$$

$$\epsilon_R \; \sigma_R = \sqrt{\sigma_{Rx}^2 + \sigma_{Ry}^2 + \sigma_{Rz}^2} = 0.045$$

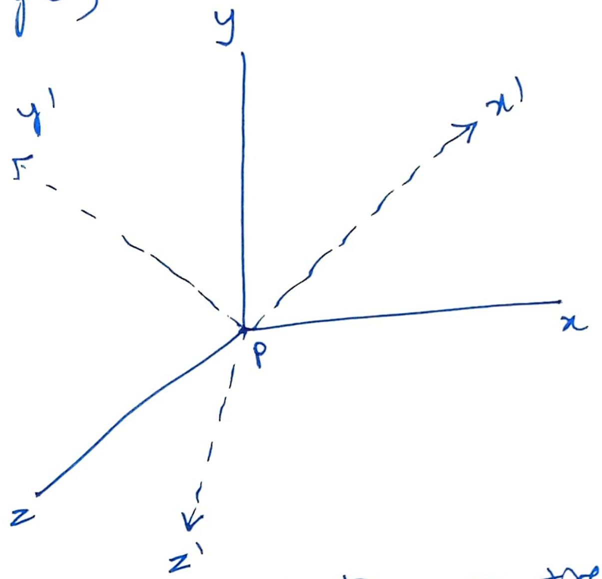
$$\epsilon \; \sigma_n = \sigma_{PQ} = \sigma_{Rx} l + \sigma_{Ry} m + \sigma_{Rz} n$$

$$= 0.8 \times 0.02 + 0.6 \times 0.032 + 0 \times 0.026 = 0.0352$$

$$\gamma \; \tau_{PQ} = \sqrt{\sigma_R^2 - \sigma_n^2} = \sqrt{0.045^2 - 0.0352^2} = 0.028$$

Stress transformation :-

Many number of planes pass through a point. On each of these planes there is a resultant stress with three stress components. All these resultant stresses together define the complete state of stress at a point. For transforming the stress from one coordinate system P(x, y, z) to another coordinate system P(x'y'z')



Counter clockwise is taken as +ve.

The transformation matrix [Q] is given as

$$[Q] = \begin{bmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{bmatrix}$$

Example :- The state of stress at a point relative to an xyz coordinate system

$$\begin{bmatrix} 50 & -25 & 15 \\ -25 & 30 & 10 \\ 15 & 10 & 20 \end{bmatrix} \text{MPa}$$

Determine the state of stress relative to an equivalent x'y'z' coordinate system if

Axes	x	y	z
x'	30°	60°	90°
y'	120°	30°	90°
z'	90°	90°	90°

The transformation matrix $[Q]$

$$[Q] = \begin{bmatrix} \cos(x'x) & \cos(x'y) & \cos(x'z) \\ \cos(y'x) & \cos(y'y) & \cos(y'z) \\ \cos(z'x) & \cos(z'y) & \cos(z'z) \end{bmatrix} = \begin{bmatrix} \cos 30^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 120^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 90^\circ \end{bmatrix}$$

$$[Q] = \begin{bmatrix} 0.87 & 0.5 & 0 \\ -0.5 & 0.87 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The stress in another coordinate system can be obtained by using the general tensor transformation rule

$$[\sigma'] = [Q][\sigma][Q]^T$$

$$= \begin{bmatrix} 0.87 & 0.5 & 0 \\ -0.5 & 0.87 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 50 & -25 & 15 \\ -25 & 30 & 10 \\ 15 & 10 & 20 \end{bmatrix} \begin{bmatrix} 0.87 & -0.5 & 0 \\ 0.5 & 0.87 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

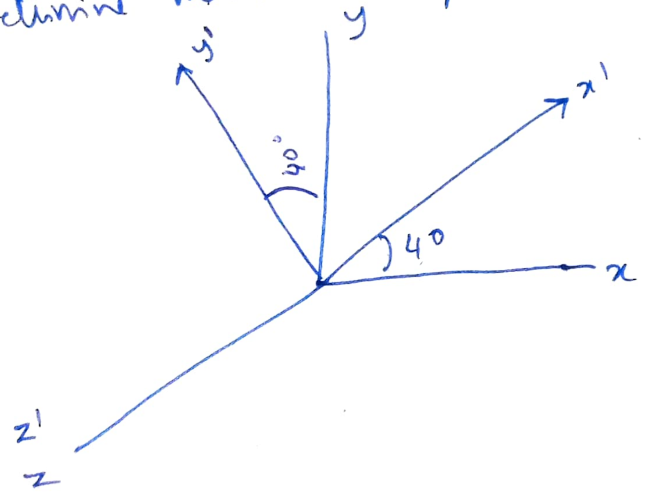
$$[\sigma'] = \begin{bmatrix} 23.6 & -21.37 & 0 \\ -21.37 & 56.96 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

Example:- Consider a state of stress at a following point

$$\begin{bmatrix} 70 & 80 & 50 \\ 80 & -60 & 40 \\ 50 & 40 & 30 \end{bmatrix} \text{MPa}.$$

Consider another set of coordinate axes in which z' coincides with z and x' is rotated counter clockwise by 40° from the x axis in new coordinate system.

Determine the stress components



The transformation matrix $[\Phi]$ is given by

$$[\Phi] = \begin{bmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{bmatrix} = \begin{bmatrix} \cos 40^\circ & \cos 50^\circ & \cos 90^\circ \\ \cos 130^\circ & \cos 40^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0.77 & 0.64 & 0 \\ -0.64 & 0.77 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The stress in another coordinate system can be obtained by using the general tensor transformation rule

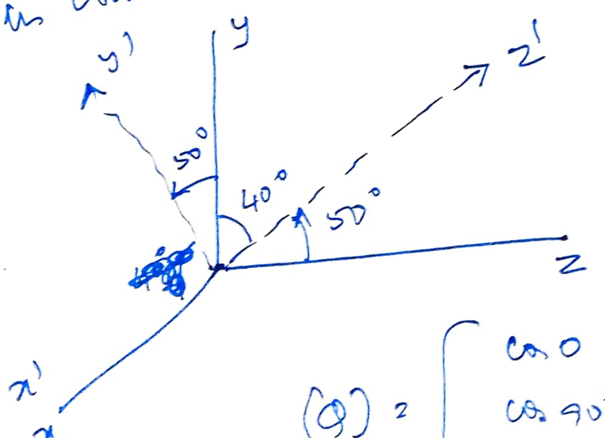
$$[\sigma'] = [\Phi][\sigma][\Phi]^T$$

$$= \begin{bmatrix} 0.77 & 0.64 & 0 \\ -0.64 & 0.77 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 70 & 80 & 50 \\ 80 & -60 & 40 \\ 50 & 40 & 30 \end{bmatrix} \begin{bmatrix} 0.77 & -0.64 & 0 \\ 0.64 & 0.77 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\sigma'] = \begin{bmatrix} 95.06 & -49.4 & 64.1 \\ -49.4 & -85.75 & -1.2 \\ 64.1 & -1.2 & 30 \end{bmatrix} \text{ MPa.}$$

Example :- The strain components at a point w.r.t x, y and z axes are $\epsilon_x = 0.5$ $\epsilon_y = 0.3$ $\epsilon_z = 0.2$ $\gamma_{xy} = 0.16$ $\gamma_{yz} = 0.20$ $\gamma_{zx} = 0.12$

If the coordinate axes are rotated about the x axis through 50° in counter clockwise direction. Det. the new strain components.



$$[\Phi] = \begin{bmatrix} \cos 0 & \cos 90^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 50^\circ & \cos(140^\circ) \\ \cos 90^\circ & \cos 40^\circ & \cos 50^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.64 & -0.77 \\ 0 & 0.77 & 0.64 \end{bmatrix}$$

The strain in another coordinate system can be obtained by using the general tensor transformation rule.

$$[\epsilon'] = [Q][\epsilon][Q]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.67 & 0.77 \\ 0 & -0.77 & 0.64 \end{bmatrix} \begin{bmatrix} 0.5 & \frac{0.16}{2} & \frac{0.12}{2} \\ \frac{0.16}{2} & 0.3 & \frac{0.20}{2} \\ \frac{0.12}{2} & \frac{0.20}{2} & 0.2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.64 & -0.77 \\ 0 & 0.77 & 0.64 \end{bmatrix}$$

$$[\epsilon'] = \begin{bmatrix} 0.5 & 0.098 & -0.02 \\ 0.098 & 0.34 & -0.067 \\ -0.02 & -0.067 & 0.16 \end{bmatrix}$$

(Note):- State of stress in xyz coordinate $= \sigma$

State of stress in $x'y'z'$ coordinate $= \sigma'$

Orthogonal Rotation matrix $= Q$

$$\sigma' = Q \sigma Q^T$$

(Note):-

Stress and Strain Tensor – 2nd order tensor → need two directions (direction and plane)

σ_{xy} – acting on plane whose normal is along x-axis and is ^ the direction of y axis.

PRINCIPAL STRESSES AND PLANES

In engineering structures safety against failure of a structure is major concern. So finding maximum normal and shear stresses are very important. These values depends on the plane under consideration. State of stress is fully known if we know the component of stress acting on three mutually perpendicular planes. If we know the nine components of stress matrix we can find the normal and shear stresses on any arbitrary plane passing through the point.

At any point in a stressed body there are atleast three planes called principal planes whose normals are called principal directions. Stresses along the principal directions has no shear stress components. The three stresses normal to these principal planes are called principal stresses.

The component of the stress tensor depends on the orientation of the coordinate systems at the point under consideration. There are certain invariants associated with the stress tensor which are independent of the coordinate systems. Every second rank tensors like stress and strains has three independent invariant quantities associated with them. One such invariant is the principal stresses of the stress tensor which are eigen values of the stress tensor and their direction vectors are the principal direction or eigen vectors.

Note :- It is important to find

(a) planes passing through the given point on which the resultant stresses are wholly normal (resultant is along normal)

(b) Plane on which normal stress is maximum

(c) Plane on which Tangential or shear stress is maximum.

The magnitude of the normal and shear stresses at any point depends on the orientation of the plane. You need to know the values of maximum and minimum normal and shear stresses to know on which plane the isotropic body fails.

We know that

$$\sigma_n = l\,\sigma_{Rx} + m\,\sigma_{Ry} + n\,\sigma_{Rz}$$

assume that resultant is along the normal.

To find the extreme values of σ_n, understand the variation of σ_n w.r.t direction cosines.

Direction cosines satisfy $l^2+m^2+n^2=1$. If l and m are independent variables then n depends on l and m which is dependent on l and m

$$n^2 = 1-l^2-m^2$$

Differentiate w.r.t l

$$2n\frac{\partial n}{\partial l} = -2l$$

$$\frac{\partial n}{\partial l} = -\frac{l}{n} \quad —①$$

Differentiate w.r.t m

$$2n\frac{\partial n}{\partial m} = -2m$$

$$\frac{\partial n}{\partial m} = -\frac{m}{n} \quad —②$$

$\left.\begin{matrix} l \\ m \end{matrix}\right\}$ independent variables

$n \rightarrow$ dependent variable on l and m

$\frac{\partial m}{\partial l} = 0$ because m and l are independent whereas $\frac{\partial n}{\partial l}$ exist because n is dependent on l [see eqn ① and ②]

For finding the extreme values of σ_n, differentiate w.r.t l and m and equate to zero.

$$\frac{\partial \sigma_n}{\partial l} = 0 \quad \text{and} \quad \frac{\partial \sigma_n}{\partial m} = 0$$

$$\frac{\partial \sigma_n}{\partial l} = \frac{\partial}{\partial l}\left(l\,\sigma_{Rx} + m\,\sigma_{Ry} + n\,\sigma_{Rz}\right) = \sigma_{Rx} + \sigma_{Rz}\left(\frac{-l}{n}\right) = 0 \quad —③$$

$$\frac{\partial \sigma_n}{\partial m} = \frac{\partial}{\partial m}\left(l\,\sigma_{Rx} + m\,\sigma_{Ry} + n\,\sigma_{Rz}\right) = \sigma_{Ry} + \sigma_{Rz}\frac{\partial n}{\partial m} = \sigma_{Ry} + \sigma_{Rz}\left(\frac{-m}{n}\right) = 0 \quad —④$$

The magnitude of the normal and shear stresses at any point depends on the orientation of the plane. you need to know the values of maximum and minimum normal and shear stresses to known on which plane the isotropic body fails.

we know that

$$\sigma_n = l\,\sigma_{Rx} + m\,\sigma_{Ry} + n\,\sigma_{Rz}$$

assume that resultant is along the normal.

To find the extreme values of σ_n, understand the variation of σ_n w.r.t direction cosines.

Direction cosines satisfy $l^2+m^2+n^2=1$. if l and m are independent variables then n depends on l and m which is dependent on l and m

$$n^2 = 1-l^2-m^2$$

Differentiate w.r.t l

$$2n\frac{\partial n}{\partial l} = -2l$$

$$\frac{\partial n}{\partial l} = -\frac{l}{n} \quad -(1)$$

Differentiate w.r.t m

$$2n\frac{\partial n}{\partial m} = -2m$$

$$\frac{\partial n}{\partial m} = -\frac{m}{n} \quad -(2)$$

l, m } independent variables

$n \rightarrow$ dependent variable on l and m

$\frac{\partial m}{\partial l} = 0$ because m and l are independent whereas $\frac{\partial n}{\partial l}$ exist because n is dependent on l [see eqn (1) and (2)]

For finding the extreme values of σ_n, differentiate w.r.t l and m and equate to zero.

$$\frac{\partial \sigma_n}{\partial l} = 0 \quad \text{and} \quad \frac{\partial \sigma_n}{\partial m} = 0$$

$$\frac{\partial \sigma_n}{\partial l} = \frac{\partial}{\partial l}\left(l\,\sigma_{Rx} + m\,\sigma_{Ry} + n\,\sigma_{Rz}\right) = \sigma_{Rx} + \sigma_{Rz}\left(\frac{-l}{n}\right) = 0 \quad -(3)$$

$$\frac{\partial \sigma_n}{\partial m} = \frac{\partial}{\partial m}\left(l\,\sigma_{Rx} + m\,\sigma_{Ry} + n\,\sigma_{Rz}\right) = \sigma_{Ry} + \sigma_{Rz}\frac{\partial n}{\partial m} = \sigma_{Ry} + \sigma_{Rz}\left(\frac{-m}{n}\right) = 0 \quad -(4)$$

from eqn (3) and (4)

$$\frac{\sigma_{Rx}}{l} = \frac{\sigma_{Ry}}{m} = \frac{\sigma_{Rz}}{n} = \sigma_p \text{ (say)}$$

let l_p, m_p and n_p are the direction cosines of σ_p

$$\sigma_{Rx} = l_p \sigma_p$$
$$\sigma_{Ry} = m_p \sigma_p$$
$$\sigma_{Rz} = n_p \sigma_p$$

This indicates that the stress resultant σ_R is parallel to the unit normal and hence contains no shear component values. It is concluded that on a plane for which τ_n has extreme values the shear stress vanishes. The normal is thus coincides with the resultant stress.

As per Cauchy's formula

$$\begin{bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l_p \\ m_p \\ n_p \end{bmatrix} = \begin{bmatrix} l_p \sigma_p \\ m_p \sigma_p \\ n_p \sigma_p \end{bmatrix}$$

$$\left.\begin{aligned} l_p \sigma_p &= \sigma_x l_p + \tau_{xy} m_p + \tau_{xz} n_p \\ m_p \sigma_p &= \tau_{yx} l_p + \sigma_y m_p + \tau_{yz} n_p \\ n_p \sigma_p &= \tau_{zx} l_p + \tau_{zy} m_p + \sigma_z n_p \end{aligned}\right\}$$

$$(\sigma_x - \sigma_p) l_p + \tau_{xy} m_p + \tau_{xz} n_p = 0$$
$$\tau_{yx} l_p + (\sigma_y - \sigma_p) m_p + \tau_{yz} n_p = 0$$
$$\tau_{zx} l_p + \tau_{zy} m_p + (\sigma_z - \sigma_p) n_p = 0$$

$$\begin{bmatrix} (\sigma_x - \sigma_p) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - \sigma_p) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z - \sigma_p) \end{bmatrix} \begin{bmatrix} l_p \\ m_p \\ n_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$l_p = m_p = n_p = 0$ is not possible because condition $l_p^2 + m_p^2 + n_p^2 = 1$ has to be satisfied. Hence the only possible non-trivial solution can be obtained as

$$\begin{vmatrix} (\sigma_x - \sigma_p) & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & (\sigma_y - \sigma_p) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z - \sigma_p) \end{vmatrix} = 0$$

evaluating the determinant gives the characteristic equation

$$\sigma_p^3 - I_1 \sigma_p^2 + I_2 \sigma_p - I_3 = 0 \quad \text{—— (5)}$$

where $I_1 = \sigma_x + \sigma_y + \sigma_z$ (Trace of the matrix)

I_2 = Cofactor of σ_x + Cofactor of σ_y + Cofactor of σ_z

$$= \begin{vmatrix} \sigma_y & \tau_{yz} \\ \sigma_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \sigma_{xz} \\ \sigma_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix}$$

$$I_3 = \text{Determinant of } [\sigma] = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

Coefficients I_1, I_2 and I_3 are called first, second and third stress invariants resp. These values are constant and do not change or do not depend on the orientation of the coordinate system. So change in axis orientation, these values are constant.

Equation (5) has 3 real roots σ_1, σ_2 and σ_3 which are the principal stresses or eigen values.

Principal planes or Direction cosines of the Principal stresses

The relationship between the particular principal stress σ_1 and Cartesian stress components in matrix form is given by

$$\begin{bmatrix} (\sigma_x-\sigma_1) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y-\sigma_1) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z-\sigma_1) \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Cofactor for $(\sigma_x-\sigma_1)$ be $a = \begin{vmatrix} (\sigma_y-\sigma_1) & \tau_{yz} \\ \tau_{zy} & \sigma_z-\sigma_1 \end{vmatrix}$

Cofactor for τ_{xy} be $b = -\begin{vmatrix} \tau_{yx} & \tau_{yz} \\ \tau_{zx} & (\sigma_z-\sigma_1) \end{vmatrix}$

Cofactor for τ_{xz} be $c = \begin{vmatrix} \tau_{yx} & (\sigma_y-\sigma_1) \\ \tau_{zx} & \tau_{zy} \end{vmatrix}$

The direction cosines of the principal stresses are given as

$l_1 = aK \quad m_1 = bK \quad n_1 = cK$

where $K = \dfrac{1}{\sqrt{a^2+b^2+c^2}}$ thus satisfying the condition $l_1^2+m_1^2+n_1^2=1$.

This process can be repeated to other principal stresses σ_2 and σ_3 also, thus obtaining (l_2, m_2, n_2) and $(l_3, m_3$ and $n_3)$

The maximum shear stress $= \tau_{max} = \dfrac{\sigma_1-\sigma_3}{2}$.

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$. This plane is inclined at 45° to the principal plane

For finding the principal strains and their directions, same procedure is adopted. Normally strain invariants are denoted as J_1, J_2 and J_3 resp.

Example:- The state of stress at a point is given by

$\sigma_x = 150$ MPa $\quad \sigma_y = 60$ MPa $\quad \sigma_z = -90$ MPa

$\tau_{xy} = 30$ (corrected: -50) MPa $\quad \tau_{yz} = -75$ MPa $\quad \tau_{xz} = 30$ MPa

Determine the three principle stresses and their directions. Also find the maximum shear stress.

$$[\sigma] = \begin{bmatrix} 150 & -50 & 30 \\ -50 & 60 & -75 \\ 30 & -75 & -90 \end{bmatrix} \text{ MPa}$$

$$\sigma_p^3 - I_1 \sigma_p^2 + I_2 \sigma_p - I_3 = 0$$

$I_1 = \sigma_x + \sigma_y + \sigma_z = 150 + 60 - 90 = 120$

$I_2 =$ Cofactor of σ_x + Cofactor of σ_y + Cofactor of σ_z

$$= \begin{vmatrix} 60 & -75 \\ -75 & -90 \end{vmatrix} + \begin{vmatrix} 150 & 30 \\ 30 & -90 \end{vmatrix} + \begin{vmatrix} 150 & -50 \\ -50 & 60 \end{vmatrix}$$

$= -11025 - 14400 + 6500 = -18925$

$$I_3 = \text{Determinant of } [\sigma] = \begin{vmatrix} 150 & -50 & 30 \\ -50 & 60 & -75 \\ 30 & -75 & -90 \end{vmatrix} = -1257750$$

Note

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\sigma_p^3 - 120\sigma_p^2 - 18925\sigma_p + 1257750 = 0 \qquad |A - \lambda I| = 0$$

$\sigma_1 = 185.47$ MPa

$\sigma_2 = 55.88$ MPa

$\sigma_3 = -121.35$ MPa

Direction cosines of σ_1:-

$$\begin{bmatrix} (150-185.47) & -50 & 30 \\ -50 & (60-185.47) & -75 \\ 30 & -75 & (-90-185.47) \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -35.47 & -50 & 30 \\ -50 & -125.47 & -75 \\ 30 & -75 & -275.47 \end{bmatrix} \begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Cofactor of the determinants on the elements of the first row

$$a = \begin{vmatrix} -125.47 & -75 \\ -75 & -275.47 \end{vmatrix} = 28938.2$$

$$b = -\begin{vmatrix} -50 & -75 \\ 30 & -275.47 \end{vmatrix} = -16023.5$$

$$c = \begin{vmatrix} -50 & -125.47 \\ 30 & -75 \end{vmatrix} = 7514.18$$

$$K = \frac{1}{\sqrt{a^2+b^2+c^2}} = 2.98 \times 10^{-5}$$

The direction cosines of the principal stress σ_1 are

$l_1 = aK = 0.853$

$m_1 = bK = -0.472$

$n_1 = cK = 0.2215$

<u>Direction cosines of σ_2 :-</u>

$$\begin{bmatrix} (150-55.88) & -50 & 30 \\ -50 & (60-55.88) & -75 \\ 30 & -75 & (-90-55.88) \end{bmatrix} \begin{bmatrix} l_2 \\ m_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 94.12 & -50 & 30 \\ -50 & 4.12 & -75 \\ 30 & -75 & -145.88 \end{bmatrix} \begin{bmatrix} l_2 \\ m_2 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a = \begin{vmatrix} 4.12 & -75 \\ -75 & -145.88 \end{vmatrix} = -6228.03$$

$$b = -\begin{vmatrix} -50 & -75 \\ 30 & -145.88 \end{vmatrix} = -9544$$

$$c = \begin{vmatrix} -50 & 4.12 \\ 30 & -75 \end{vmatrix} = 3628.4$$

$$K = \frac{1}{\sqrt{a^2+b^2+c^2}} = 8.36\times10^{-5}$$

Direction cosines of σ_2

$l_2 = aK = -0.52$

$m_2 = bK = -0.798$

$n_2 = cK = 0.303$

Direction cosines of σ_3

$$\begin{bmatrix} (150+121.35) & -50 & 30 \\ -50 & (60+121.35) & -75 \\ 30 & -75 & (-90+121.35) \end{bmatrix} \begin{bmatrix} l_3 \\ m_3 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 271.35 & -50 & 30 \\ -50 & 181.35 & -75 \\ 30 & -75 & 31.35 \end{bmatrix} \begin{bmatrix} l_3 \\ m_3 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a = \begin{vmatrix} 181.35 & -75 \\ -75 & 31.35 \end{vmatrix} = 60.32$$

$$b = -\begin{vmatrix} -50 & -75 \\ 30 & 31.35 \end{vmatrix} = -682.5$$

$$c = \begin{vmatrix} -50 & 181.35 \\ 30 & -75 \end{vmatrix} = -1690.5$$

$$K = \frac{1}{\sqrt{a^2+b^2+c^2}} = 5.48\times10^{-4}$$

The direction cosines of the principal stress σ_3 are

$l_3 = ak = 0.33$

$m_3 = bk = -0.374$

$n_3 = ck = -0.926$

Maximum shear stress = $\frac{\sigma_1 - \sigma_3}{2} = \frac{(185.47) - (-121.35)}{2}$

$= 153.41$ MPa

Example:-

The state of strain at a point is given by

$\epsilon_x = 0.01 \quad \epsilon_y = 0.03 \quad \epsilon_z = 0.05$

$\gamma_{xy} = 0.024 \quad \gamma_{yz} = 0.016 \quad \gamma_{zx} = 0.030$

Det. the three principal strains. Also find the directions of the major principal strain.

The given strain state is

$$[\epsilon] = \begin{bmatrix} 0.01 & \frac{0.024}{2} & \frac{0.030}{2} \\ \frac{0.024}{2} & 0.03 & \frac{0.016}{2} \\ \frac{0.030}{2} & \frac{0.016}{2} & 0.05 \end{bmatrix} = \begin{bmatrix} 0.01 & 0.012 & 0.015 \\ 0.012 & 0.03 & 0.008 \\ 0.015 & 0.008 & 0.050 \end{bmatrix}$$

$$\epsilon_p^3 - J_1 \epsilon_p^2 + J_2 \epsilon_p - J_3 = 0$$

J_1, J_2 and J_3 are strain invariants are given by

$J_1 = \epsilon_x + \epsilon_y + \epsilon_z = 0.01 + 0.03 + 0.05 = 0.09$

J_2 = Cofactor of ϵ_x + Cofactor of ϵ_y + cofactor of ϵ_z

$$= \begin{vmatrix} 0.03 & 0.008 \\ 0.008 & 0.05 \end{vmatrix} + \begin{vmatrix} 0.01 & 0.015 \\ 0.015 & 0.05 \end{vmatrix} + \begin{vmatrix} 0.01 & 0.012 \\ 0.012 & 0.03 \end{vmatrix}$$

$= 1.867 \times 10^{-6}$

$$J_3 = \text{Det. of } [\epsilon] = \begin{vmatrix} 0.01 & 0.012 & 0.015 \\ 0.012 & 0.03 & 0.008 \\ 0.015 & 0.008 & 0.05 \end{vmatrix} = 3.29 \times 10^{-6}$$

$$\epsilon_p^3 - 0.09\epsilon_p^2 + 1.867\times10^{-6}\epsilon_p - 3.29\times10^{-6} = 0$$

$\epsilon_1 = 0.059$

$\epsilon_2 = 0.0284$

$\epsilon_3 = 0.00193$

Major principal strain is ϵ_1

minor principal strain is ϵ_3

Direction cosine of ϵ_1

$$\begin{bmatrix} (0.01-0.059) & 0.012 & 0.015 \\ 0.012 & (0.03-0.059) & 0.008 \\ 0.015 & 0.008 & (0.050-0.059) \end{bmatrix} \begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -0.049 & 0.012 & 0.015 \\ 0.012 & -0.029 & 0.008 \\ 0.015 & 0.008 & -0.009 \end{bmatrix} \begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a = \begin{vmatrix} -0.029 & 0.008 \\ 0.008 & -0.009 \end{vmatrix} = 1.97\times10^{-4}$$

$$b = -\begin{vmatrix} 0.012 & 0.008 \\ 0.015 & -0.009 \end{vmatrix} = 2.28\times10^{-4}$$

$$c = \begin{vmatrix} 0.012 & -0.029 \\ 0.015 & 0.008 \end{vmatrix} = 5.31\times10^{-4}$$

$$K = \frac{1}{\sqrt{a^2+b^2+c^2}} = 1637.9$$

The direction cosines of the principal strain ϵ_1 are

$l_1 = aK = 0.323$

$m_1 = bK = 0.373$

$n_1 = cK = 0.869$

Normal and shear stresses on a plane inclined with respect to the principal plane :-

For a given stress tensor there exists an orthogonal set of axes 1, 2, and 3 (principal axes) in which all the stress tensor elements except the diagonal elements are zero (shear stress components are zero on principal planes).

Transformation matrix will give eigen values.

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \xrightarrow{\text{eigen values}} [\sigma'] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

The principal axes can be taken as new coordinate system (1, 2, 3).

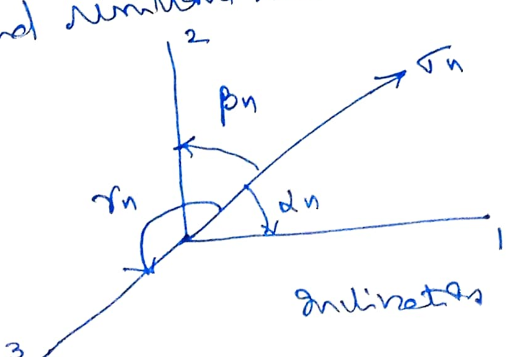
The angles between these coordinate system (1, 2, 3) and the existing coordinate system (x, y, z) are known as the eigen vectors.

Axes	x	y	z
1	cos(1, x)	cos(1, y)	cos(1, z)
2	cos(2, x)	cos(2, y)	cos(2, z)
3	cos(3, x)	cos(3, y)	cos(3, z)

$\approx$

Axes	x	y	z
1	l_1	m_1	n_1
2	l_2	m_2	n_2
3	l_3	m_3	n_3

The tensor transformation rule can be applied w.r.t the principal axis also. Hence if the inclinations of the normal stress acting on any oblique plane w.r.t the principal axes are known, by applying Cauchy's rule we can find the normal, tangential and resultant stresses on this oblique plane.



$\cos\alpha_n = l_n$

$\cos\beta_n = m_n$

$\cos\gamma_n = n_n$

Inclinations of the normal stress σ_n w.r.t principal axes.

$$\begin{bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} l_n \\ m_n \\ n_n \end{bmatrix}$$

$\sigma_{Rx} = \sigma_1 l_n$

$\sigma_{Ry} = \sigma_2 m_n$

$\sigma_{Rz} = \sigma_3 n_n$

$$\sigma_R = \sqrt{\sigma_1^2 l_n^2 + \sigma_2^2 m_n^2 + \sigma_3^2 n_n^2}$$

$$\sigma_n = l_n \sigma_{Rx} + m_n \sigma_{Ry} + n_n \sigma_{Rz}$$

Normal stress } $\sigma_n = \sigma_1 l_n^2 + \sigma_2 m_n^2 + \sigma_3 n_n^2$

shear stress } $\tau_n = \sqrt{\sigma_R^2 - \sigma_n^2}$

<u>Example</u>:- A stress system has three principal values $\sigma_1 = 220$ MPa, $\sigma_2 = 169$ MPa and $\sigma_3 = 70$ MPa. find the normal and shear stresses on a plane the normal of which is inclined at 50° and 60° w.r.t σ_1 and σ_2 directions.

$\cos 50° = l_n = 0.643$

$\cos 60° = m_n = 0.5$

$n_n = \sqrt{1 - l_n^2 - m_n^2} = 0.58$

Apply Cauchy's rule,

$$\begin{bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{bmatrix} = \begin{bmatrix} 220 & 0 & 0 \\ 0 & 169 & 0 \\ 0 & 0 & 70 \end{bmatrix} \begin{bmatrix} 0.643 \\ 0.5 \\ 0.58 \end{bmatrix}$$

$\sigma_{Rx} = 141.46$ MPa

$\sigma_{Ry} = 84.5$ MPa

$\sigma_{Rz} = 40.6$ MPa

$$\sigma_R = \sqrt{141.46^2 + 84.5^2 + 40.6^2} = 169.7 \text{ MPa}$$

$$\sigma_n = \sigma_1 l_n^2 + \sigma_2 m_n^2 + \sigma_3 n_n^2 \text{ or } l_n \sigma_{Rx} + m_n \sigma_{Ry} + n_n \sigma_{Rz}$$

$$= 0.643 \times 141.46 + 0.5 \times 84.5 + 40.6 \times 0.58$$

$$= 156.76 \text{ MPa}$$

$$\tau_n = \sqrt{\sigma_R^2 - \sigma_n^2} = \sqrt{169.7^2 - 156.76^2} = 65 \text{ MPa.}$$

Example :- For the above case, find the normal, shear and resultant strains acting on a plane the normal strains of which is inclined at 64° and 73° with the major and minor principal strains.

$$\cos 64^\circ = l_n = 0.438$$

$$\cos 73^\circ = n_n = 0.29$$

$$m_n = \sqrt{1 - l_n^2 - n_n^2} = 0.85$$

Applying cauchy's rule

$$\begin{bmatrix} \epsilon_{Rx} \\ \epsilon_{Ry} \\ \epsilon_{Rz} \end{bmatrix} = \begin{bmatrix} 0.059 & 0 & 0 \\ 0 & 0.0284 & 0 \\ 0 & 0 & 0.00193 \end{bmatrix} \begin{bmatrix} 0.438 \\ 0.85 \\ 0.29 \end{bmatrix}$$

$$\epsilon_{Rx} = 0.0258$$

$$\epsilon_{Ry} = 0.0241$$

$$\epsilon_{Rz} = 0.000566$$

$$\epsilon_R = \sqrt{0.0258^2 + 0.0241^2 + 0.000566^2} = 0.0353$$

$$\epsilon_n = 0.438 \times 0.0258 + 0.85 \times 0.0241 + 0.29 \times 0.000566$$

$$= 0.0319$$

$$\gamma_n = \sqrt{0.0353^2 - 0.0319^2} = 0.015$$

SS

Note:- $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$ $|A - \lambda I| = 0$ Calculate eigen values.

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & -3-\lambda \end{vmatrix} = 0$$

$\lambda_1, \lambda_2, \lambda_3$ are principle stresses.

$\Rightarrow 2-\lambda \left[(3-\lambda)(-3-\lambda) - 16 \right] = 0$

$(\lambda = 2)(5-\lambda)(5+\lambda) = 0$

$\lambda = 5, 2, -5$ principle stresses.

Construction of Mohr's Circle

It is a graphical method to carry out stress analysis. This is a two-dimensional graphical representation of the transformation law for the Cauchy stress tensor. The representation of a 3D state of stress is obtained by the superposition of 3 Mohr's circles as shown

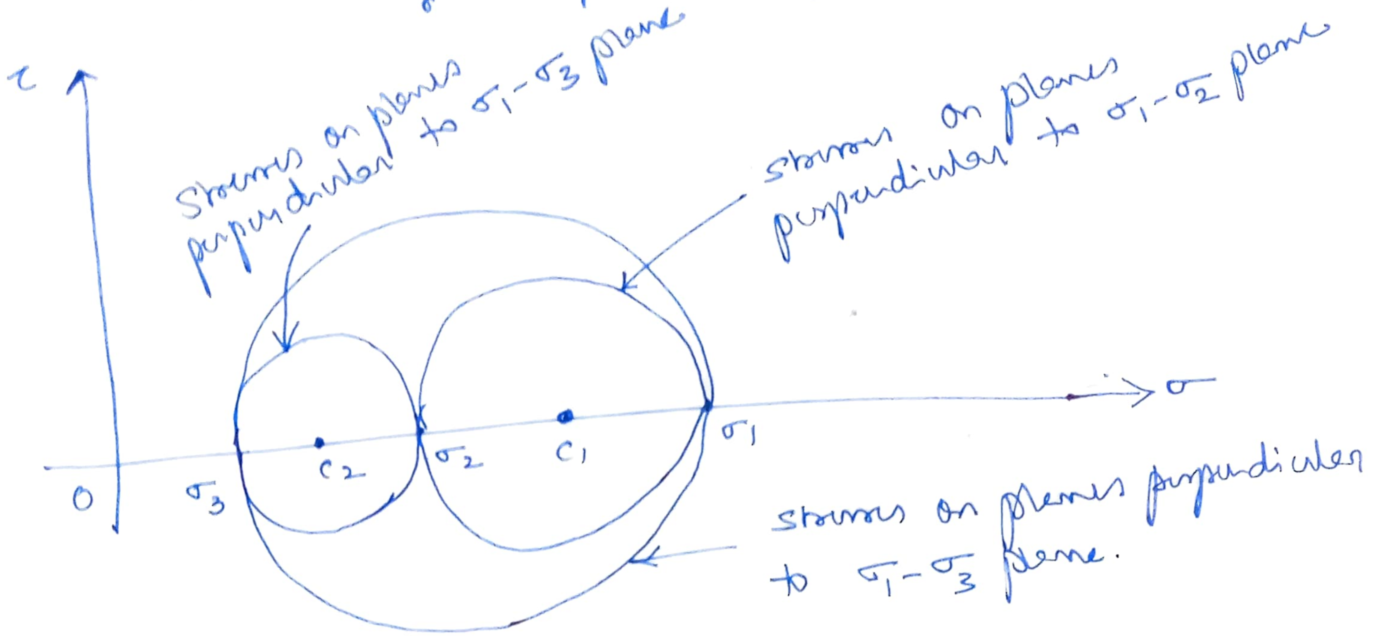


Fig:- Representation of 3D state of stress.

Steps involved in the construction of Mohr's circle in 3D :-

① Draw the normal and shear axes perpendicular to each other. Mark $\sigma_1, \sigma_2, \sigma_3$ on the normal axes (normally tension is taken as positive).

② Construct the Mohr's circle for the normal stresses σ_1 and σ_2. Mark C_1 as the centre of this circle.

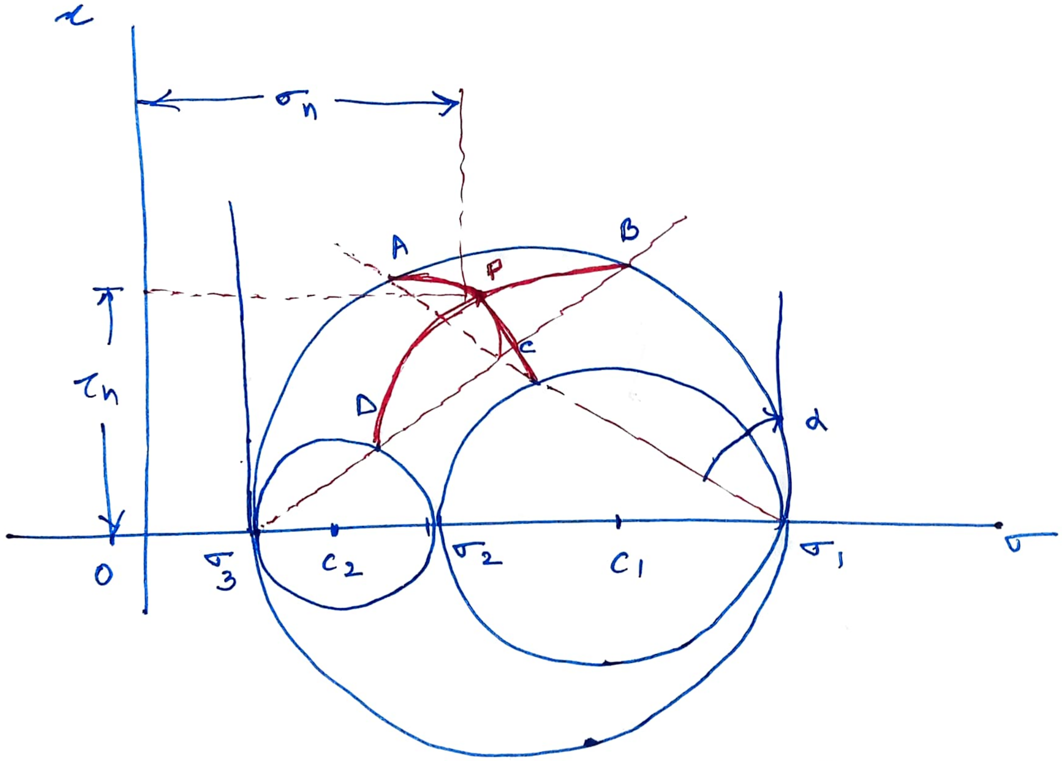
③ Draw the Mohr's circle connecting the normal stresses σ_2 and σ_3 marking C_2 as the centre.

④ Draw the Mohr's circle connecting the normal stresses σ_1 and σ_3.

⑤ Construct the line $\sigma_1 - A$ at an inclination of angle α w.r.t. vertical drawn at σ_1, and draw the line $\sigma_3 - B$ at an inclination of angle γ w.r.t vertical drawn at σ_3.

(6) With centres C_1 and C_2 draw arcs BD and AC which intersect at point 'P'.

(7) The coordinate of P along the normal axis gives the normal stress and the coordinate of P along the shear axes gives the shear stress. OP gives the resultant stress.



<u>Construction</u> of <u>Mohr's circle in 3D</u>

The same procedure can be adopted for strains also.

Stress Invariants

Invariants mean those quantities that are unexchangeable and do not vary under different conditions. In the context of stress tensor, invariants are such quantities that do not change with rotation of axes or which remain unaffected under transformation, from one set of axes to another. Therefore the combination of stresses at a point that do not change with the orientation of coordinate axes is called stress-invariants.

First invariant of stress $= I_1 = \sigma_x + \sigma_y + \sigma_z$

Second invariant of stress $= I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_y - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$

Third invariant of stress $= I_3 = \sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}$

Example:- $\sigma = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$ Cal. stress invariants.

$I_1 = -5 + 2 + 3 = 0$

$I_2 = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} -5 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} -5 & 1 \\ 1 & 2 \end{vmatrix}$

$I_2 = -33$

$I_3 = \text{Det}|\sigma| = -5\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$

$= 16$

Example:- When the stress tensor at a point with reference to axes $\begin{bmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix}$ MPa. Show that the stress invariants remain unchanged by transformation of the axes by 45° about the z-axis.

The stress invariants are

$I_1 = 4 - 6 + 8 = 18 MPa$

$I_2 = \begin{vmatrix} 6 & 0 \\ 8 & 8 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 1 & 6 \end{vmatrix}$

$= 48 + 28 + 23 = 99 MPa$

$I_3 = 4 \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 2 & 8 \end{vmatrix} + 2 \begin{vmatrix} 1 & 6 \\ 2 & 0 \end{vmatrix}$

$= 4 \times 48 - 1 \times 8 + 2(-12) = 160 MPa$

$$\sigma = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix}$$

$\sigma' = Q \sigma Q^T$ (see pg 49).

$$\sigma' = \begin{bmatrix} 6 & 1 & \sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 8 \end{bmatrix} MPa$$

$I_1' = 18 MPa$

$I_2' = 99 MPa$

$I_3' = 160 MPa$

} remains unchanged. Hence proved.

Cube roots of Cubic equation

$x^3 + 7x^2 + 14x + 8 = 0$

↳ product of three roots

↳ sum/ difference of three roots

8 cannot be root

8 factors are 1, 2, 4, ~~8~~

$1+2+4 = 7$

sum of roots

$x+1 = 0$

$x+2 = 0$

$x+4 = 0$

$x = -1, -2, -4$ are roots of cubic equation

$x^3 + 10x^2 + 27x + 18 = 0$

1, ~~2~~, 3, 6, ~~9~~, ~~18~~

$1+3+6 = 10$

$x+1 = 0$

$x+3 = 0$

$x+6 = 0$

$x = -1, -3, -6$

$x^3 - 4x^2 + 5x - 2 = 0$

roots

factors of 2 are 1, 2

since there is no third root add 1 as root

1, 1, 2

-1, -1, -2

$1+1+2 = 4$

$-1-1-2 = -4$

$x-1 = 0$

$x-1 = 0$

$x-2 = 0$

$x = 1, 1, 2$

Matrix algebra

2x2 matrix

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

($ad-bc$ ↓ determinant)

$A \times A^{-1} = I$

↓

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Identity matrix

Reciprocal of number 8 is $1/8 \simeq 8^{-1}$

Inverse of matrix 'A' is the same idea A^{-1} because $1/A$ dont exist. In matrix there is no division (No concept of division of matrix) So we can inverse a matrix.

$8 \times \frac{1}{8} = 1$ or $\frac{1}{8} \times 8 = 1$

$A \times A^{-1} = I$ $\quad A^{-1} \times A = I$

↓

Identity matrix equivalent to number '1'.

1. Determinant cannot be zero
2. Inverse of matrix must be square (same no. of rows and columns)
3. If determinat is zero Inverse of matrix dont exist. (Matrix is called singular)

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{Adj}(A)$$

Matrix of minors

Ex:-

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \qquad A^{-1} = ?$$

Matrix of Minors:-

$$\begin{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 2 & -2 \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} +2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$$

Matrix of Cofactors

$$\begin{bmatrix} +2 & +2 & +2 \\ -2 & +3 & +3 \\ 0 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & -3 \\ 0 & 10 & 0 \end{bmatrix}$$

Adjugate (Adjoint) (Transpose)

$$Adj(A) = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

Determinant

$$= 3\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 3\times 2 - 0\times 2 + 2\times 2 = 10$$

$$A^{-1} = \frac{1}{Det\,A} \cdot Adj(A) = \frac{1}{10}\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

(#)

$[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 0 & 4 \\ 2 & 4 & 3 \end{bmatrix}$ What are the eigenvalues of [A]?

$|A - \lambda I| = 0$ $\quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Determinant $\begin{vmatrix} 5-\lambda & 1 & 2 \\ 1 & 0-\lambda & 4 \\ 2 & 4 & 3-\lambda \end{vmatrix} = 0$

$5-\lambda\left[(-\lambda)(3-\lambda) - 4\times4\right] - 1\left[(3-\lambda) - 8\right] + 2\left[4 + 2\lambda\right] = 0$

$\lambda^3 - 8\lambda^2 - 6\lambda + 67 = 0$

$\lambda = -2.786,\ 7.637,\ 3.149$

(eigen values)

(#) $u = -6x^2 e_1 + 3xy\, e_2 - 5xyz\, e_3$

$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -6x^2 & 3xy & -5xyz \end{vmatrix}$

$= e_1\left[\frac{\partial}{\partial y}(-5xyz) - \frac{\partial}{\partial z}(3xy)\right] - e_2\left[\frac{\partial}{\partial x}(-5xyz) + \frac{\partial}{\partial z}6x^2\right]$

$+ e_3\left[\frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial y}(6x^2)\right]$

$\nabla \times u = -5xz\, e_1 + 5yz\, e_2 + 3y\, e_3$

(#) $u = -6x^3 e_1 + 3xy^2 e_2 - 5xyz\, e_3$

$\nabla \cdot u = \left(\frac{\partial}{\partial x} e_1 + \frac{\partial}{\partial y} e_2 + \frac{\partial}{\partial z} e_3\right) \cdot \left(-6x^3 e_1 + 3xy^2 e_2 - 5xyz\, e_3\right)$

$= -18x^2 + 6xy - 5xy$

$\nabla \cdot u = -18x^2 + xy$

$\phi = x^3 - xy^2z$

$\nabla^2\phi = \nabla \cdot (\nabla\phi)$

$= \left(\frac{\partial}{\partial x} e_1 + \frac{\partial}{\partial y} e_2 + \frac{\partial}{\partial z} e_3\right) \cdot \left[(3x^2 - y^2z)e_1 + (-2xyz)e_2 + (-xy^2)e_3\right]$

↑

$\left(\frac{\partial}{\partial x} e_1 + \frac{\partial}{\partial y} e_2 + \frac{\partial}{\partial z} e_3\right)(x^3 - xy^2z)$

$\nabla^2\phi = 6x - 2xz$

Note:- Divergence theorem - relates volume integral to surface integral

Stoke's theorem - relates contour/ Line integral to surface integral.

(#) $z_1 = 2+3i \quad z_2 = 1-5i \qquad i^2 = -1$

$$z_1 \times z_2 = (2+3i)(1-5i) = 2 - 10i + 3i - 15i^2$$

$$= \underline{\underline{17 - 7i}}$$

(#)

$z_1 = 2+3i \quad z_2 = 1-5i$

$$\frac{z_1}{z_2} = \frac{2+3i}{1-5i} = \frac{(2+3i)(1+5i)}{(1-5i)(1+5i)} = \frac{2+15i^2+3i+10i}{\sqrt{26}}$$

$$\frac{z_1}{z_2} = \frac{-13+13i}{\sqrt{26}}$$

(#) $f(z) = (x^2-y^2) + v(x,y)i \quad \simeq f(z) = u(x,y) + v(x,y)i$

$f(z)$ is an analytic function if $\left.\begin{aligned} \frac{du}{dx} &= +\frac{dv}{dy} \\ \frac{du}{dy} &= -\frac{dv}{dx} \end{aligned}\right\}$

function $f(x) = u(r,\theta) + v(r,\theta)i$ to be analytic in polar coordinate.

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial v}{r\partial\theta} \\ \frac{\partial u}{r\partial\theta} &= -\frac{\partial v}{\partial r} \end{aligned}\right\}$$

(#) $z = x+iy$

$f(z) = a/z + bz^2$

$f(\bar{z}) = \frac{a}{\bar{z}} + b\bar{z}^2$

$$\frac{a}{\bar{z}} = \frac{a}{x-iy} = \frac{a(x+iy)}{\sqrt{x^2+y^2}} = \frac{ax}{\sqrt{x^2+y^2}} + \frac{ayi}{\sqrt{x^2+y^2}}$$

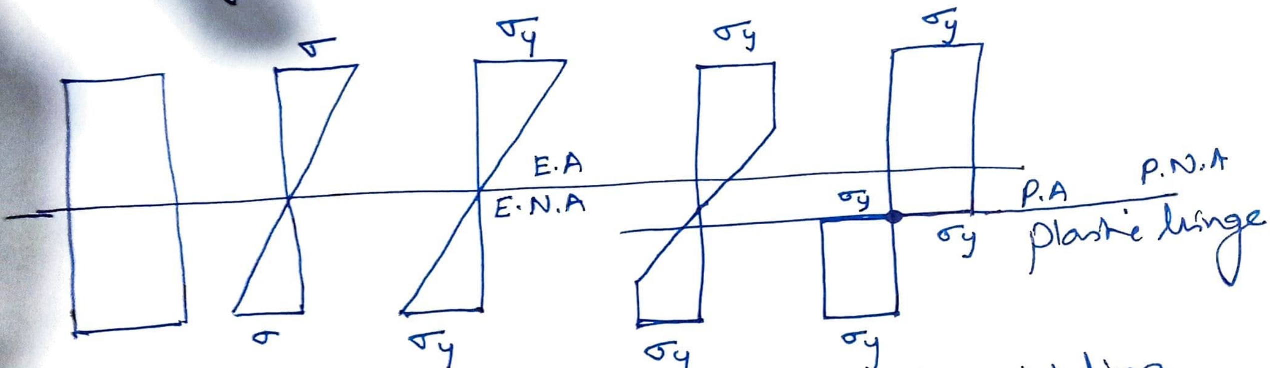
$$b\bar{z}^2 = b(x-iy)^2 = b(x^2-y^2-2xyi)$$

Stress-strain behaviour

In linear elasticity approach, the stress-strain relationship is defined in undeformed configuration. In non-linear elasticity approach, the stress-strain relationship is defined in both deformed and undeformed configuration.

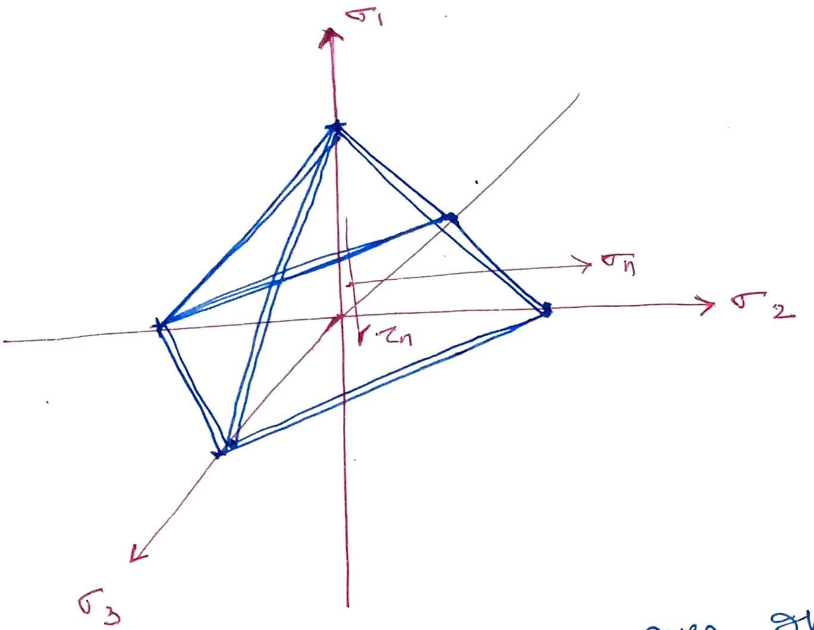
Non-linear elasticity problems has large deformations or small deformations but large rotation/displacement problems.

Plastic hinge Mechanism:-



Upper yield point is the load required to initiate yielding.
Lower yield point is the minimum load required to maintain yield.
normally lower yield point is used to determine the yield strength of the material because the upper yield point is momentary cannot be calculated easily where as lower yield point is more defined.

Octahedral planes



σ_1, σ_2 and σ_3 are reference axes. There exists a plane that is equally inclined to these axes. Such a plane is called octahedral plane. The direction cosines of this plane will be $l = m = n$. Since $l^2+m^2+n^2=1$ we get $l = m = n = \pm \frac{1}{\sqrt{3}}$.

There are eight such planes as shown in figure. The normal and shearing stresses on these planes are called the octahedral normal and shear stresses resp.:

$$\begin{bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{Rx} \\ \sigma_{Ry} \\ \sigma_{Rz} \end{bmatrix} = \begin{bmatrix} \sigma_1/\sqrt{3} \\ \sigma_2/\sqrt{3} \\ \sigma_3/\sqrt{3} \end{bmatrix}$$

The normal stress on the octahedral plane is

$$\sigma_{oct} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{I_1}{3} \quad \text{—} \ (1)$$

$$\sigma_R = \sqrt{\frac{\sigma_1^2}{3} + \frac{\sigma_2^2}{3} + \frac{\sigma_3^2}{3}}$$

$$\tau_{oct} = \sqrt{\sigma_R^2 - \sigma_{oct}^2}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} \quad \text{—— (2)}$$

or $\frac{\sqrt{2}}{3}\sqrt{I_1^2 - 3I_2}$

Example The state of stress at a point is given by

$$\begin{bmatrix} 175 & -20 & 40 \\ -20 & 75 & 30 \\ 40 & 30 & 50 \end{bmatrix} \text{MPa.}$$

find the octahedral normal and shear stresses for this system.

I_1 = Trace of the matrix = 175 + 75 + 50 = 300

I_2 = Cofactor of σ_x + Cofactor of σ_y + cofactor of σ_z

$$= \begin{vmatrix} 75 & 30 \\ 30 & 50 \end{vmatrix} + \begin{vmatrix} 175 & 40 \\ 40 & 50 \end{vmatrix} + \begin{vmatrix} 175 & -20 \\ -20 & 75 \end{vmatrix} = 22725$$

$$\sigma_{oct} = \frac{I_1}{3} = 100 \text{ MPa}$$

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sqrt{I_1^2 - 3I_2} = 69.64 \text{ MPa}$$

Example - The state of strains at a point is given by $\epsilon_x = 0.03$, $\epsilon_y = -0.008$, $\epsilon_z = 0.06$, $\gamma_{xy} = 0.034$, $\gamma_{yz} = -0.09$, $\gamma_{xz} = 0.02$. find the octahedral normal and shear strains for this systems

$$\begin{bmatrix} 0.03 & 0.017 & 0.01 \\ 0.017 & -0.008 & -0.045 \\ 0.01 & -0.045 & 0.06 \end{bmatrix}$$

J_1 = Trace of matrix = 0.03 − 0.008 + 0.06 = 0.082

$$I_2 = \begin{vmatrix} -0.008 & -0.045 \\ -0.045 & 0.06 \end{vmatrix} + \begin{vmatrix} 0.03 & 0.01 \\ 0.01 & 0.06 \end{vmatrix} + \begin{vmatrix} 0.03 & 0.017 \\ 0.017 & -0.008 \end{vmatrix}$$

$= -1.33 \times 10^{-3}$

$$\epsilon_{oct} = \frac{J_1}{3} = 0.027$$

$$\gamma_{oct} = \frac{\sqrt{2}}{3}\sqrt{J_1^2 - 3J_2} = 0.048.$$

State of Pure shear :-

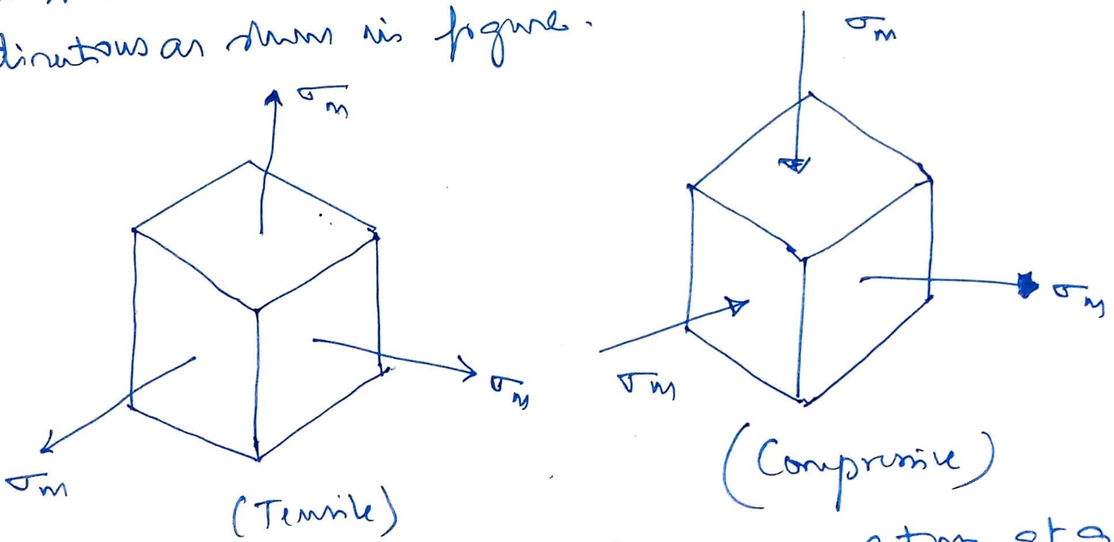
The state of a stress at a point is expressed by the six rectangular stress components. If $\sigma_x = \sigma_y = \sigma_z = 0$ then a state of pure shear exist at that point.

$$\begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & 0 & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & 0 \end{bmatrix}$$

For this system, first stress invariant $I_1 = 0$. I_1 is an invariant, that is true for any coordinate system at the point. Hence the necessary condition for the state of pure shear to exist is $I_1 = 0$. When $I_1 = 0$, an octahedral plane is subjected to pure shear with no normal stress.

Hydrostatic state of stress :-

The state of stress consisting only of the hydrostatic stress is called the hydrostatic stress state. The hydrostatic stress is the average of the normal stresses $\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{I_1}{3}$. This is similar to three equal normal stresses acting in the three directions as shown in figure.



(Tensile) (Compressive)

This is equivalent to hydrostatic pressure acting at a point in a fluid, the only difference being that the hydrostatic pressure acting on fluids is only compressive in nature, where as σ_m can be either be tensile or compressive in nature. The hydrostatic stress σ_m does not cause any plastic deformation. It causes only elastic volume change.

$$\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

For plastic deformation to occur, shear stresses are required to cause the shearing of atomic planes. As shear stresses are absent in the hydrostatic state of stress, no plastic deformation can be induced and only elastic volume change occurs. The terms hydrostatic, spherical, volumetric, mean, dilatational and octahedral normal stresses all indicate the same quantity.

Deviatoric state of stress :-

The state of stress that causes plastic deformation is called deviatoric state of stress. This component can be obtained by reducing the normal component by the hydrostatic stress.

$$\begin{bmatrix} (\sigma_x - \sigma_m) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - \sigma_m) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z - \sigma_m) \end{bmatrix}$$

Here first stress invariant $I_1 = 0$. Hence the deviatoric state of stress is also known as pure shear state of stress or distortional state of stress or stress deviator.

Decomposition into hydrostatic and Deviatoric stress states

Any arbitrary state of stress can be resolved into a hydrostatic and deviatoric stress states.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\text{Hydrostatic state}} + \underbrace{\begin{bmatrix} (\sigma_x - \sigma_m) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - \sigma_m) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z - \sigma_m) \end{bmatrix}}_{\text{Deviatoric state}}$$

If the given state is the principal axis, then

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_1-\sigma_m & 0 & 0 \\ 0 & \sigma_2-\sigma_m & 0 \\ 0 & 0 & \sigma_3-\sigma_m \end{bmatrix}$$

Example:-
The components of stress at a point are given by the $\begin{bmatrix} 5 & -1 & -5 \\ -1 & 9 & 3 \\ -5 & 3 & 7 \end{bmatrix}$
Determine the spherical and deviatoric stress components.

Mean stress $\sigma_m = \dfrac{\sigma_x+\sigma_y+\sigma_z}{3} = \dfrac{5+9+7}{3} = 7$ MPa

Spherical stress tensor $\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ MPa

Deviatoric stress tensor

$$\begin{bmatrix} \sigma_x-\sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y-\sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z-\sigma_m \end{bmatrix} = \begin{bmatrix} -2 & -1 & -5 \\ -1 & 2 & 3 \\ -5 & 3 & 0 \end{bmatrix}$$

~~Note: The above concept is applicable to~~

Example :- The principal strains at a point are $\epsilon_1 = 0.005$, $\epsilon_2 = 0.003$ and $\epsilon_3 = 0.001$ resp. Determine the spherical and deviatoric strain components.

The principal axis is
The mean strain is $\epsilon_m = \dfrac{\epsilon_1+\epsilon_2+\epsilon_3}{3} = \dfrac{0.005+0.003+0.001}{3} = 0.003$

Spherical strain tensor is $\begin{bmatrix} \epsilon_m & 0 & 0 \\ 0 & \epsilon_m & 0 \\ 0 & 0 & \epsilon_m \end{bmatrix} = \begin{bmatrix} 0.003 & 0 & 0 \\ 0 & 0.003 & 0 \\ 0 & 0 & 0.003 \end{bmatrix}$

Deviatoric strain tensor is

$$\begin{bmatrix} \epsilon_1-\epsilon_m & 0 & 0 \\ 0 & \epsilon_2-\epsilon_m & 0 \\ 0 & 0 & \epsilon_3-\epsilon_m \end{bmatrix} = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.002 \end{bmatrix}$$

Example Consider a fish under two conditions

(a) depth of 100m, with hydrostatic pressure of 1.0 MPa acting on it

(b) squeezed between our fingers, with an applied stress (uniaxial) of about 0.75 MPa.

Compare these two conditions and give your observations.

(a) stress acting on the fish = $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ -ve due to compressive force.

(b) Stress acting on the fish is

$$\begin{bmatrix} -0.75 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.25 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -0.25 \end{bmatrix} + \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

In case (a) fish is subjected to only hydrostatic pressure, the deviatoric stress is zero which will not be uncomfortable for the fish. In case (b), addition to hydrostatic stress, deviatoric stress component also acts on the fish making very uncomfortable.

Example:-

$$\sigma = \begin{bmatrix} 6 & 5 & 7 \\ 5 & 3 & 4 \\ 7 & 4 & -3 \end{bmatrix}$$

Calculate the deviatoric stress. (σ_D)

$$\sigma_m = \frac{1}{3}\sigma_{ii} \approx \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(6+3-3) = 2$$

$$\sigma_D = \begin{bmatrix} 6-\sigma_m & 5 & 7 \\ 5 & 3-\sigma_m & 4 \\ 7 & 4 & -3-\sigma_m \end{bmatrix} = \begin{bmatrix} 4 & 5 & 7 \\ 5 & 1 & 4 \\ 7 & 4 & -5 \end{bmatrix}$$

Example:-

$$\sigma = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

Find out traction vector on a plane whose normal is given by $n = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]^T$

Traction $(T_i) = \sigma_{ij} n_j$

$$\sigma_{ij} = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix} \qquad n = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]^T$$

$$T_i = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -6/\sqrt{2} \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(#)

Lamé's constants :-

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \; ; \; \mu = \frac{E}{2(1+\nu)}$$

$$K = \frac{E}{3(1-2\nu)} \; ; \; G = \frac{E}{2(1+\nu)}$$

Relation between elastic constants.

Example :- $\epsilon_{xx} = 0.5\times10^{-3}$ $\epsilon_{yy} = -0.4\times10^{-3}$ $\epsilon_{zz} = 0.7\times10^{-3}$

$E = 2GPa$ $\nu = 0.18$ calculate $\sigma_{hydrostatic}$?

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 476.6 \text{ MPa}$$

$$\sigma_{hydrostatic} = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

$$\mu = \frac{E}{2(1+\nu)} = 847.45 \text{ MPa}$$

$$\left.\begin{aligned} \sigma_x &= \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu\,\epsilon_x \\ \sigma_y &= \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu\,\epsilon_y \\ \sigma_z &= \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu\,\epsilon_z \end{aligned}\right\} \text{ pg (23)}$$

$$\sigma_x + \sigma_y + \sigma_z = 3\lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu(\epsilon_x + \epsilon_y + \epsilon_z)$$

$$\sigma_{hydrostatic} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \left(\frac{3\lambda + 2\mu}{3}\right)(\epsilon_x + \epsilon_y + \epsilon_z)$$

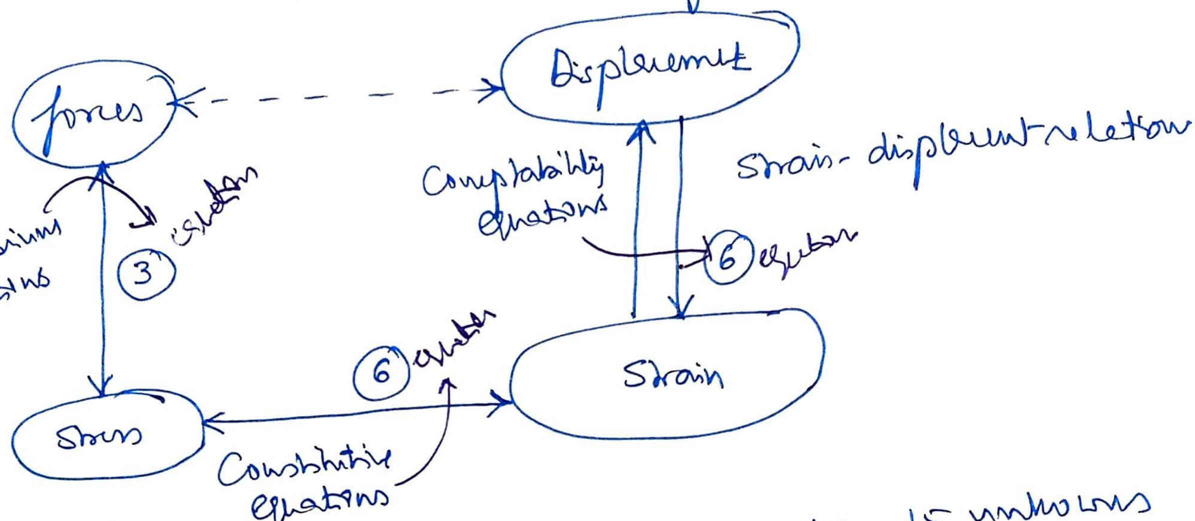
Substitute values

$$\sigma_{hydrostatic} = 0.833 \text{ MPa}$$

Constitutive Equations or Relations

In elasticity, four equations are important

1. Equilibrium equations → relating forces and stresses
2. Strain-displacement relationships → relating displacement field with strains
3. Compatibility equations → for smooth differential displacement field.
4. Constitutive equations → relating stress and strain..



Total 15 equations (3+6+6) are able to solve 15 unknowns (6 stresses, 6 strains, 3 displacements). The constitutive equation for a solid material assuming a linear relationship between stress and strain is the Hooke's law.

The microscopic behaviour (based on internal constitution) of solids is normally defined by the constitutive stress-strain relations.

These equations should be independent of the coordinate system
These equations should project material symmetries i.e,
isotropic materials have infinite planes of symmetry whereas orthotropic materials have three mutually perpendicular planes of symmetry.

For most general case, the stress-strain relationship has 81 elastic constants. If you consider symmetry the number of elastic constants are reduced to 36.

$$\underset{9\times1}{\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yx}\\ \tau_{zy}\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{zx}\end{bmatrix}} = \underset{9\times9}{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & C_{17} & C_{18} & C_{19}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} & C_{28} & C_{29}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} & C_{39}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} & C_{47} & C_{48} & C_{49}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} & C_{57} & C_{58} & C_{59}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} & C_{67} & C_{68} & C_{69}\\ C_{71} & C_{72} & C_{73} & C_{74} & C_{75} & C_{76} & C_{77} & C_{78} & C_{79}\\ C_{81} & C_{82} & C_{83} & C_{84} & C_{85} & C_{86} & C_{87} & C_{88} & C_{89}\\ C_{91} & C_{92} & C_{93} & C_{94} & C_{95} & C_{96} & C_{97} & C_{98} & C_{99}\end{bmatrix}} \underset{9\times1}{\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \gamma_{xy}\\ \gamma_{yx}\\ \gamma_{zy}\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xz}\end{bmatrix}}$$

$$\underset{6\times1}{\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix}} = \underset{6\times6}{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot\\ C_{31} & \cdot & \cdot & \cdot & \cdot & \cdot\\ C_{41} & \cdot & \cdot & \cdot & \cdot & \cdot\\ C_{51} & \cdot & \cdot & \cdot & \cdot & \cdot\\ C_{61} & \cdot & \cdot & \cdot & \cdot & C_{66}\end{bmatrix}} \underset{6\times1}{\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ e_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}}$$

Anisotropy or triclinic Materials

↳ three planes of symmetry.

Number of elastic constants for anisotropic or triclinic materials are reduced to 21.

anisotropic - having a physical property which has a different value when measured in different directions.

example:- wood (strong is along the grain direction than across it)

different crystallographic orientations. opposite is isotropic.

orthotropic - subset of anisotropic - material properties that differ along three mutually orthogonal. ex wood

of the symmetry of the material increases the number of elastic constants decreases. In order to determine the number of independent elastic constants for various materials, a certain sequence of rotation of the coordinate axis can be carried out.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & \text{Symmetric} & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

21 constants

Monoclinic materials - materials with one plane of symmetry. The no. of independent elastic constants is 13.

Anisotropic materials - no. of independent elastic constants are 21.

Orthotropic materials - exhibit symmetry about three diagonal planes. The properties are different in all three orthogonal directions. The no. of independent elastic constants are 9.

Note: Tensorial representation of strain at a point with (ϵ_{ij}) displacement field $u = [u_1, u_2, u_3]^T$

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

→ The order of constitutive tensor is 4th order ($3^4 = 81$ elements) for general case

→ Isotropic material has independent elastic constants of 2.

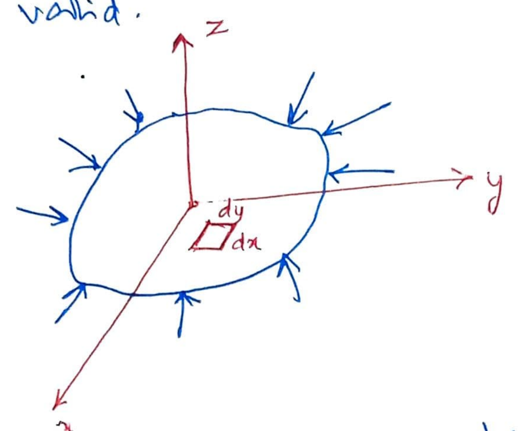
Transverse Isotropic material has independent elastic constants of 5.

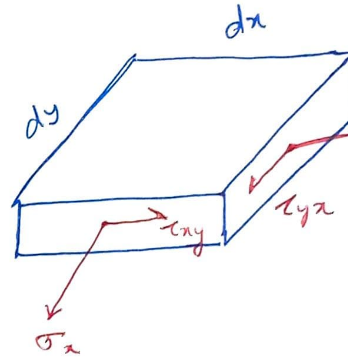
Plane stress and Plane strain problems

3D equations in elasticity are very complex. Many problems in elasticity are solved using simplified 3D equations. Many plane problems do not require 3D equations to solve them. These problems can be reduced to 2D or 2 variable problems.

Plane stress Problems :-

A flat thin sheet (plate) loaded at its mid-plane is an example of plane stress problem if the following assumptions are valid.





1. The plate is flat and has a plane of symmetry.
2. Loads and support conditions are symmetric about the mid-plane.
3. Thickness of the plate is small compared to its plane dimensions.
4. In-plane displacements, strains and stresses are uniform throughout the thickness.
5. Normal and shear stresses in the transverse directions are zero. $\epsilon_z \neq 0$

namely σ_z, τ_{zx} and τ_{zy} are zero.

Equation for Plane stress :-

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + f_y = 0 \end{aligned}\right\} \text{equilibrium equations} \quad (1)$$

$$\left[\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2}\right] = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$ compatability equations. (2)

stress - strain relationships — (3)

$$\epsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y)$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x)$$

$$\epsilon_z = \frac{1}{E}(-\mu(\sigma_x + \sigma_y))$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Differentiate eqn (1) w.r.t x

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial F_x}{\partial x} = 0$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial F_x}{\partial x} \quad \text{—— (4)}$$

Differentiat eqn (1) w.r.t y

$$\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{yx}}{\partial x \partial y} + \frac{\partial F_y}{\partial y} = 0$$

$$\frac{\partial^2 \tau_{yx}}{\partial x \partial y} = -\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial F_y}{\partial y} \quad \text{—— (5)}$$

Add (4) and (5)

$$2\frac{\partial^2 \tau_{yx}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} \quad \text{—— (6)}$$

Substitute equation (3) in eqn (2)

$$\frac{1}{G}\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{1}{E}\left[\frac{\partial^2(\sigma_x - \mu\sigma_y)}{\partial y^2} + \frac{\partial^2(\sigma_y - \mu\sigma_x)}{\partial x^2}\right]$$

$$\therefore G = \frac{E}{2(1+\mu)}$$

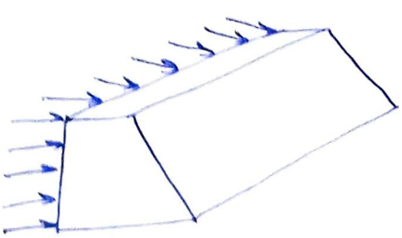
$$\frac{2(1+\mu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{1}{E}\left[\frac{\partial^2(\sigma_x - \mu\sigma_y)}{\partial y^2} + \frac{\partial^2(\sigma_y - \mu\sigma_x)}{\partial x^2}\right] \quad \text{—— (7)}$$

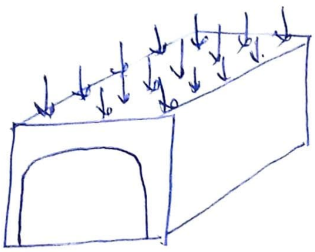
From (6) and (7)

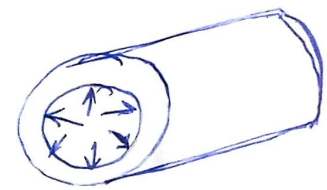
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\mu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \quad \text{--- (8)}$$

Equation for plane strain :-

examples of plane strain problems are Box culverts, tunnels, retaining walls, long cylindrical tubes, control rods etc. are subjected to uniform loads along the length and also the dimensions along the z-direction is very large.







$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

} equilibrium equations (1)

$$\left[\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2}\right] = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Compatability equations (2)

$$\epsilon_x = \frac{1}{E}[\sigma_x - \mu\sigma_y - \mu\sigma_z]$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \mu\sigma_x - \mu\sigma_z]$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \mu\sigma_x - \mu\sigma_y] = 0$$

$$\sigma_z = \mu(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

} stress-Strain relationships (3)

substitute σ_z in ϵ_x

$$\epsilon_x = \frac{1}{E}[\sigma_x - \mu\sigma_y - \mu(\mu(\sigma_x + \sigma_y))]$$

$$= \frac{1}{E}[\sigma_x - \mu\sigma_y - \mu^2(\sigma_x + \sigma_y)]$$

$$\epsilon_x = \frac{1}{E}\left[(1-\mu^2)\sigma_x - \mu(1+\mu)\sigma_y\right]$$

$$\epsilon_x = \frac{(1+\mu)}{E}\left[(1-\mu)\sigma_x - \mu\sigma_y\right] \quad \text{——— (4)}$$

Substitute σ_z in ϵ_y

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \mu\sigma_x - \mu(\mu(\sigma_x+\sigma_y))\right]$$

$$= \frac{1}{E}\left[\sigma_y - \mu\sigma_x - \mu^2(\sigma_x+\sigma_y)\right]$$

$$= \frac{1}{E}\left[(1-\mu^2)\sigma_y - \mu(1+\mu)\sigma_x\right]$$

$$\epsilon_y = \frac{(1+\mu)}{E}\left[(1-\mu)\sigma_y - \mu\sigma_x\right] \quad \text{——— (5)}$$

Differentiate (1) w.r.t x and y

$$\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\tau_{xy}}{\partial x\partial y} + \frac{\partial f_x}{\partial x} = 0 \; ; \quad \frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\tau_{yx}}{\partial x\partial y} + \frac{\partial f_y}{\partial y} = 0$$

$$\left.\begin{aligned} \frac{\partial^2\tau_{xy}}{\partial x\partial y} &= -\frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial f_x}{\partial x} \\ \frac{\partial^2\tau_{xy}}{\partial x\partial y} &= -\frac{\partial^2\sigma_y}{\partial y^2} - \frac{\partial f_y}{\partial y} \end{aligned}\right\} \quad (6)$$

Add equation (6) terms

$$2\frac{\partial^2\tau_{xy}}{\partial x\partial y} = -\frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_y}{\partial y^2} - \frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}$$

Substitute γ_{xy}, ϵ_x, ϵ_y in eqn (2)

$$\frac{1}{G}\frac{\partial^2\tau_{yx}}{\partial y\partial x} = \frac{1+\mu}{E}\left[\frac{\partial^2((1-\mu)\sigma_x - \mu\sigma_y)}{\partial y^2} + \frac{\partial^2((1-\mu)\sigma_y - \mu\sigma_x)}{\partial x^2}\right] \quad \text{——— (7)}$$

$\because \; G = \frac{E}{2(1+\mu)}$ eqn (7) becomes

$$\frac{(1+\mu)}{E}\frac{\partial^2\tau_{xy}}{\partial x\partial y} = \frac{(1+\mu)}{E}\left[\frac{\partial^2((1-\mu)\sigma_x - \mu\sigma_y)}{\partial y^2} + \frac{\partial^2((1-\mu)\sigma_y - \mu\sigma_x)}{\partial x^2}\right] \quad \text{——— (8)}$$

From eqn (6) and (8) we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y) = -\frac{1}{(1-\mu)}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) \quad \text{——— (9)}$$

In case of absence of body forces, the equations of plane stress and plane strain reduces to the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0 \quad \text{——— (10)}$$

compatibility equation.

Therefore the stress distribution is same for both cases of plane strain and plane stress problems, provided the shape of the boundary and the external forces are the same.

* Solutions for 2D problems :- (Stress function polynomial)

From equations (8) (pg 88) and equations (9) (pg 88) back it is observed that the solution of 2D problems reduces to the integration of the differential equations of equilibrium, compatibility and boundary conditions. The following are few methods of solution proposed for 2D problems

(a) Airy's stress function method
(b) Strain energy function method
(c) Displacement function method
(d) Integral equation method
(e) Betti's method
(f) potential function method
(g) Numerical method
(h) Fourier transform method
(i) Inverse method or Semi-inverse method

* Airy's stress function Method :-

Airy's stress function method considers an arbitrary function $\phi=\phi(x,y)$ such that it satisfies the relation:

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2} \qquad \sigma_y=\frac{\partial^2\phi}{\partial x^2} \qquad \tau_{xy}=-\frac{\partial^2\phi}{\partial x\,\partial y}$$

Substituting in eqn (10) (pg 89)

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Substitute in eqn (10) (pg 89)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=0$$

$$\frac{\partial^4 \phi}{\partial x^4}+2\frac{\partial^4 \phi}{\partial x^2 \partial y^2}+\frac{\partial^4 \phi}{\partial y^4}=0$$

$$\nabla^4=\frac{\partial^4}{\partial x^4}+\frac{2\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$$

$$\nabla^4 \phi = 0$$

This is called biharmonic equation and its solutions are known as biharmonic functions. (pg 96)

The biharmonic equation can be satisfied by expressing Airy's stress function ϕ in the form of homogeneous polynomials. The triangular pattern of numbers known as the pascal's triangle can be used to form polynomial equations.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal's triangle

1
First Degree: x y
Second Degree: x^2 $2xy$ y^2
Third Degree: x^3 $3x^2y$ $3xy^2$ y^3
Fourth Degree: x^4 $4x^3y$ $6x^2y^2$ $4xy^3$ y^4
Fifth Degree: x^5 $5x^4y$ $10x^3y^2$ $10x^2y^3$ $5xy^4$ y^5

$(x+y)^0$ zero degree
$(x+y)^1$ first degree
$(x+y)^2$ second degree
$(x+y)^3$ third degree
$(x+y)^4$ fourth degree
$(x+y)^5$ fifth degree

$$(x+y)^n = x^n + x^{n-1}y + x^{n-2}y^2 + x^{n-3}y^3 + \dots + xy^{n-1} + y^n$$

Polynomial of First Degree (linear function)

$$\phi = C_1 x + C_2 y$$

It satisfies the biharmonic function $\nabla^4 \phi = 0$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

This shows that by adding or subtracting a linear function to Airy's function, there will be no change to the stress as $(\sigma, \tau) = 0$

Polynomial of Second Degree (Quadratic function)

$$\phi = c_1 x^2 + C_2 xy + C_3 y^2$$

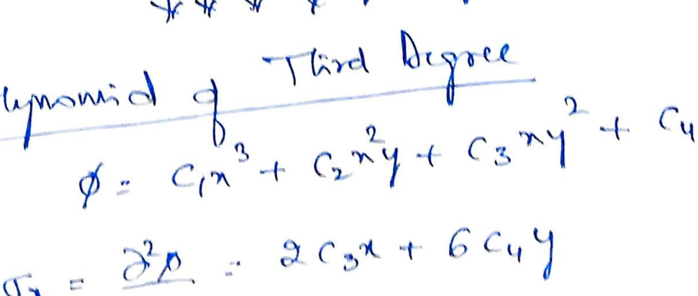
This satisfies biharmonic function $\nabla^4 \phi = 0$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2C_3$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 2C_1$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -C_2$$

This represents a constant stress field



Polynomial of Third Degree

$$\phi = C_1 x^3 + C_2 x^2 y + C_3 x y^2 + C_4 y^3$$

→ Mathematically developed based on experience or Developed using Pascal's triangle.

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2C_3 x + 6C_4 y$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 6C_1 x + 2C_2 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -2(C_2 x + C_3 y)$$

When C_1, C_2 and C_3 equal to zero then

$$\sigma_x = 6 c_4 y$$

$$\sigma_y = \tau_{xy} = 0$$

In other words $\sigma_x = ky$

proportional to y.

This shows that σ_x is a linear function of y. This represents the case of a beam subjected to pure bending.

Example:-

Check if the function $10x^4 + 15x^2y^2 - 15y^4$ is an Airy's stress function.

Airy's stress function should satisfy the biharmonic equation

$$\nabla^4 \phi = 0$$

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

$$\frac{\partial^4 \phi}{\partial x^4} = 240 \; ; \quad \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 60 \; ; \quad \frac{\partial^4 \phi}{\partial y^4} = -360$$

$$240 + 2 \times 60 - 360 = 0$$

Hence the given function is Airy's stress function.

Example:- Examine the stress distribution represented by the following Airy's stress function

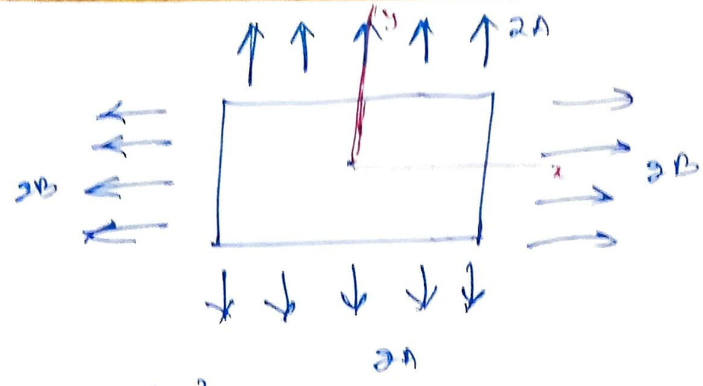
1. $\phi = Ax^2 + By^2$
2. $\phi = By^3$
3. $\phi = A[x^4 - 2x^2y^2]$

(1) $\phi = Ax^2 + By^2$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2B \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 2A$$

This represents a uniform two-dimensional stress field as shown below.



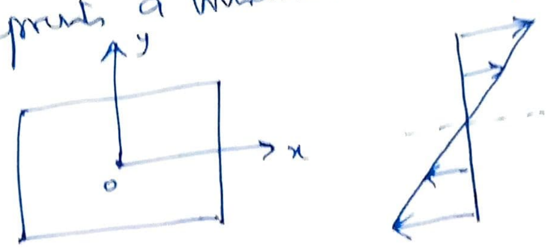
(2) $\phi = By^3$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 6By$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

This represents a uniaxial linearly varying stress field



(3) $\phi = A\left[x^4 - 2x^2y^2\right]$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = -4Ax^2$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = A\left(12x^2 - 4y^2\right)$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 8Axy$$

This represents a complex stress field.

Example:-

A large thin plate is subjected to certain boundary conditions on its thin edges (with zero stress vector on its large faces) leading to the stress function $\phi = Ax^2y^3 - By^5$

1. Use the biharmonic equation to express A in terms of B.
2. Calculate all the stress components.

3. Calculate all the strain components in terms of E, B and μ
4. Find the volumetric strain
5. Check if the compatibility equation is satisfied.
6. Check if the equilibrium equation is satisfied.

This problem can be idealized as a plane stress problem as it is a large thin plate with zero stress vector on its large face.

Hence $\sigma_z = 0$, $\tau_{xz} = 0$, $\tau_{yz} = 0$

① Biharmonic equation

$$\nabla^4 \phi = 0$$

$$\left(\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}\right) = 0$$

$$\frac{\partial^4 \phi}{\partial x^4} = 0 \quad ; \quad \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 12Ay \quad ; \quad \frac{\partial^4 \phi}{\partial y^4} = -120By$$

$$24Ay - 120By = 0$$

$$A = 5B$$

$$\phi = 5Bx^2y^3 - By^5 = B[5x^2y^3 - y^5]$$

② Stress components

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = B[30x^2y - 20y^3]$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 10By^3$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -30Bxy^2$$

③ Strain components

$$\epsilon_x = \frac{1}{E}(\sigma_x - \mu(\sigma_y + \sigma_z)) = \frac{B}{E}((30x^2y - 20y^3) - \mu(10y^3 + 0))$$

$$= \frac{B}{E}[30x^2y - (20 + \mu 10)y^3]$$

$$\epsilon_y = \frac{B}{E}\{(10 + 20\mu)y^3 - \mu(30x^2y)\}$$

$$\epsilon_z = \frac{B}{E}\left(10\mu y^3 - \mu 30x^2 y\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad G = \frac{E}{2(1+\mu)}$$

$$= \frac{-60Bxy^2(1+\mu)}{E}$$

$$\gamma_{yz} = \gamma_{xz} = 0$$

(4) Volumetric strain $\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$

$$= \frac{B}{E}\left(30x^2y(1-2\mu) + 10y^3(2\mu-1)\right)$$

(5) Compatibility equation

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left[\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2}\right]$$

$$\frac{-120By(1+\mu)}{E} = \frac{B(-60\mu y - 120y - 60\mu y)}{E}$$

$$LHS = \frac{-120By(1+\mu)}{E} = RHS$$

Hence it is compatible.

(6) Equilibrium equation in the absence of body forces

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\Rightarrow B[60xy] - B(60xy) = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0$$

$$\Rightarrow -30By^2 + 30By^2 = 0$$

equilibrium equation is satisfied.

Example

A very thick component has the same boundary conditions on any given cross-section, leading to the following stress function $\phi = x^5 - xy^4 - 4x^3y^2$

1. Check if this is a valid stress function
2. Calculate all the stress components ($\mu = 0.25$)
3. Calculate all the strain components
4. Find the displacements.

This problem can be idealized as a plane strain problem as it is a thick component with the same boundary conditions on any given cross-section. Hence $\epsilon_z = 0$, $\gamma_{xz} = 0$, $\gamma_{yz} = 0$

(1) To check if it a valid stress function the biharmonic equation $\nabla^4 \phi = 0$ should be satisfied.

$$\left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) = 0$$

$$\frac{\partial^4 \phi}{\partial x^4} = 120x$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = -48x$$

$$\frac{\partial^4 \phi}{\partial y^4} = -24x$$

$120x - 96x - 24x = 0$

$0 = 0$

As it satisfies the biharmonic equation it is a valid stress function.

(2) Stress components

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = -12xy^2 - 8x^3$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 20x^3 - 24xy^2$$

$$\sigma_z = \mu(\sigma_x + \sigma_y) = -8xy^2 + 3x^3$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 24x^2y + 4y^3$$

$\tau_{xz} = \tau_{yz} = 0$

(3) Strain components

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \mu(\sigma_y + \sigma_z)\right)$$ Substituting $\sigma_x, \sigma_y, \sigma_z$

$$= \frac{1}{E}\left(-3.75xy^2 - 13.75x^3\right)$$

$$\epsilon_y = \frac{1}{E}\left(21.25x^3 - 18.75xy^2\right)$$ $\epsilon_z = 0$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \Rightarrow G = \frac{E}{2(1+\mu)}$$

$$\gamma_{xy} = \frac{60x^2y + 10y^3}{E}$$

$$\gamma_{yz} = \gamma_{xz} = 0$$

(4) Displacement $\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E}\left(-3.75xy^2 - 13.75x^3\right)$

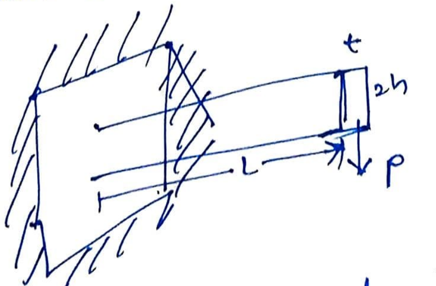
$$u = \frac{1}{E}\left[-1.875x^2y^2 - 3.4375x^4\right] + C_1$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E}\left(21.25x^3 - 18.75xy^2\right)$$

$$v = \frac{1}{E}\left(21.25x^3y - 6.25xy^3\right) + C_2$$

Example:-

Find the expression for the vertical deflection curve for the thin cantilever beam loaded as shown. (Bending of beam)



This problem can be idealized as a plane stress problem as it is a thin plate with only a point load at the free end. Body forces are not given. Hence $\sigma_z = 0$, $\tau_{xz} = 0$, $\tau_{yz} = 0$.

and hence they are taken as zero.

The bending stress $\sigma_b = \frac{My}{I} = -\frac{Pxy}{I}$

(−ve, load acting downwards)

$\frac{M}{I} = \frac{f}{y}$

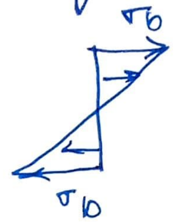
$f = \frac{My}{I}$

$\sigma_y = 0$ and the shear stress, τ_{xy} is a function of x and y.

Applying the equilibrium equation

$\sigma_b = \sigma_x$

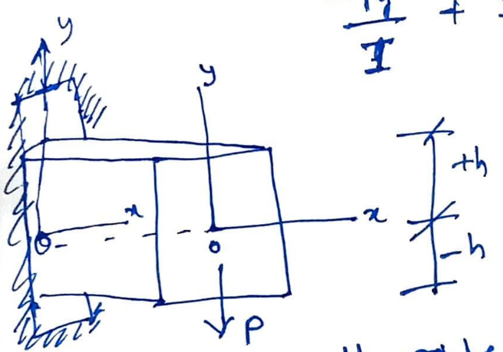
$\sigma_y = 0$



$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$-\frac{Py}{I} + \frac{\partial \tau_{yx}}{\partial y} = 0 \Rightarrow \frac{\partial \tau_{yx}}{\partial y} = \frac{Py}{I}$$

$$\tau_{yx} = \frac{Py^2}{2I} + C$$

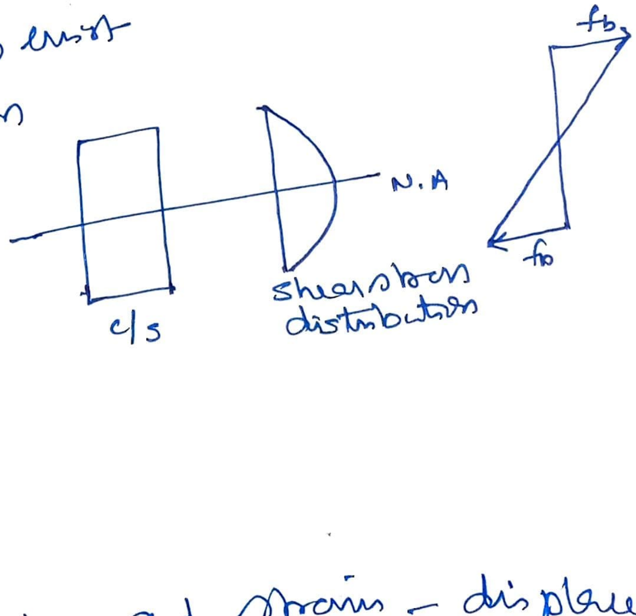


Shear stress will be zero at the top and bottom faces

only on the c/s face shear stress exist

So apply boundary condition

$\tau_{xy} = 0$ @ $y = \pm h$



$$0 = \frac{Ph^2}{2I} + C$$

$$C = -\frac{Ph^2}{2I}$$

$$\tau_{xy} = \frac{-P(h^2 - y^2)}{2I}$$

From stress-strain relationship and strain-displacement relationship

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \mu\sigma_y)$$

$\sigma_y = 0$

$\sigma_x = \sigma_b$

$$\frac{\partial u}{\partial x} = -\frac{Pxy}{IE}$$

$$u = -\frac{Px^2 y}{2EI} + f(y) \quad \text{—— (1)}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \mu\sigma_x)$$

$$\frac{\partial v}{\partial y} = \frac{\mu P x y}{IE}$$

$$v = \frac{\mu P x y^2}{2IE} + g(x) \quad \text{—— (2)}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G}$$

$$= -\frac{Px^2}{2IE} + g'(x) + \mu\frac{Py^2}{2IE} + f'(y) \quad \text{—— (3)}$$

$$\gamma_{xy} = \frac{-P(h^2 - y^2)}{2IG} \quad \text{—— (4)}$$

equating (3) and (4)

$$\left\{-\frac{Px^2}{2IE} + g'(x)\right\} + \left\{\frac{\mu P y^2}{2IE} - \frac{Py^2}{2IG} + f'(y)\right\} = \frac{-Ph^2}{2IG}$$

$$F(x) + G(y) = K$$

This means that $F(x)$ and $G(y)$ must also be some constant. Otherwise taking $F(x)$ and $G(y)$ to be varying with x and y respectively, the equality will be violated. Therefore $F(x)$ must be some constant d_1 and $G(y)$ must be some constant e_1

$$\left.\begin{aligned} d_1 = F(x) &= \frac{-Px^2}{2IE} + g'(x) \\ e_1 = G(y) &= \frac{\mu P y^2}{2IE} - \frac{Py^2}{2IG} + f'(y) \end{aligned}\right\} \quad \text{—— (4)}$$

$$K = \frac{-Ph^2}{2IG}$$

$$d_1 + e_1 = K \Rightarrow d_1 + e_1 = \frac{Ph^2}{2IG} \quad \text{—— (4)(a)}$$

(-ve is removed as d_1 and e_1 are constants)

$$g'(x) = \frac{Px^2}{2IE} + d_1$$

$$g(x) = \frac{Px^3}{6IE} + d_1 x + m_1 \quad \text{—— (5)}$$

$$f'(y) = -\frac{\mu P y^2}{2IE} + \frac{P y^2}{2IG} + e_1$$

$$f(y) = -\frac{\mu P y^3}{6EI} + \frac{P y^3}{6IG} + e_1 y + n_1 \quad \text{—} \; (6)$$

Substituting (5) and (6) in (1) and (2) equations.

$$u = -\frac{P x^2 y}{2IE} - \frac{\mu P y^3}{6IE} + \frac{P y^3}{6IG} + e_1 y + n_1$$

$$v = \frac{\mu P x y^2}{2IE} + \frac{P x^3}{6IE} + d_1 x + m_1$$

The constants d_1, e_1, m_1 and n_1 can be evaluated by applying the boundary conditions. The boundary conditions are u and v are zero for $x = l$, $y = 0$ and we get

$$n_1 = 0, \quad m_1 = -\frac{P l^3}{6IE} - d_1 l$$

To find the constant d_1, we can use the third constant which is eliminating the possibility of rotation of the beam in the xy plane about a fixed point. The constraint can be realized in the various ways. The two cases are -

(1) $\left(\frac{\partial v}{\partial x}\right)_{x=l, y=0} = 0$ (2) $\left(\frac{\partial u}{\partial y}\right)_{x=l, y=0} = 0$

Applying the first case we get $d_1 = -\frac{P l^2}{2IE}$

from eqn (4) (2) $e_1 = \frac{P l^2}{2IE} - \frac{P h^2}{2IG}$

So

$$u = -\frac{P x^2 y}{2IE} - \frac{\mu P y^3}{6IE} + \frac{P y^3}{6IG} + \left(\frac{P l^2}{2IE} - \frac{P h^2}{2IG}\right) y$$

$$v = \frac{\mu P x y^2}{2IE} + \frac{P x^3}{6IE} - \frac{P l^2 x}{2IE} + \frac{P l^3}{3IE}$$

The vertical deflection curve is obtained by substituting $y = 0$ in the equation for v

$$(v)_{y=0} = \frac{Px^3}{6IE} - \frac{Pl^2x}{2IE} + \frac{Pl^3}{3IE} \quad \text{——} \quad (7)$$

Which gives a value of $\frac{Pl^3}{3IE}$ at the free end $x=0$

This value coincides with the value derived in elementary strength of materials.

If the vertical element of the c/s is fixed, i.e. if the second case is considered we get

$$(v)_{y=0} = \frac{Px^3}{6IE} - \frac{Pl^2x}{2IE} + \frac{Pl^3}{3IE} + \frac{Ph^2}{2IG}(l-x)$$

The additional terms is the effect of shearing force on the deflection of the beam.

Biharmonic equation :- (Stress function)

$\nabla^4 \phi = 0$ $\qquad \nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

Since the biharmonic function satisfies all the equilibrium and compatibility equations, a solution to this equation is also the solution for the 2D problem. But in addition to satisfying the biharmonic equation, solution has to satisfy the boundary conditions also.

To solve the desired equations of elasticity, it is suggested that the polynomial functions, inverse functions or semi-inverse functions are to be used.

Solution of the two-dimensional problems reduces to the integration of the differential equations of equilibrium together with the compatibility equations and the boundary conditions. If the body forces present is the weight of the body only, then

The equations to be satisfied are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g = 0$$

} equilibrium equations in terms of stress components — (1)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

} compatibility equations in terms of stress components. — (2)

$$F_x = \sigma_x n_x + \tau_{xy} n_y$$

$$F_y = \sigma_y n_y + \tau_{xy} n_x$$

} Boundary conditions — (3)

F_x, F_y are surface forces per unit area.

The usual method of solving these equations is by introducing a new function called "stress function", in the solution of 2D problems.

equation (1) is satisfied by taking any function 'ϕ' of x and y and using stress component expressions.

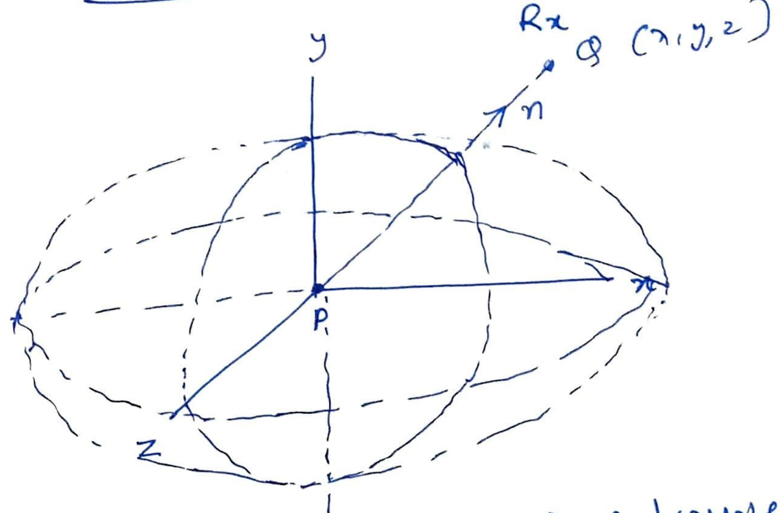
i.e. $\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \rho g y$; $\sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \rho g y$; $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$ ——— (4)

In this manner, we get variety of solutions of the equations of equilibrium (eqn (1)). The true solution of the problem is that which satisfies compatibility equations also (eqn (2)). Substitute eqn (4) in eqn (2) we find the stress function ϕ must satisfy the equation,

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad \text{——— (5)}$$

Thus the solution of the 2D problems, when weight of the body is the only body force, reduces to finding a solution of eqn (5) that satisfies the boundary conditions (3) of the problem.

Lame's Ellipsoid (Stress Ellipsoid)



P_{xyz} be the coordinate frame of reference at point 'P' parallel to the principal axes at 'P'. On plane passing through P with normal 'n', the resultant stress vector is R^n and its components are -

$\sigma_1, \sigma_2, \sigma_3$ are principal stresses along x, y, z axes at P.

$$\left.\begin{aligned} R^n_x &= \sigma_1 n_x \\ R^n_y &= \sigma_2 n_y \\ R^n_z &= \sigma_3 n_z \end{aligned}\right\} \quad — (1)$$

$PQ = |R_n|$

PQ is the resultant stress vector so magnitude of the coordinates (x, y, z) of the point Q are

$$x = R^n_x, \quad y = R^n_y, \quad z = R^n_z \quad — (2)$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

so from eqn (1) and (2)

$$\boxed{\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} + \frac{z^2}{\sigma_3^2} = 1}$$

This is the equation of an ellipsoid referred to the principle axes. This ellipsoid is called stress ellipsoid or Lame's ellipsoid. If two of the principal stresses are equal ($\sigma_1 = \sigma_2$)

then Lami's ellipsoid is an ellipsoid of revolution. If all the principal stresses are equal then ($\sigma_1 = \sigma_2 = \sigma_3$) Lami's ellipsoid becomes a sphere. Stress represented by a radius vector (PQ) of the stress ellipsoid acts on the plane parallel to tangent plane to the surface called the stress-director surface defined by

$$\frac{x^2}{\sigma_1} + \frac{y^2}{\sigma_2} + \frac{z^3}{\sigma_3} = 1$$

Lami's ellipsoid and the stress-director surface together completely define the state of stress at a point.

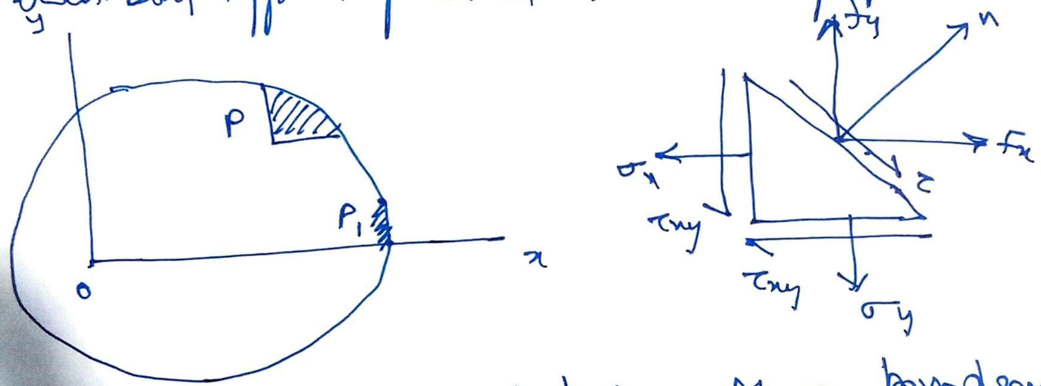
Boundary Conditions

Equations of equilibrium for the plane stress state are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

} these equations must be satisfied throughout the volume of the body.

When the stresses vary over the plate (body having plane stress state) the stress components σ_x, σ_y and τ_{xy} must be consistent with externally applied forces at a boundary point.



Consider a two dimensional body. At a boundary point P, the outward normal is n. F_x and F_y be the components of the surface forces per unit area at this point.

F_x and F_y must be the continuation of the stresses σ_x, σ_y and τ_{xy} at the boundary. Hence using Cauchy's equations

$$R_x^n = F_x = \sigma_x n_x + \tau_{xy} n_y$$
$$R_y^n = F_y = \sigma_y n_y + \tau_{xy} n_x$$

} B.C's @ P

If the boundary of the plate happens to be parallel to y axis at point P_1, the boundary condition becomes

$$F_x = \sigma_x$$
$$F_y = \tau_{xy}$$

} B.C's @ P_1

Solution of 2D problems by the use of Polynomials

Any 2D problem is elasticity can be expressed in the form of polynomial (this process is called problem definition). Once problem is defined in the form of polynomial (called stress function) the solution to the stress function is found and this solution has to satisfy the boundary conditions of the specific problems to represent the problem in form of stress components. Polynomial or problem definition or stress function is generalised equation defining different practical problems.

Solutions of the biharmonic equation $\nabla^4 \phi = 0$ can be expressed in the form of polynomials of various degrees and suitably adjusting their coefficients can be found for many number of practical problems.

"Biharmonic equation"

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Polynomial of First Degree :- (linear function)

Let $\phi = a_1 x + b_1 y$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 0 \; ; \; \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0 \; ; \; \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

Stress function gives a stress free body (stress distribution)

Polynomial of Second Degree (Quadratic function)

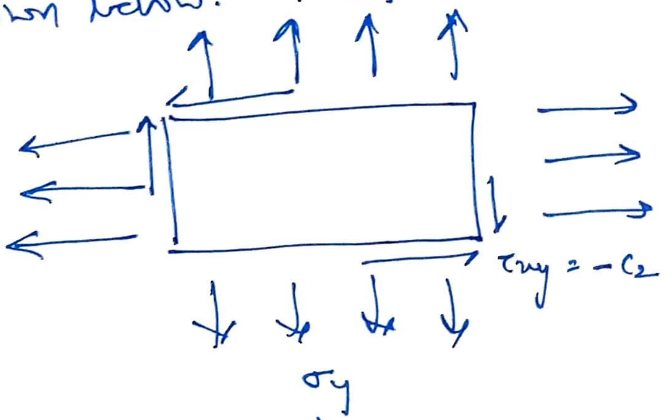
A quadratic polynomial is the lowest order polynomials that yield non-zero stresses from an Airy's stress function.

$$\phi = c_1 x^2 + c_2 xy + c_3 y^2$$ c_1, c_2, c_3 are constants

This Airy's stress function is satisfied the equation $\nabla^4 F = 0$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2c_3 \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 2c_1 \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -c_2$$

This shows that the above stress components do not depend upon the coordinates x and y i.e, they are constant throughout the body representing a constant stress field. Thus the stress function ϕ represents a state of uniform tension or compression in two perpendicular directions accompanied with uniform shear as shown below.



Polynomial of Third Degree

$$\phi = C_1x^3 + C_2x^2y + C_3xy^2 + C_4y^3 \quad \text{--- (1)}$$

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\phi}{\partial y^2} = 2C_3x + 6C_4y \\ \sigma_y &= \frac{\partial^2\phi}{\partial x^2} = 6C_1x + 2C_2y \end{aligned}\right\} \quad \text{--- (2)}$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -2(C_2x + C_3y)$$

This stress function gives a linearly varying stress field. It should be noted that the magnitude of the coefficients C_1, C_2, C_3 and C_4 are chosen freely since the expression ϕ is satisfied irrespective of values of these coefficients.

If $C_1 = C_2 = C_3 = 0$ except C_4 we get the stress components

$$\left.\begin{aligned} \sigma_x &= 6C_4y \\ \sigma_y &= 0 \quad \tau_{xy} = 0 \end{aligned}\right\} \quad \text{--- (3)}$$

This corresponds to the pure bending on the face perpendicular to the x-axis.

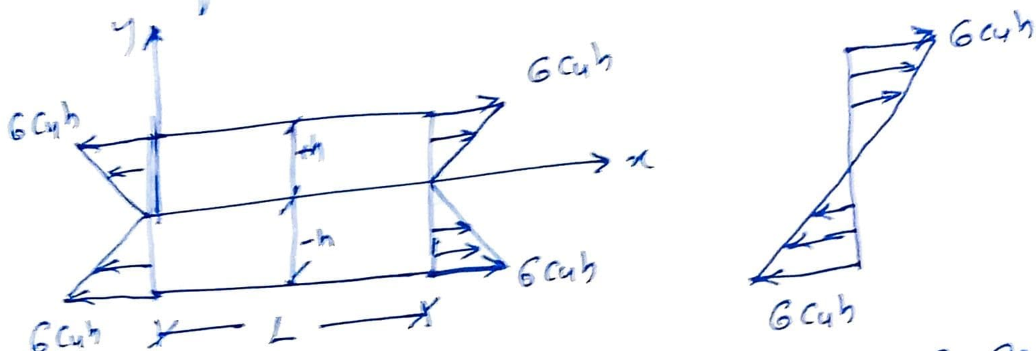
Polynomial (1) represents different practical problems and eqn (2) represents the solutions to the problems. eqn ③ represents boundary conditions.

At $y = -h$ $\quad \sigma_x = -6c_4h$

at $y = +h$ $\quad \sigma_x = 6c_4h$

} Stress boundary condition represents a state of the normal stresses due to a pure bending applied at the ends of the beam.

The variation of σ_x with y is linear.

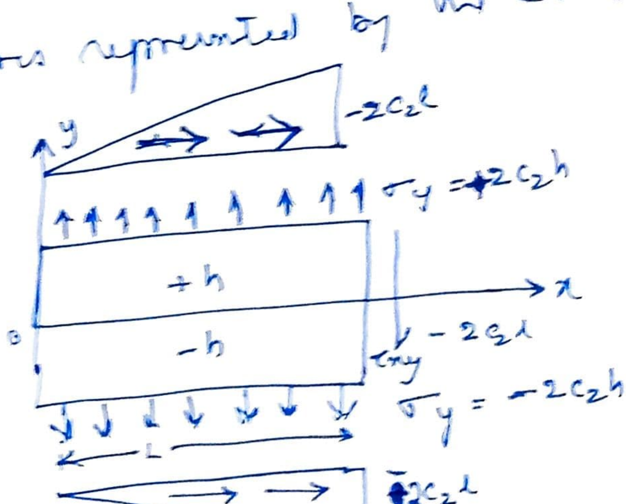


Similarly if all the coefficients are zero except c_2 are zero

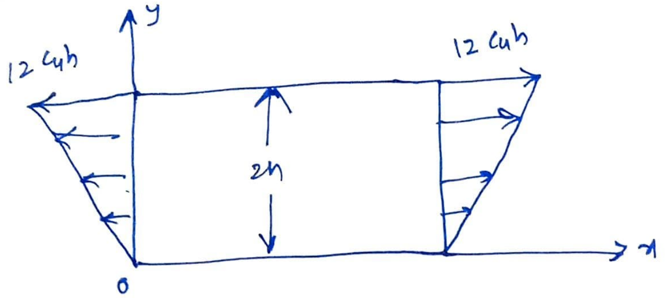
$\sigma_x = 0 \quad \sigma_y = 2c_2y$

$\tau_{xy} = -2c_2x$

The stresses represented by the above stress field will vary as shown.



The stress σ_y is constant with x (ie constant along the span x of the beam) but varies with y at a particular section. At $y = +h$, $\sigma_y = 2c_2h$ (tensile), @ $y = -h$, $\sigma_y = -2c_2h$ (compressive), σ_x is zero throughout. Shear stress τ_{xy} is zero at $x = 0$ and is equal to $-2c_2l$ @ $x = l$. At any other section, the shear stress is proportional to x.



$C_1 = C_2 = C_3 = 0$

$\sigma_x = 6C_4 y$, $y = 2h$

The stress boundary condition represents a state of the normal stresses due to bending plus axial load applied to the ends of the beam.

In the case of polynomials of higher degree, the equation $\nabla^4 F = 0$ is satisfied only if certain relations between the coefficients are satisfied.

Polynomial of 4th Degree :-

Let us consider the stress function in the form of a polynomial of the fourth degree

$$\phi = C_1 x^4 + C_2 x^3 y + C_3 x^2 y^2 + C_4 x y^3 + C_5 y^4$$

function is satisfied the equation $\nabla^4 F = 0$ only if $C_5 = -(2C_3 + C_1)$

The stress components in this case are

$$\sigma_x = C_3 x^2 + C_4 xy - (2C_3 + C_1) y^2$$

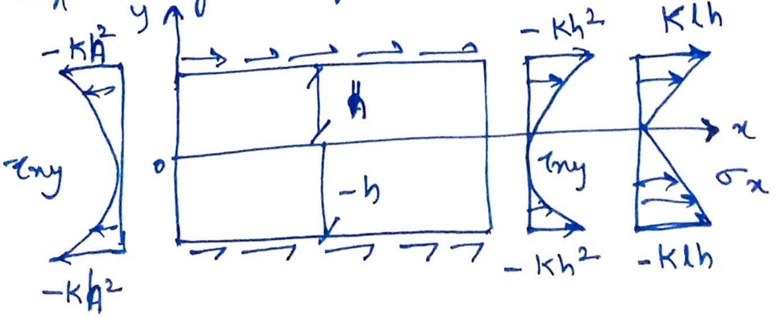
$$\sigma_y = C_1 x^2 + C_2 xy + C_3 y^2$$

$$\tau_{xy} = -C_2 x^2 - 2C_3 xy - C_4 y^2$$

If all coefficients except C_4 are zero and C_4 = a constant $= K$

$$\sigma_x = Kxy, \quad \sigma_y = 0, \quad \tau_{xy} = -Ky^2$$

$$\phi = Kxy^3$$



On the longitudinal sides $y = \pm h$ are uniformly distributed shearing forces. At the ends, the shearing forces are distributed according to a parabolic distribution. The shearing forces acting on the boundary of the beam are reduced to the couple.

$\tau = \frac{F}{A} \Rightarrow F = \tau \times A$

$F \times h = \tau \times A \times h \quad \because A = \underline{l \times 1}$ unit thickness

area on which shear force acts.

This couple balances the couple produced by the normal forces along the side $x = l$ of the beam.

$$M = 2\left[Kh^2 \times (l \times h)\right] - 2\left[Kxy \times \frac{h}{3}\right]$$

($l \times h$ → Area; h → ⊥r distance; Kxy → Klh)

Area $= l \times 1 = l$

$$M = 2Kh^3 l - \frac{2Klh^2}{3}$$

$\sigma_x = \frac{f}{A}$

$f = \sigma_x A$

$f \times \frac{h}{3} = \sigma_x \times A \times \frac{h}{3}$

$= Kxy \times A \times \frac{h}{3}$

$= Kxy \times \frac{h}{3}$

$= Klh \times \frac{h}{3}$

$= \frac{Klh^2}{3}$

Polynomial of the fifth Degree

Let $\phi = c_1 x^5 + c_2 x^4 y + c_3 x^3 y^2 + c_4 x^2 y^3 + c_5 x y^4 + c_6 y^5$

The corresponding stress components are given by

$\sigma_x = c_3 x^3 + c_4 x^2 y - (2c_3 + 3c_1) x y^2 - (c_2 + 2c_4) y^3$

$\sigma_y = c_1 x^3 + c_2 x^2 y + c_3 x y^2 + \frac{c_4}{3} y^3$

$\tau_{xy} = -c_2 x^3 - c_3 x^2 y - c_4 x y^2 + (2c_3 + 3c_1) y^3$

Here the coefficients c_1, c_2, c_3, c_4 are arbitrary and in adjusting them we obtain solutions for various loading conditions of the beam.

If all coefficients except d_3 are zero

$\sigma_x =$

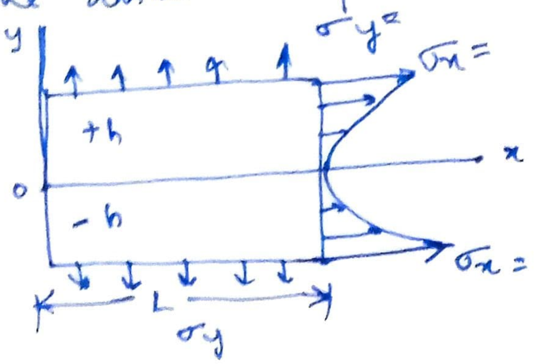
$\sigma_y =$

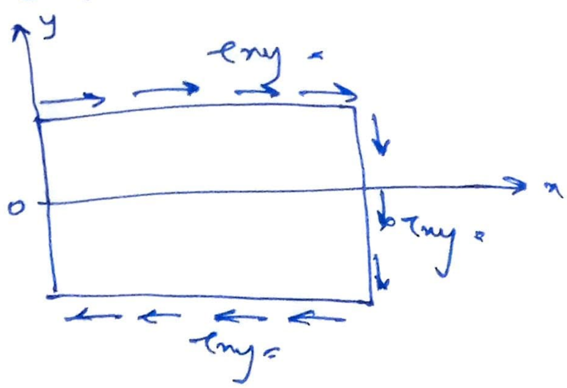
$\tau_{xy} =$

(i) The normal forces are uniformly distributed along the longitudinal sides of the beam.

(ii) Along the side $x = L$, the normal forces consists of two parts — one following a linear law and the other following the law of a cubic parabola. The shearing forces are proportional to x on the longitudinal sides of the beam and follow a parabolic law along the side $x = L$.

The distribution of stresses in case (i) and (ii) are





On the longitudinal sides $y = \pm h$ are uniformly distributed shearing forces. At the ends, the shearing forces are distributed according to a parabolic distribution. The shearing forces acting on the boundary of the beam are reduced to the couple.

$\tau = \frac{F}{A} \Rightarrow F = \tau \times A$

$F \times h = \tau \times A \times h \quad \because A = l \times 1$ unit thickness

area on which shear force acts.

This couple balances the couple produced by the normal forces along the side $x = l$ of the beam.

$$M = 2\left[Kh^2 \times (l \times h)\right] - 2\left[Kxy \times \frac{h}{3}\right]$$

(Area: $l \times h$; ⊥ distance: h; $Kxy \to Klh$)

Area $= l \times 1 = l$

$$M = 2Kh^3 l - \frac{2Klh^2}{3}$$

$\sigma_x = \frac{f}{A}$

$f = \sigma_x A$

$f \times \frac{h}{3} = \sigma_x \times A \times \frac{h}{3}$

$= Kxy \times A \times \frac{h}{3}$

$= Kxy \times \frac{h}{3}$

$= Klh \times \frac{h}{3}$

$= \frac{Klh^2}{3}$

let $\phi = c_1 x^5 + c_2 x^4 y + c_3 x^3 y^2 + c_4 x^2 y^3 + c_5 x y^4 + c_6 y^5$

The corresponding stress components are given by

$\sigma_x = c_3 x^3 + c_4 x^2 y - (2c_3 + 3c_1) x y^2 - (c_2 + 2c_4) y^3$

$\sigma_y = c_1 x^3 + c_2 x^2 y + c_3 x y^2 + \frac{c_4}{} y^3$

$\tau_{xy} = -c_2 x^3 - c_3 x^2 y - c_4 x y^2 + (2c_3 + 3c_1) y^3$

Here the coefficients c_1, c_2, c_3, c_4 are arbitrary and in adjusting them we obtain solutions for various loading conditions of the beam.

If all coefficients except d_3 are zero

$\sigma_x =$

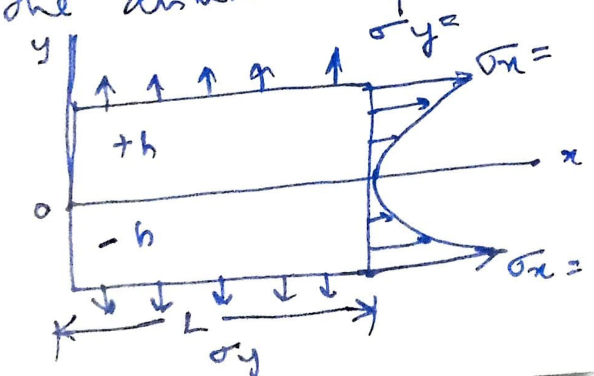
$\sigma_y =$

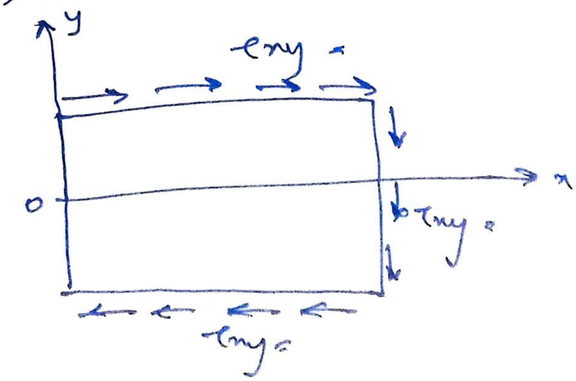
$\tau_{xy} =$

(i) The normal forces are uniformly distributed along the longitudinal sides of the beam.

(ii) Along the side $x = L$, the normal forces consists of two parts - one following a linear law and the other following the law of a cubic parabola. The shearing forces are proportional to x on the longitudinal sides of the beam and follow a parabolic law along the side $x = l$.

The distribution of stresses in case (i) and (ii) are





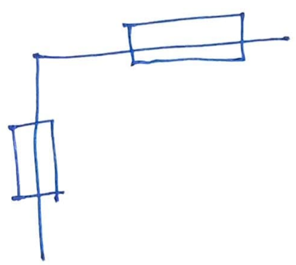
Surface strains

or

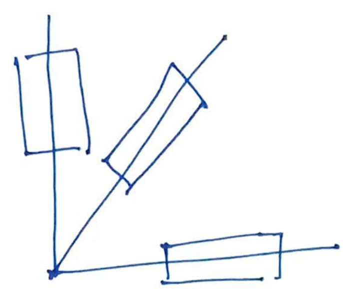
Strain Rosettes

Strain gauge is a device used to measure the strains on the free surface of a structure. Strain gauges are employed to measure the linear deformation over a given gauge length to sense the change in length.

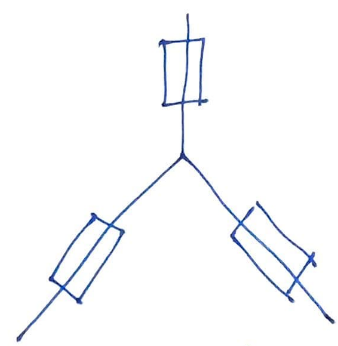
Since a single gauge can measure the strain in only a single direction, two gauges are needed to determine the strains ϵ_x and ϵ_y. Strain gauge rosettes consist of two or more co-located strain gauges oriented at a fixed angle w.r.t each other. Rosettes typically involve 2, 3 or 4 strains gauges with relative orientations of 30°, 45°, 60° or 90°. The different types of strain rosettes are



Tee rosette
(0/90°)



Rectangular rosette
(0/45°/90°)

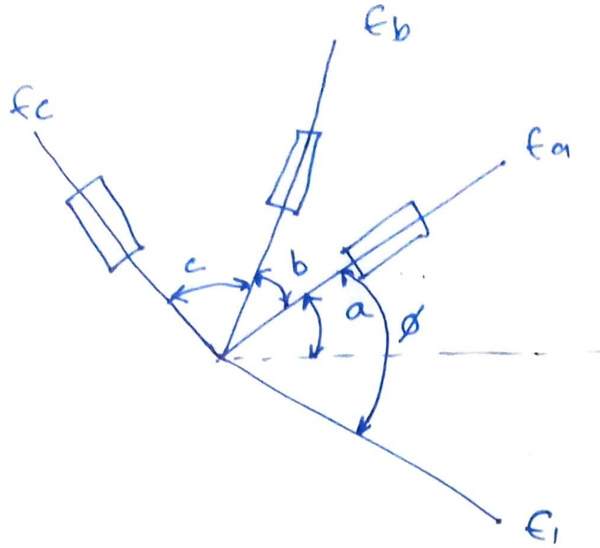


Delta rosette
(0/60°/120°)

Types of strain rosettes

The Tee rosette is used only when principal strain directions are known in advance. The Rectangular rosette and the Delta rosette are the most commonly used 3-gauge rosettes because of their simple geometry.

Consider a strain rosette with the gauges oriented at a, (a+b) and (a+b+c) with the horizontal axis. Let the strains measured using these rosettes be ϵ_a, ϵ_b and ϵ_c respectively. ϵ_1 is the major principal strain which is oriented at ϕ w.r.t ϵ_a.



Orientation of strains

Using the transformation matrix, we can write

$$\epsilon_a = \epsilon_x \cos^2 a + \epsilon_y \sin^2 a + \gamma_{xy} \sin a \cos a$$

$$\epsilon_b = \epsilon_x \cos^2(a+b) + \epsilon_y \sin^2(a+b) + \gamma_{xy} \sin(a+b)\cos(a+b)$$

$$\epsilon_c = \epsilon_x \cos^2(a+b+c) + \epsilon_y \sin^2(a+b+c) + \gamma_{xy} \sin(a+b+c)\cos(a+b+c)$$

Solve $\epsilon_a, \epsilon_b, \epsilon_c$ to get ϵ_x, ϵ_y and γ_{xy}

Principal strains are

$$\epsilon_1 = \left(\frac{\epsilon_x+\epsilon_y}{2}\right) + \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\epsilon_2 = \left(\frac{\epsilon_x+\epsilon_y}{2}\right) - \sqrt{\left(\frac{\epsilon_x-\epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

The inclination of the principal plane ϕ w.r.t the strain gauge marked ϵ_a is given by

$$\tan 2\phi = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

The principal stresses can be calculated from Hooke's law

$$\sigma_1 = \frac{E}{(1-\mu^2)}(\epsilon_1 + \mu\epsilon_2)$$

$$\sigma_2 = \frac{E}{(1-\mu^2)}(\epsilon_2 + \mu\epsilon_1)$$

Example

Let us consider three strain rosette in xy plane as a, b and c, the measured strains in these rosette are $e_a = 0.5 \times 10^{-3}$, $e_b = 0.4 \times 10^{-3}$, $e_c = 0.3 \times 10^{-3}$ resp. The angles of the rosette w.r.t the positive x-axis as $\theta_a = 45^\circ$, $\theta_b = 90^\circ$ and $\theta_c = 135^\circ$ resp. If $\lambda = 140.6$ GPa and $\mu = 75$ GPa. Calculate τ_{xy}.

$\theta_a = 45^\circ$ $\quad e_a = 0.5 \times 10^{-3}$ $\quad \lambda = 140.6$ GPa

$\theta_b = 90^\circ$ $\quad e_b = 0.4 \times 10^{-3}$ $\quad \mu = 75$ GPa

$\theta_c = 135^\circ$ $\quad e_c = 0.3 \times 10^{-3}$

$$e'_x = e_x \cos^2 45^\circ + e_y \sin^2 45^\circ + 2 e_{xy} \sin 45^\circ \cos 45^\circ$$

or

e_a

$$0.5 \times 10^{-3} = \frac{e_x}{2} + \frac{e_y}{2} + e_{xy} \quad \text{---(1)}$$

$$e_b = e_x \cos^2 90^\circ + e_y \sin^2 90^\circ + 2 e_{xy} \cos 90^\circ \sin 90^\circ$$

$$e_y = 0.4 \times 10^{-3} \quad \text{---(2)}$$

$$e_c = e_x \cos^2 135^\circ + e_y \sin^2 135^\circ + 2 e_{xy} \sin 135^\circ \cos 135^\circ$$

$$e_x + e_y - 2 e_{xy} = 0.6 \times 10^{-3} \quad \text{---(3)}$$

(1) − (3)

$$4 e_{xy} = 0.4 \times 10^{-3} \Rightarrow \underline{e_{xy} = 0.1 \times 10^{-3}}$$

$$\tau_{xy} = G \text{ or } \mu \; e_{xy}$$

$$= 75 \times 0.1 \times 10^{-3} \times 10^{3}$$

= ~~15 MPa~~

Note:

$$G = \frac{\tau_{xy}}{e_{xy}} = \frac{E}{2(1+\nu)} = \mu$$

μ = <u>Lamé constant</u>

$$G = \frac{E}{2(1+\nu)} = \mu$$

Two dimensional problems in Polar Coordinates (Pg 26) Contd.

* **Stress-strain relationship in Polar Coordinate system**

$$\epsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_\theta)$$

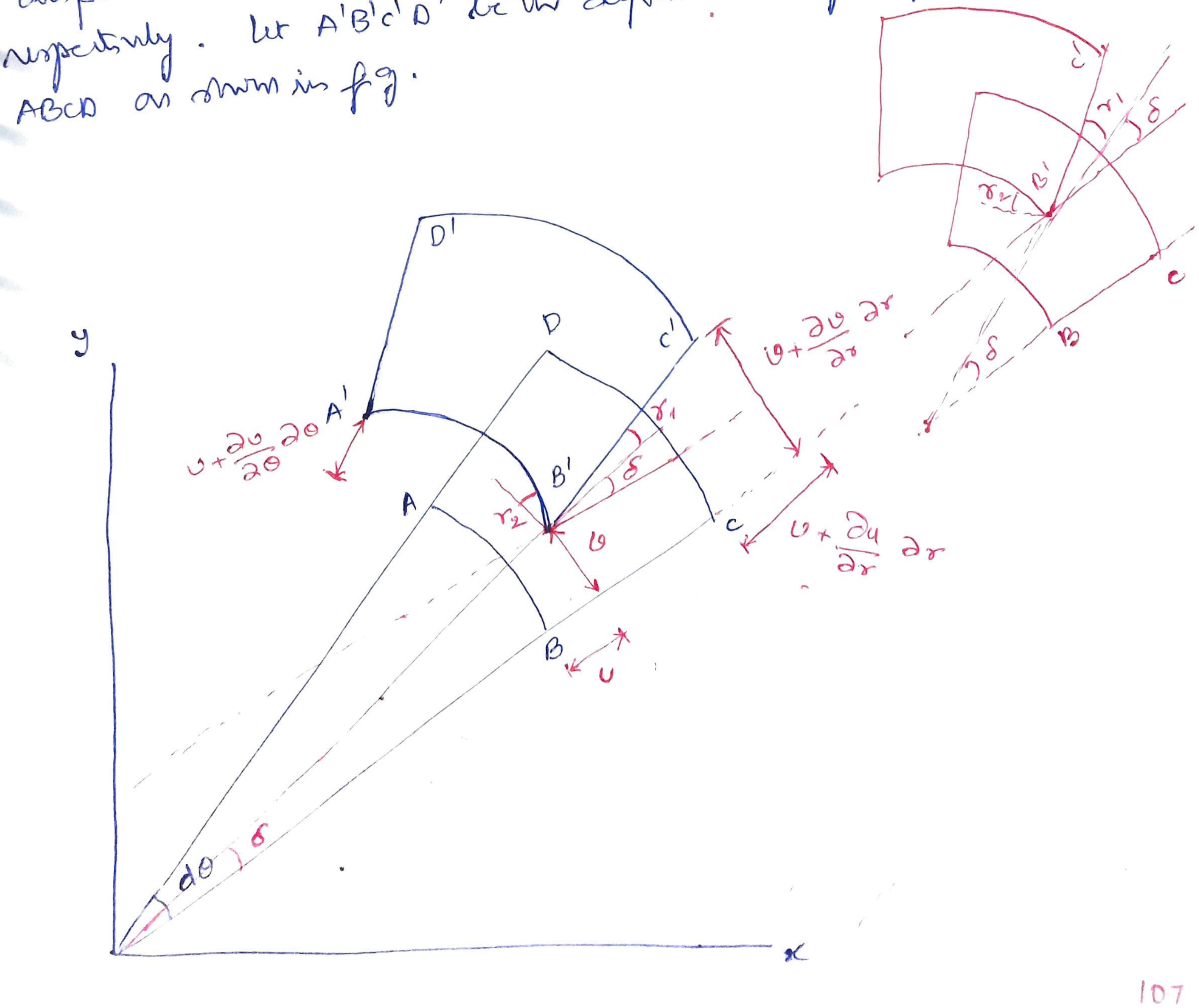
$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_r)$$

$$\gamma_{r\theta} = \frac{\tau_{r\theta}}{G}$$

* **Strain-displacement relationship in Polar Coordinate system**

Strain-displacement relationship in cylindrical polar. The strain-displacement relationship can be derived by considering undeformed coordinates (r, θ, z) and deformed elements.

Consider the deformation of the infinitesimal element ABCD, both the radial and tangential directions. Displacement u and v in the radial and tangential directions respectively. Let A'B'C'D' be the deformed shape of the element ABCD as shown in fig.



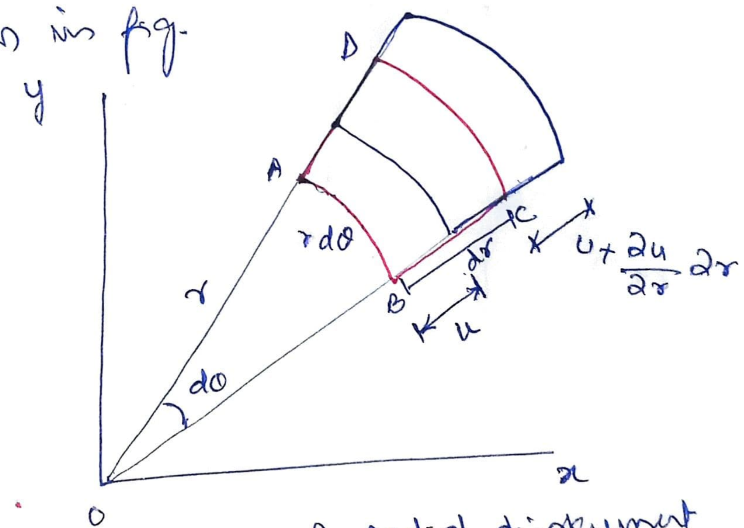
$$\gamma_1 = \frac{v + \frac{\partial v}{\partial r} dr - v}{dr} - \delta = \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$\gamma_2 = \frac{u + \frac{\partial u}{\partial \theta} d\theta - u}{r\, d\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta}$$

Shear strain $\gamma_{r\theta} = \gamma_1 + \gamma_2 = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}$

To find the radial and circumferential strains we will consider the effect of the radial and tangential displacements separately.

First let us consider the effect of the radial displacement 'u' as shown in fig.

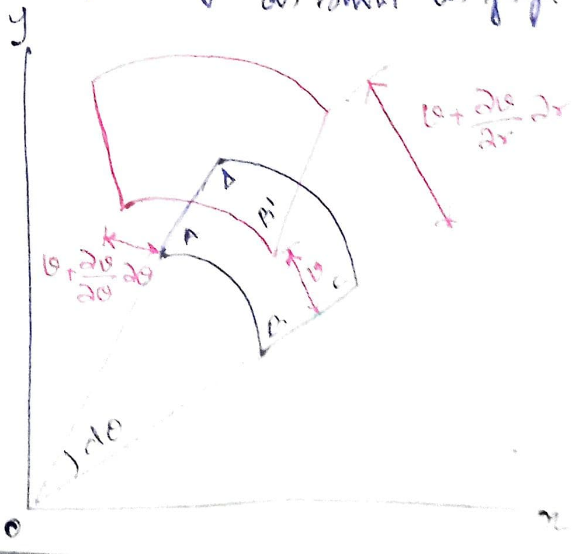


Effect of radial displacement

Radial strain $\epsilon_r = \frac{u + \frac{\partial u}{\partial r} dr - u}{dr} = \frac{\partial u}{\partial r}$

Tangential strain $\epsilon_\theta = \frac{(r+u)d\theta - r\,d\theta}{r\,d\theta} = \frac{u}{r}$

Now considering the effect of tangential displacement 'v' as shown in fig.



Effect of Tangential displacement

Radial strain $\epsilon_r = 0$

Circumferential strain $\epsilon_\theta = \dfrac{0 + \frac{\partial v}{\partial \theta} d\theta - 0}{r d\theta} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$

Combining the effects, we get

$$\left.\begin{aligned} \text{Radial strain } \epsilon_r &= \frac{\partial u}{\partial r} \\ \text{Circumferential strain } \epsilon_\theta &= \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \text{Shear strain } \gamma_{r\theta} &= \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \end{aligned}\right\} b$$

Example :-

If the displacement components are

$u = 5r^2 \sin 2\theta - r\theta^3$ and $v = 10r^3 \cos 2\theta - r^2\theta^2$

Determine the radial, circumferential and shear strain at the point $\left(-5, \frac{\pi}{3}\right)$

Radial strain $\epsilon_r = \dfrac{\partial u}{\partial r} = 10 r \sin 2\theta - \theta^3$

$= 10(-5) \sin 2\left(\frac{\pi}{3}\right) - \left(\frac{\pi}{3}\right)^3 = -44.45$

Circumferential strain $\epsilon_\theta = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = 5r \sin 2\theta - \theta^3 - 20 \sin 2\theta r^2 - 2r\theta$

$\epsilon_\theta = 5(-5) \sin 2\left(\frac{\pi}{3}\right) - \left(\frac{\pi}{3}\right)^3 - 20 \sin 2\left(\frac{\pi}{3}\right)(-5)^2 - 2(-5)\left(\frac{\pi}{3}\right)$

$= -445.34$

Shear strain $\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$

$= 30r^2 \cos 2\theta - 2r\theta^2 + 10 r \cos 2\theta - 3\theta^2 - 10r^2 \cos 2\theta + r\theta$

put $r = -5$, $\theta = \pi/3$

$\gamma_{r\theta} = -222.56$

Biharmonic Equation in Polar Coordinate system

Biharmonic equation in Cartesian system is given as

$$\nabla^4 \phi = 0$$

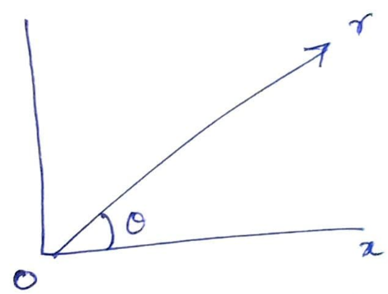
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0 \Rightarrow \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

In order to transform these equations in polar form we need to apply the chain rule

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

The cartesian stress components can be expressed in terms of polar components using the stress transformation formulae applying rotation as shown in fig.



representation of cartesian and polar components

We know that $x = r\cos\theta \quad y = r\sin\theta$

$r^2 = x^2 + y^2 \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\left(\frac{x}{r}\right) + \frac{\partial \phi}{\partial \theta}\left(\frac{-y}{r^2}\right) = \frac{\partial \phi}{\partial r}\cos\theta + \frac{\partial \phi}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r}\left(\frac{y}{r}\right) + \frac{\partial \phi}{\partial \theta}\left(\frac{x}{r^2}\right) = \frac{\partial \phi}{\partial r}\sin\theta + \frac{\partial \phi}{\partial \theta}\left(\frac{\cos\theta}{r}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \left\{\frac{\partial}{\partial r}\cos\theta + \frac{\partial}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)\right\}\left\{\frac{\partial \phi}{\partial r}\cos\theta + \frac{\partial \phi}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)\right\}$$

$$= \left\{\frac{\partial^2 \phi}{\partial r^2}\cos^2\theta + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial \phi}{\partial \theta} - \frac{\partial^2 \phi}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial \phi}{\partial r}\frac{\sin^2\theta}{r}\right.$$

$$\left. 2\sin\theta\cos\theta + \frac{\partial^2 \phi}{\partial \theta^2}\frac{\sin^2\theta}{r^2}\right\}$$

$$= \frac{\partial^2 \phi}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 \phi}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial \phi}{\partial r}\frac{\sin^2\phi}{r} + \frac{2}{r^2}\sin\theta\cos\theta\frac{\partial \phi}{\partial \theta}$$

$$+ \frac{\partial^2 \phi}{\partial \theta^2}\frac{\sin^2\theta}{r^2} \quad \text{--- (1)}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \left\{\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\right\}\left\{\frac{\partial \phi}{\partial r}\sin\theta + \frac{\partial \phi}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\right\}\phi$$

$$= \frac{\partial^2 \phi}{\partial r^2}\sin^2\theta + 2\frac{\partial^2 \phi}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 \phi}{\partial \theta^2}\frac{\cos^2\theta}{r^2}$$

$$- \frac{2}{r^2}\sin\theta\cos\theta\frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial r}\frac{\cos^2\theta}{r^2} \quad \text{--- (2)}$$

Adding (1) and (2) we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\nabla^4 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = \left(\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}\right) = 0$$

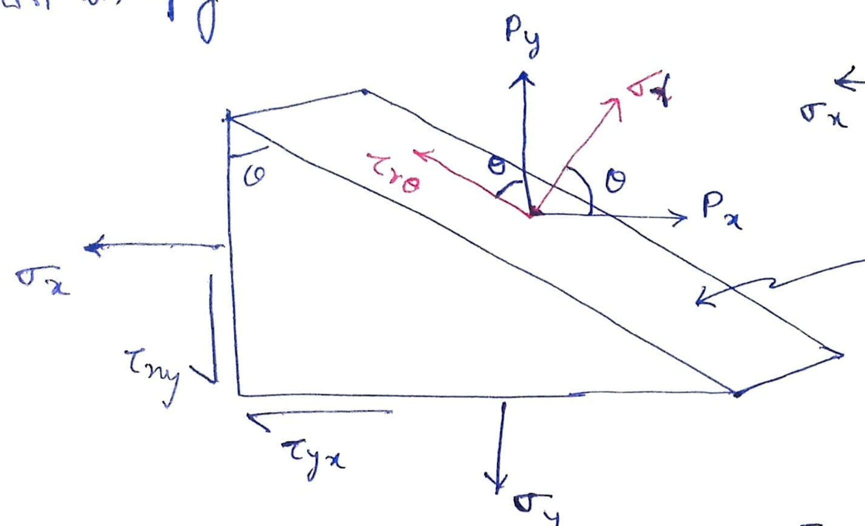
$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0 \quad \text{--- (3)}$$

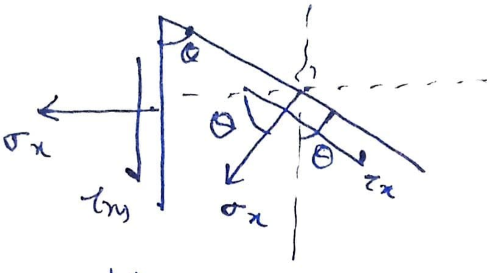
From various solutions of eqn (3) we obtain the solutions of two-dimensional problems in polar coordinates for various boundary conditions.

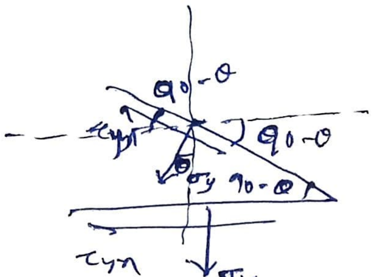
equation (3) is Biharmonic equation in Polar coordinate system

Airy's Stress Function in Polar Coordinate System

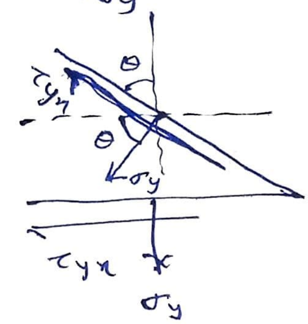
The Airy's stress function in polar coordinate system is a function of $\phi(r,\theta)$. Consider the infinitesimal element shown in fig.







Stresses in polar coordinate system



Taking equilibrium of forces along x-direction

$$P_x\, dA = \sigma_x (dA \cos\theta) + \tau_{xy} (dA \sin\theta)$$

$$P_x = \sigma_x \cos\theta + \tau_{xy} \sin\theta$$

Taking equilibrium of forces along y-direction

$$P_y\, dA = \sigma_y (dA \sin\theta) + \tau_{yx} (dA \cos\theta)$$

$$P_y = \sigma_y \sin\theta + \tau_{yx} \cos\theta$$

Similarly for stress along σ_r

$$\sigma_r = P_x \cos\theta + P_y \sin\theta$$

$$= (\sigma_x \cos\theta + \tau_{xy} \sin\theta) \cos\theta + (\sigma_y \sin\theta + \tau_{yx} \cos\theta) \sin\theta$$

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{yx} \sin\theta \cos\theta$$

Since the stresses are mutually perpendicular

σ_θ can be got by putting $(\theta + \frac{\pi}{2})$

$$\sigma_\theta = \sigma_x \cos^2\left(\theta + \frac{\pi}{2}\right) + \sigma_y \sin^2\left(\theta + \frac{\pi}{2}\right) + 2\tau_{yx} \sin\left(\theta + \frac{\pi}{2}\right) \cos\left(\theta + \frac{\pi}{2}\right)$$

$$= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{yx} \sin\theta \cos\theta$$

$\tau_{r\theta} = \tau_y \cos\theta - \tau_x \sin\theta$

$= -(\sigma_y \sin\theta + \tau_{yx} \cos\theta)\cos\theta - (\sigma_x \cos\theta + \tau_{xy} \sin\theta)\sin\theta$

$= -\sigma_y \sin\theta\cos\theta + \tau_{yx}\cos^2\theta - \sigma_x \sin\theta\cos\theta - \tau_{xy}\sin^2\theta$

$= \frac{\sigma_y}{2}\sin 2\theta + \tau_{yx}\cos 2\theta - \frac{\sigma_x}{2}\sin 2\theta$

$2\cos^2\theta - 1 = \cos 2\theta$

$$\tau_{r\theta} = \left(\frac{\sigma_y - \sigma_x}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

where $\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$ $\quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}$ $\quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

To transform equations from cartesian to polar coordinates we use the following relations

$x = r\cos\theta \qquad r^2 = x^2 + y^2$

$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

The cartesian partial derivative is

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\cdot\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\cdot\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} - \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}$$

The second partial derivatives are

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\cdot\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)\left(\cos\theta\cdot\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)$$

$$= \cos\theta\cdot\frac{\partial}{\partial r}\left(\cos\theta\cdot\frac{\partial}{\partial r}\right) - \cos\theta\cdot\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)$$

$$-\left(\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\cos\theta\cdot\frac{\partial}{\partial \theta}\right) + \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\left(\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta\cdot\frac{\partial^2}{\partial r^2} + \sin^2\theta\left(\frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\right)$$

$$-\sin 2\theta\left(\frac{1}{r^2}\cdot\frac{\partial}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2}{\partial r\partial \theta}\right)$$

Similarly.

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \cdot \frac{\partial^2}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\right)$$

$$- \sin 2\theta\left(\frac{1}{r^2}\cdot\frac{\partial}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2}{\partial r\,\partial \theta}\right)$$

$$\frac{\partial^2}{\partial x\,\partial y} = -\sin\theta\cos\theta\left(-\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$$

$$- \cos 2\theta\left(\frac{1}{r^2}\cdot\frac{\partial}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2}{\partial r\,\partial \theta}\right)$$

In order to solve problems in polar coordinates using the stress functions can be transformed using the above relationships and are given as

$$\left.\begin{aligned}\sigma_r &= \frac{1}{r}\cdot\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 \phi}{\partial \theta^2}\\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2}\\ \tau_{r\theta} &= \frac{1}{r^2}\cdot\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2 \phi}{\partial r\,\partial \theta}\end{aligned}\right\}$$

Example:- Find the stress component $\sigma_r, \sigma_\theta, \tau_{r\theta}$ when $\phi = \frac{\sigma_b y^2}{2}$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \sigma_b \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\,\partial y} = 0$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\sigma_r = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{yx}\sin\theta\cos\theta = \sigma_b\cos^2\theta$$

$$\sigma_\theta = \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - 2\tau_{yx}\sin\theta\cos\theta = \sigma_b\sin^2\theta$$

$$\tau_{r\theta} = \left(\frac{\sigma_y - \sigma_x}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta = -\frac{\sigma_b}{2}\sin 2\theta$$

Example Det. The stress components σ_r, σ_θ and $\tau_{r\theta}$ from the following stress functions. Calculate their values when $\alpha = 30°$ and $\theta = 40°$

$$\ldots r^2\sin\theta\cos\theta - r^2\cos^2\theta\tan\alpha$$

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = -0.04186 - 1.7691 = -1.81096$$

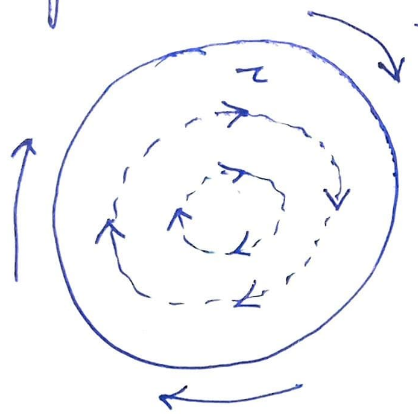
$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = 2(\alpha - \theta) + 2\sin\theta\cos\theta - 2\cos^2\theta\tan\alpha = -0.04186$$

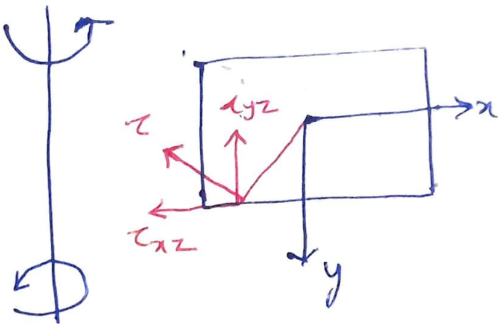
$$\tau_{r\theta} = \frac{1}{r^2}\cdot\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2 \phi}{\partial r\,\partial \theta} = -0.02093 - 0.51555$$

$$= -0.53648$$

TORSION

The equations of torsion developed by coulomb gives exact solution for the torsion problems of circular shafts. In circular sections, the shear stress 'τ' developed due to the applied torque 'T' is uniformly distributed along the circumferential lines as shown in fig.





Shear stress in a circular section.

But when Navier tried to apply the coulomb equation for torsion problems of non-circular cross-sections, he got erroneous results. The reasons for errors in non-circular sections are due to following reasons -

① The shear stress is not constant at a given distance from the axis of rotation. Hence section perpendicular to the axis of the member warp due to the out of plane displacement.

② The maximum shear stress and maximum shear strain are not at the farthest distance from the axis of rotation of the section.

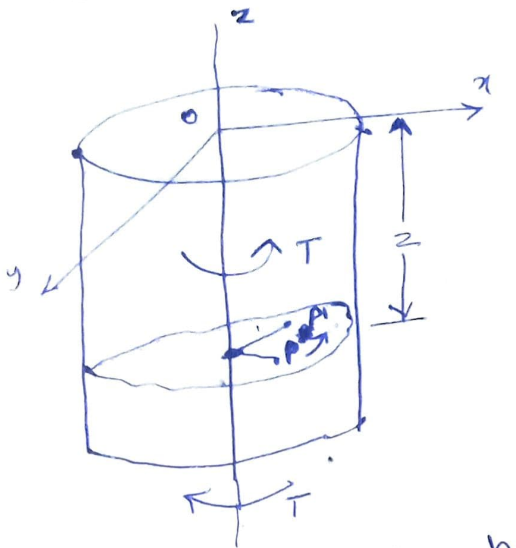
③ The shear stress is zero at the corners.

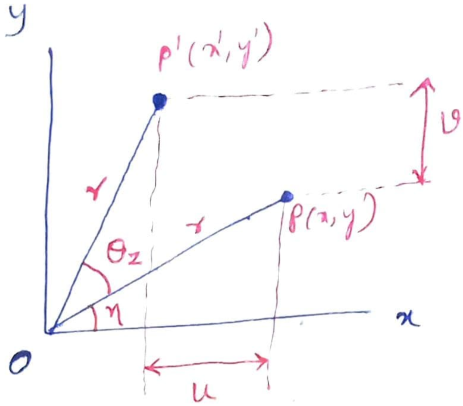
④ Plane sections do not remain plane in non-circular sections.

Therefore to obtain exact solutions for torsion problems of non-circular sections a warping function to account for the out of plane displacement has to be used. St. Venant was the first to accurately suggest the correct solution for the torsion problems of non-circular solid sections. He was semi-inverse method.

Torsion of general prismatic bars - solid sections

St. Venant's semi-inverse method or St. Venant's approach





Consider a prismatic bar of any c/s subjected to torque 'T' at the ends as shown in fig. Let a point P(x,y) at a distance r from the origin and making an angle 'η' with the x-axis rotate through an angle θ_z to the point P'(x',y'). The x displacement is 'u' and y displacement is 'v'.

The following assumptions are made.

(1) Deformation of the twisted shaft consists of c/s rotations as in the case of circular sections and

(2) Warping is same at all cross-sections.

In addition to x and y displacements, the point 'p' may undergo a displacement 'w' in z-direction. This is called warping.

We assume that the z-displacement is a function of only x, y and is independent of z. This means that warping is same for all c/s.

St. Venant's displacement components are

$$-u = r\cos\eta - r\cos(\theta_z+\eta)$$

$$= r\cos\eta - r[\cos\theta_z\cos\eta - \sin\theta_z\sin\eta]$$

if θ_z is very small $\cos\theta_z = 1$, $\sin\theta_z = \theta_z$

$$= r\cos\eta - r\cos\eta + r\sin\eta\,\theta_z = r\theta_z\sin\eta$$

$$u = -r\theta_z \sin\eta \qquad \sin\eta = \frac{y}{r}$$

$$u = -\theta_z y \quad \text{——} \quad (1) \qquad y = r\sin\eta$$

$$\cos\eta = \frac{x}{r}$$

$$x = r\cos\eta$$

$$v = r\sin(\theta_z + \eta) - r\sin\eta$$

$$= r(\cos\theta_z \sin\eta + \sin\theta_z \cos\eta) - r\sin\eta$$

$$v = r\sin\eta + r\,\theta_z\cos\eta - r\sin\eta = r\,\theta_z\cos\eta$$

$$v = x\,\theta_z \quad \text{——} \quad (2)$$

$$w = \theta_z\,\varphi(x,y) \quad \text{——} \quad (3)$$

or

$$w = \theta\varphi(x,y)$$

$\varphi(x,y)$ is called the warping function.

From the displacements u, v and w, the strain components can be found to be

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x}; \quad \epsilon_y = \frac{\partial v}{\partial y}; \quad \epsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}; \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned}\right\}$$

The above are normal strains and shearing strains.

from (1), (2) and (3)

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0$$

$$\gamma_{xy} = -\theta_z + \theta_z = 0$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta\frac{\partial \varphi}{\partial y} + x\theta = \theta\left(\frac{\partial \varphi}{\partial y} + x\right)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta\frac{\partial \varphi}{\partial x} - y\theta = \theta\left(\frac{\partial \varphi}{\partial x} - y\right)$$

From the stress-strain relationships, we get

$$\left.\begin{aligned} \sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \\ \tau_{xz} &= G\gamma_{xz} = G\theta\left(\frac{\partial \varphi}{\partial x} - y\right) \\ \tau_{yz} &= G\gamma_{yz} = G\theta\left(\frac{\partial \varphi}{\partial y} + x\right) \end{aligned}\right\} \quad \text{——} \quad (4)$$

Note: $G = \frac{E}{2(1+\nu)}$ $K = \frac{E}{3(1-2\nu)}$

$\epsilon = \epsilon_x + \epsilon_y + \epsilon_z$ = cubical dilatation or volumetric strains

The above stress components are the ones corresponding to the assumed displacement components. These stress components should satisfy the equations of equilibrium.
It is seen from the assumptions that normal stresses are absent between the longitudinal fibres of the shaft and in the longitudinal direction of the fibres.
As $\epsilon_x, \epsilon_y, \epsilon_z$ and γ_{xy} vanish there will not be any distortions in the planes of the c/s. At each point, pure shears depicted by the components τ_{xz} and τ_{yz} acts. Next is determination of the warping function $\varphi(x,y)$ of the c/s such that equilibrium equations are satisfied. Neglecting body forces and substituting equation (4) in equilibrium equations as given below

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + f_z = 0 \end{aligned}\right\} (5)$$

$f_x = f_y = f_z = 0$ body forces are neglected.
First two equations are satisfied identically

$$\frac{\partial \tau_{xz}}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \tau_{zy}}{\partial z} = 0$$ (first two equations of equilibrium) → (5a)

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0$$ (from third equation of equilibrium)

substitute eqn (4) then

$$G\theta\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \nabla^2 \varphi = 0 \quad \text{— 5(b)}$$

This is called as Laplace equation.

Warping function φ is harmonic and satisfies Laplace equation.

Differentiate eqn (4) w.r.t x and y

$$\frac{\partial \tau_{yz}}{\partial x} = G\theta \quad \text{and} \quad \frac{\partial \tau_{xz}}{\partial y} = -G\theta$$

$$\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -G\theta - G\theta = -2G\theta$$

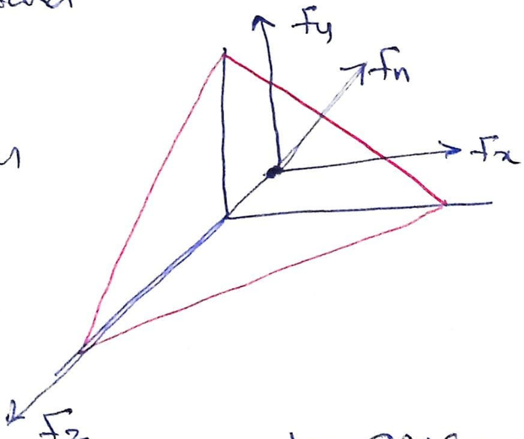
$$\boxed{\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{zy}}{\partial x} = -2G\theta} \quad \text{— 5(c)}$$

Also known as "Poisson's equation".

Therefore stress in a bar of arbitrary section may be determined by solving eqn (a) and (b) along with boundary conditions.

NB:- Harmonic function is the function of two variables having value at any point is equal to the average of its values at any point along the circle around that point.

Boundary conditions:- If F_x, F_y and F_z are the components of the stress on a plane with outward normal $F_n (n_x, n_y, n_z)$ at a point on the surface.



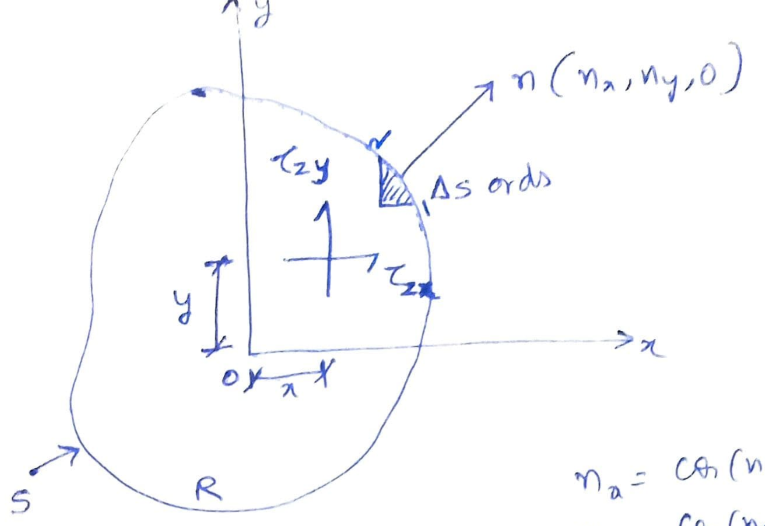
F_x, F_y, F_z are the distributed force per unit area (traction force) under which body deforms. These are components of surface forces per unit area.

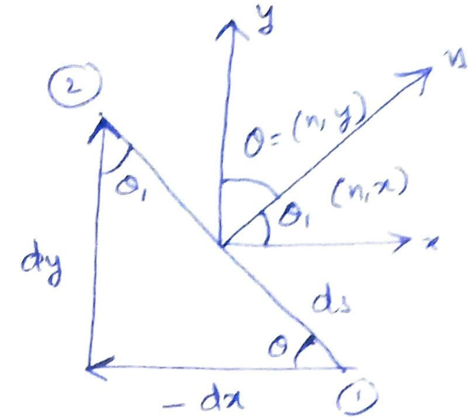
τ_{zx} and τ_{zy} are the shear stresses over triangular area.

n_x, n_y, and n_z are direction cosines.

$$\left.\begin{aligned} n_x \sigma_x + n_y \tau_{xy} + n_z \tau_{xz} &= F_x \\ n_x \tau_{xy} + n_y \sigma_y + n_z \tau_{yz} &= F_y \\ n_x \tau_{xz} + n_y \tau_{yz} + n_z \sigma_z &= F_z \end{aligned}\right\}$$

Boundary conditions or Cauchy's stress formulae





$n_x = \cos(n,x)$

$n_y = \cos(n,y)$

Point moves from ① to ② so x is -ve and y is +ve.

In this case there are no external forces acting on the boundary and normal n to the surface is perpendicular to z axis $n_z = 0$.

From 5(a) and eqn (4)

$$a\theta\left(\frac{\partial\varphi}{\partial x} - y\right)n_x + a\theta\left(\frac{\partial\varphi}{\partial y}+x\right)n_y = 0$$

$n_x = \cos(n,x) = \frac{dy}{ds}$ (θ_1) $n_y = \cos(n,y) = -\frac{dx}{ds}$ (θ)

The boundary condition to be satisfied is

$$\left(\frac{\partial\varphi}{\partial x}-y\right)\frac{dy}{ds} - \left(\frac{\partial\varphi}{\partial y}+x\right)\frac{dx}{ds} = 0 \quad (6)$$

Therefore each problem of torsion is reduced to the problem of finding a function φ which is harmonic i.e. satisfies eqn 5(b) within the body and eqn (6) on the boundary.

Expression for Torque:-

Shear stresses τ_{yz} and τ_{zx} causes torque. The resultant forces in x, y directions should vanish. The resultant force in x direction is

$$\iint \tau_{zx}\,dx\,dy = a\theta\iint\left(\frac{\partial\varphi}{\partial x}-y\right)dx\,dy = 0 \quad (\Sigma H = 0)$$

Similarly $\iint \tau_{yz}\, dx\,dy = 0$ $(\Sigma H = 0)$

$\Sigma M = 0$, Torque 'T' required to give twist 'θ' is

$$T = \iint (\tau_{yz}\, x - \tau_{zx}\, y)\, dx\,dy$$

$$T = G\theta \iint \left(x^2 + y^2 + x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) dx\,dy$$

Let $J = \iint \left(x^2 + y^2 + x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) dx\,dy$

$$\boxed{T = GJ\theta}$$

Torque 'T' is proportional to the angle of twist per unit length with a proportionality constant GJ (called torsional rigidity of the shaft).

For circular section J reduces to I (polar moment of inertia) for non-circular shafts, the product GJ is called as torsional rigidity.

Prandtl's torsion stress function Method

(Alternative approach to find torsion in prismatic bars)

Prandtl's approach leads to simpler boundary condition as compared to eqn (6). In this method the principal unknowns are the stress components rather than the displacement components as in St. venant's approach.

Based on the result of the torsion of the circular shaft, let the non-vanishing stress components be τ_{zx} and τ_{yz}. The remaining stress components are $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$

From eqn (5) (a)

$$\frac{\partial \tau_{zx}}{\partial z} = 0 \;;\quad \frac{\partial \tau_{yz}}{\partial z} = 0 \;;\quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad \text{— (1)}$$

τ_{xz} and τ_{yz} are independent of z. so first two equations are zero's.

If it is assumed that in case of pure torsion (no warping) the stresses are same in every c/s ie independent of z, then first two conditions of the above are automatically satisfied.

In order to satisfy the third condition, we assume a function $\phi(x,y)$ called the stress function such that

$$\tau_{zx} = \frac{\partial \phi}{\partial y} \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x} \quad \text{— (2)}$$

Assuming this stress function also called as Prandtl torsion stress function, the third condition is also satisfied. The assumed stress components have to satisfy the compatibility conditions if they are to be proper elasticity solutions so substitute these stress component into the equations of compatibility. alternatively we can determine the strains corresponding to the assumed stresses and then apply the strain compatibility conditions.

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0 \qquad \left.\begin{aligned} \gamma_{yz} &= \frac{1}{G}\tau_{yz} \\ \gamma_{zx} &= \frac{1}{G}\tau_{zx} \end{aligned}\right\} \text{— (3)}$$

$$\gamma_{yz} = -\frac{1}{G}\frac{\partial \phi}{\partial x}$$

$$\gamma_{zx} = \frac{1}{G}\frac{\partial \phi}{\partial y}$$ — (4)

Strain compatibility relations are - (six)

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \epsilon_z}{\partial z^2} + \frac{\partial^2 \epsilon_x}{\partial x^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \epsilon_z}{\partial x \partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) = 2\frac{\partial^2 \epsilon_x}{\partial y \partial z}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \epsilon_y}{\partial x \partial z}$$ — (5)

Strain compatibility equations are also called as "St. Venant's equations of compatibility".

Substitute equation (3) in eqn (5)

First three equations of strain compatibility equations will be zero and fourth one will also be zero because ϕ is independent of z. Thus fifth and sixth equation of (5) becomes

$$\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) = 0$$ — (6)

substitute eqn (4) in eqn (6)

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0 \;;\quad \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0$$

Integrate on both sides

Hence, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi$ = a constant 'F' —— (7)

This is known as poisson's equation. The stress function 'ϕ' should satisfy it. Constant F is unknown.

Boundary conditions:-

If F_x, F_y and F_z are the components of stress on a plane at a point on the surface with outward normal n (n_x, n_y, n_z) (pg 115)

$$n_x \sigma_x + n_y \tau_{xy} + n_z \tau_{xz} = f_x$$
$$n_x \tau_{xy} + n_y \sigma_y + n_z \tau_{yz} = f_y$$
$$n_x \tau_{xz} + n_y \tau_{yz} + n_z \sigma_z = f_z$$

substitute (4) in BCs

The first two equations are identically satisfied

$n_z \frac{\partial \phi}{\partial y} = 0$, $n_z \frac{\partial \phi}{\partial x} = 0$ } identically satisfied

since $n_z = 0$ both are satisfied

The third equation gives

$$n_x \frac{\partial \phi}{\partial y} - n_y \frac{\partial \phi}{\partial x} = 0$$

$n_x = \frac{dy}{ds}$, $n_y = -\frac{dx}{ds}$ (pg 116)

$$\frac{\partial \phi}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \cdot \frac{dx}{ds} = 0$$

ie, $\frac{d\phi}{ds} = 0$ —— (8)

ϕ = constant 'c'

Therefore ϕ is constant around the boundary. —— (9)

$\phi = 0$ on s (for simple shapes) (with no holes)

Since τ_{yz} and τ_{zx} causes Torque. The resultants in x and y directions should vanish so that moment of the shear stresses τ_{yz} and τ_{zx} about 'O' causes Torque 'T'.

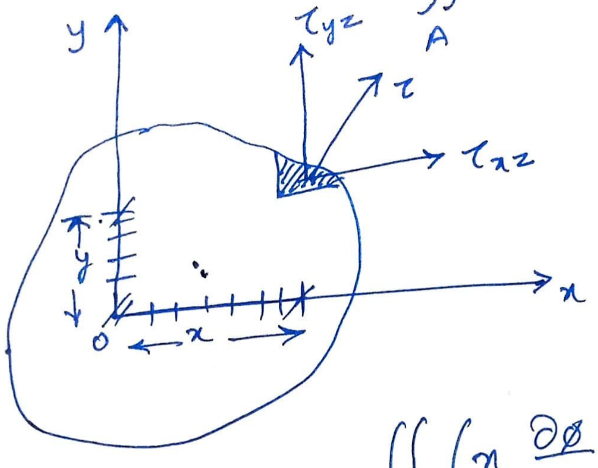
$$\iint \tau_{zx} \, dx dy = 0 \; ; \; \iint \tau_{yz} \, dx dy = 0$$

Since ϕ is constant around the boundary.

$\phi = 0$ on surface 's' because the resultant of forces distributed over the ends is zero and these forces represent a couple 'T'.

Expression for Torque 'T':-

Applied torque $T = \iint_A \tau_{yz}\, x\, dA - \tau_{xz}\, y\, dA$ $\qquad dA = dx\,dy$



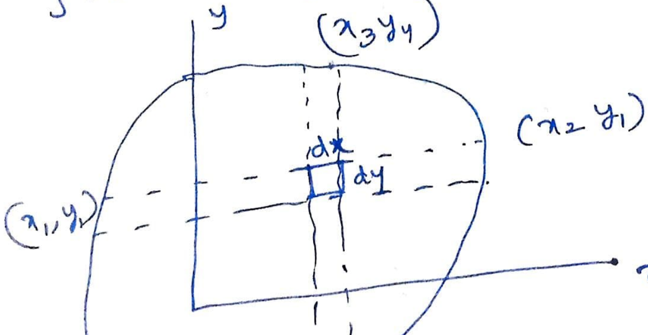
$$T = -\iint_A \left(x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right) dx\,dy$$

$$= -\iint_A x \frac{\partial \phi}{\partial x} dx\,dy - \iint_A y \frac{\partial \phi}{\partial y} dx\,dy$$

$$= -\int dy \int x \frac{\partial \phi}{\partial x} dx - \int dx \int y \frac{\partial \phi}{\partial y} dy$$

$$= -\int dy \left[x\phi - \int \phi\, dx \right]_{x_1}^{x_2} - \int dx \left[y\phi - \int \phi\, dy \right]_{y_3}^{y_4}$$

$$\int u v\, dx = u \int v\, dx - \int u' \left(\int v\, dx \right) dx$$



$(x_1 y_1)$ $(x_2 y_1)$ $(x_3 y_3)$ $(x_3 y_4)$ are the points on the boundary.

$$T = -\int dy \left[x_2\phi_2 - x_1\phi_1 - \int_{x_1}^{x_2} \phi\, dx \right]$$

$$= -\int dx \left[y_4\phi_4 - y_3\phi_3 - \int_{y_3}^{y_4} \phi\, dy \right]$$

$$\boxed{T = 2 \iint_A \phi\, dx\, dy} \quad \text{——— (10)}$$

Since $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$ on the boundary.

Hence the half of the torque is due to the stress component τ_{xz} and the other half due to τ_{yz}. We see that all the equations of elasticity are satisfied and the solution obtained in this manner is the exact solution of the torsion problem. The exact solution is satisfied if the stress all differential equations and BC's are satisfied if the stress function ϕ obeys eqn (7) to (10).

constant 'β' can be determined as follows by equation (5)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial \tau_{zx}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x}$$

$$= G\left(\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right)$$

$$= G\left(\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)\right)$$

$$= G\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$$= G\frac{\partial}{\partial z}\left(-2\omega_z\right)$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

$$\tau_{zx} = G\gamma_{zx}, \quad \tau_{yz} = G\gamma_{yz}$$

$$\gamma_{zx} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

ω_z is the rotation of the element at (x, y) about the z-axis.

$\partial/\partial z\,(\omega_z)$ is the rotation per unit length (i.e., twist θ)

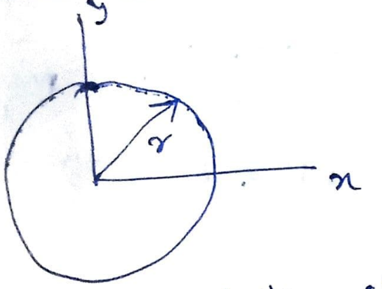
$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \nabla^2\phi = -2G\theta} \quad \text{——— (11)}$$

Shear stress acting in the x-direction is equal to the slope of the stress function $\phi(x,y)$ in the y-direction. The shear stress acting in the y-direction is equal to the negative of the slope of the stress function in the x-direction.
Shear stress in any direction at a point is given by the magnitude of slope (x,y) measured normal to the tangent line i.e normal to the contour line at the concerned point.

$$\tau_{zs} = -\frac{\partial \phi}{\partial n}$$

TORSION OF DIFFERENT CROSS-SECTIONAL BARS

TORSION OF CIRCULAR CROSS-SECTION :-



The boundary of the circular c/s is given by the equation

$$x^2+y^2=r^2$$

The poisson's equation and the boundary condition are satisfied by taking the stress function in the form.

$$\phi = \rho(x^2+y^2-r^2)$$

poisson's equation $\Rightarrow \nabla^2\phi = -2G\theta$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} \Rightarrow \quad \rho(2+2) = -2G\theta$$

$$\rho = -\frac{G\theta}{2}$$

hence $\phi = -\frac{G\theta}{2}(x^2+y^2-r^2)$ —— (12)

Applied Torque $T = 2\iint \phi\, dx\,dy$

$$= -G\theta \iint (x^2+y^2-r^2)\, dx\,dy$$

$$= -G\theta\left(\iint_A x^2 dA + \iint_A y^2 dA - \iint r^2 dA\right)$$

$r^2 A = r^2 \pi r^2 = \pi r^4$

Areal integral $\int_A$ or $\iint$

$\int_S$ Surface integral

we know that

$$\iint x^2 dA = I_{yy} = \frac{\pi r^4}{4}$$

$$\iint y^2 dA = I_{xx} = \frac{\pi r^4}{A}$$

$$T = -G\theta\left(\frac{\pi r^4}{4}+\frac{\pi r^4}{4}-\pi r^4\right) = G\theta\left(\frac{\pi r^4}{2}\right)$$

where

Polar moment of inertia $J = \frac{\pi r^4}{2}$

so $\boxed{T = G\theta J}$ —— (13)

Shear stresses are τ_{xz} and τ_{yz}

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -G\theta y$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = G\theta x$$

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = G\theta\sqrt{(x^2+y^2)} = G\theta\sqrt{r^2} = G\theta r$$

$$\frac{\tau}{r} = G\theta$$

Coulomb's equation of torsion for circular sections is

θ = angle of twist per unit length

$$\frac{T}{J} = \frac{\tau}{r} = G\theta$$

Warping constant

$$\tau_{xz} = G\theta\left(\frac{\partial \varphi}{\partial x} - y\right) \quad \text{pg 114}$$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -G\theta y$$

$$\left.\begin{aligned} &\text{where } \phi = e\,(x^2+y^2-r^2) \\ &e = \frac{-G\theta}{2} \end{aligned}\right\} 6$$

$$-G\theta y = G\theta\left(\frac{\partial \varphi}{\partial x} - y\right)$$

$$\frac{\partial \varphi}{\partial x} - y = -y \Rightarrow \frac{\partial \varphi}{\partial x} = 0 \Rightarrow \varphi = C$$

for circular cross section, $\varphi = 0$; the only c/s with zero warping.

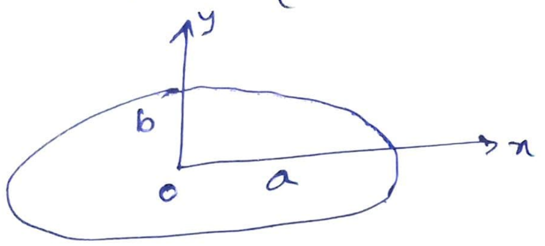
TORSION OF ELLIPTICAL CROSS-SECTION

The boundary of the elliptical c/s is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The poisson's equation and the boundary conditions are satisfied by taking the stress function in the form

$$\phi = e\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$



$$\nabla^2 \phi = -2G\theta$$

$$e\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = -G\theta$$

$$e = -\frac{G\theta a^2 b^2}{a^2+b^2}$$

Hence $\phi = -\frac{G\theta a^2 b^2}{a^2+b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

Torque applied $T = 2\iint \phi \, dx\, dy$

$$= -2G\theta \iint \frac{a^2 b^2}{a^2+b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) dx\, dy$$

$$= -2G\theta \frac{a^2 b^2}{a^2+b^2}\left(\iint \frac{x^2}{a^2} dA + \iint \frac{y^2}{b^2} dA - \iint dA\right)$$

$$\iint x^2 dA = I_{yy} = \frac{\pi a^3 b}{4}$$

$$\iint y^2 dA = I_{xx} = \frac{\pi b^3 a}{4}$$

$$\iint dA = A = \pi ab$$

Polar moment of inertia $J = \dfrac{\pi a^3 b^3}{a^2+b^2}$

$$T = -2G\theta \frac{a^2b^2}{a^2+b^2}\left(\frac{\pi ab}{4} + \frac{\pi ab}{4} - \pi ab\right)$$

$$= 2G\theta \frac{a^2b^2}{a^2+b^2}\left(\frac{\pi ab}{2}\right)$$

$$T = \frac{G\theta\pi a^3b^3}{a^2+b^2} = G\theta J \Rightarrow \boxed{T = G\theta J}$$

Shear stresses :-

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = -\frac{2G\theta a^2 y}{a^2+b^2}$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = \frac{2G\theta b^2 x}{a^2+b^2}$$

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \frac{2G\theta}{a^2+b^2}\sqrt{a^4y^2 + b^4x^2}$$

Maximum shear stress occurs at $(0, b)$

$$\tau_{max} = \frac{2G\theta a^2 b}{a^2+b^2}$$

Relation between τ_{max} and T is

$$\tau_{max} = \frac{2Ta^2b}{J(a^2+b^2)} = \frac{2T}{\pi ab^2}$$

Warping constant

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = -\frac{2G\theta a^2 y}{a^2+b^2}$$

$$\tau_{xz} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right)$$ pg 114

$$G\theta\left(\frac{\partial\psi}{\partial x} - y\right) = \frac{-2G\theta a^2 y}{(a^2+b^2)}$$

$$\frac{\partial\psi}{\partial x} = \frac{-2a^2y}{a^2+b^2} + y = \left(\frac{b^2-a^2}{a^2+b^2}\right)y$$

$$\varphi = \left(\frac{b^2 - a^2}{a^2 + b^2}\right) xy + c$$

At the centre, warping is zero i.e @ $x = y = 0$, $\varphi = 0$

So $c = 0$

$$\varphi = \left(\frac{b^2 - a^2}{a^2 + b^2}\right) xy$$

We know that

$$u = -\theta z y$$

$$v = -\theta z x$$

$$w = \theta_z \varphi(x, y)$$

$$w = \theta_z \left(\frac{b^2 - a^2}{a^2 + b^2}\right) xy$$

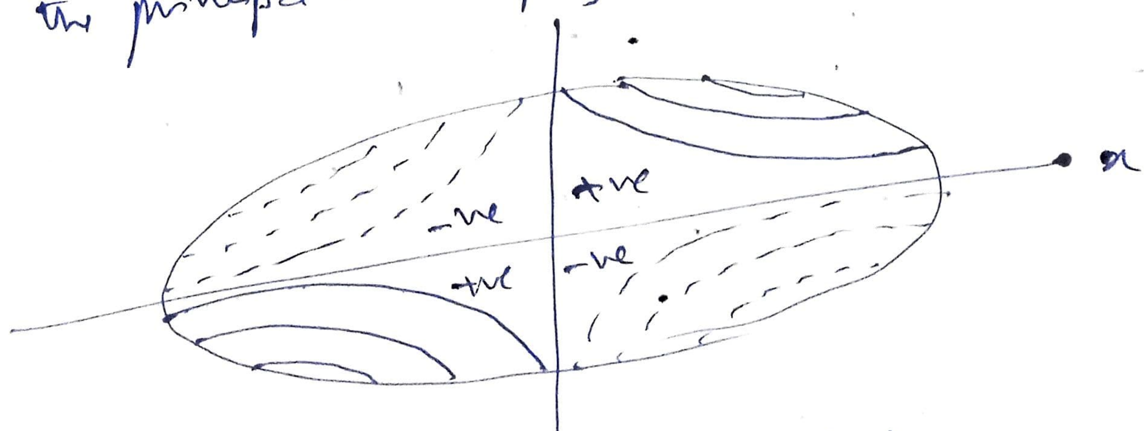
θ_z is the angle rotated during torsion.
θ_z is the angle of twist

$$\theta = \frac{T}{GJ} \Rightarrow \theta_z = \frac{T(a^2 + b^2)}{G\pi a^3 b^3}$$

$$w = \text{warping} = \frac{T(a^2 + b^2)}{G\pi a^3 b^3} \left(\frac{b^2 - a^2}{a^2 + b^2}\right) xy$$

$$w = \frac{T(b^2 - a^2)}{G\pi a^3 b^3} xy$$

Hence the contour lines for the warped c/s are hyperbolas having the principal axes of the ellipse as asymptote.



contour lines for warped elliptical c/s.

Example of the warping function $\varphi = 20(y^3 - 3x^2y)$ for a non-circular section under torsion determine τ_{xz} at the point (10, -5)

$$\tau_{xz} = G\theta\left(\frac{\partial\varphi}{\partial x} - y\right)$$

$$= G\theta\left(20(-6xy) - y\right)$$

$$\tau_{xz} = 6005\,G\theta$$

Example :- Det. the shear stress and the angle of twist in an elliptical bar at the point (10, 20) where semi-axes are (100, 0) mm and (0, 50) mm respectively. Given T = 10 kNm and G = 70 kN/mm².

Angle of twist per unit length $\theta = \frac{T}{GJ}$

Polar M.I $\quad J = \frac{\pi a^3 b^3}{a^2 + b^2} = \frac{\pi \times 100^3 \times 50^3}{100^2 + 50^2} = 31415926 \text{ mm}^4$

Angle of twist per unit length $\theta = \frac{10 \times 10^6}{70 \times 10^3 \times 31415926} = 4.55 \times 10^{-6}$ rad

For elliptical c/s $\quad \varphi = \left(\frac{b^2 - a^2}{b^2 + a^2}\right) xy$

$$\varphi = \left(\frac{50^2 - 100^2}{100^2 + 50^2}\right) xy = -0.6xy$$

$$\tau_{xz} = G\theta\left(\frac{\partial\varphi}{\partial x} - y\right) = -1.6y\,G\theta = -1.6 \times 20 \times 70 \times 10^3 \times 4.55 \times 10^{-6}$$

$$= -10.19 \text{ MPa}$$

$$\tau_{yz} = G\theta\left(\frac{\partial\varphi}{\partial y} + x\right) = 0.4\,G\theta = 0.4 \times 10 \times 70 \times 10^3 \times 4.55 \times 10^{-6}$$

$$= 1.27 \text{ MPa}$$

$$\tau_{max} = \frac{2T}{\pi a b^2} = 25.46 \text{ MPa}$$

Bending of Prismatic Bars

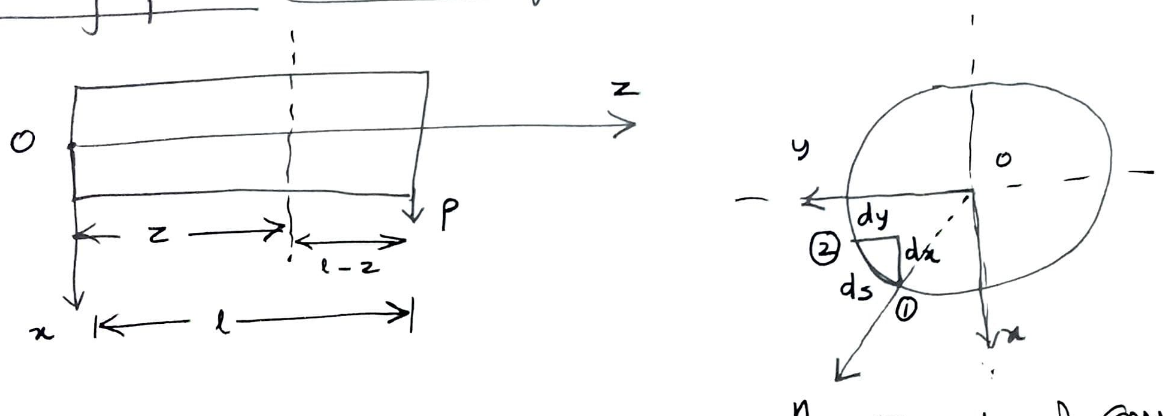
(Prismatic bar → having same c/s area along the length)

The problems of bending of a prismatic bar of symmetrical c/s under action of bending moments only can be solved using SOM principles using well known formulae.

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R} \quad \text{and} \quad \frac{d^2y}{dx^2} = -M \quad \text{— (1)}$$

equation (1) is not applicable for prismatic bars of general c/s.

Bending of a Cantilever (unsymmetrical c/s) by terminal load



Consider the bending of a cantilever of constant c/s of any shape by a force 'P' applied at the end and parallel to one of the principal axes of the c/s as shown in fig.

z axis is along the length of the bar through the centre while the x axis and y axis are orthogonal axes at the centroid of the end z = 0. The lateral surface of the bar is free from external forces and the body forces are assumed to vanish.

$\sigma_x = \sigma_y = \tau_{xy} = 0$ no external forces or stresses in x, y direction.

The stresses σ_z, τ_{zx}, τ_{zy} will be chosen in such a way that equations of equilibrium, compatibility and Boundary conditions are satisfied. We assume that normal stresses over a c/s at a distance 'z' from the fixed end are distributed in the same manner as in the case of pure bending

$$\frac{M}{I} = \frac{f}{y} \Rightarrow \sigma_z = \frac{-P(l-z)x}{I} \quad \text{— (2)}$$

$\because M = -P(l-z)$; $y = x$

$\sigma_z = \dfrac{-P(l-z)x}{I}$ —— (2)

hogging so -ve

$\curvearrowleft (-) \curvearrowright$
M M

$\sigma_x = \sigma_y = \tau_{xy} = 0$ and body forces are neglected.

With these assumptions, the equations of equilibrium becomes

$$\frac{\partial \tau_{zx}}{\partial z} = 0 , \quad \frac{\partial \tau_{yz}}{\partial z} = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{Px}{I} \quad \text{—— (3)}$$

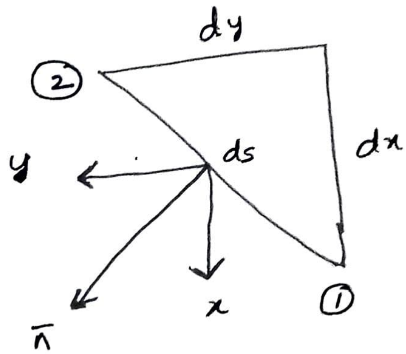
since $\dfrac{\partial \sigma_z}{\partial z} = +\dfrac{Px}{I}$

from eqn (3) we conclude that shearing stresses do not depend on 'z' and are same in all c/s of the bar.

The boundary conditions are

$$\sigma_x l + \tau_{xy} m + \tau_{xz} n = 0$$
$$\tau_{xy} l + \sigma_y m + \tau_{yz} n = 0$$
$$\tau_{zx} l + \tau_{yz} m + \sigma_z n = 0$$

($n \to 0$)



The first two equations are identically satisfied and third equation gives

$$\tau_{zx} l + \tau_{yz} m = 0$$

$$\tau_{zx} \frac{dy}{ds} - \tau_{yz} \frac{dx}{ds} = 0 \quad \text{—— (4)}$$

$l = \cos(\bar{n}, x) = \dfrac{dy}{ds}$

$m = \cos(\bar{n}, y) = -\dfrac{dx}{ds}$

$n = \cos(\bar{n}, z) = 0$ (90°)

ds is an element of the boundary curve of the c/s

Now the Beltrami - Michell equations (compatibility equations) eqn (5) reduced to

$$\nabla^2 \tau_{yz} = 0$$
$$\nabla^2 \tau_{zx} = -\frac{P}{I(1+\nu)} \quad \text{—— (5)}$$

$\sigma_x = \sigma_y = \tau_{xy} = 0$

$\sigma_z = \dfrac{-P(l-z)x}{I}$

Thus the solution of the problem of bending of a prismatic cantilever of any c/s reduces to finding for τ_{zx} and τ_{yz} functions of x and y that satisfy the equation of equilibrium eqn (3) the boundary condition equ (4) and equ (5)

By taking

$$\left.\begin{aligned} \tau_{zx} &= \frac{\partial \phi}{\partial y} - \frac{P x^2}{2I} + f(y) \\ \tau_{yz} &= -\frac{\partial \phi}{\partial x} \end{aligned}\right\} \quad (6)$$

→ function of y only

The equ (6) satisfies equilibrium equations. ϕ is the stress function of x and y and f(y) is a function of y only to be determined from the boundary conditions.

Substitute equation (6) in equ (5) we find

$$\nabla^2 \tau_{yz} = 0$$

$$\frac{\partial^2 (\tau_{yz})}{\partial x^2} + \frac{\partial^2 (\tau_{yz})}{\partial y^2} + \frac{\partial^2}{\partial z^2}(\tau_{yz}) = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This part becomes zero because ϕ is the function of x and y.

$$\frac{\partial^2}{\partial z^2}\left(-\frac{\partial \phi}{\partial x}\right) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial z^2}\right) = 0$$

Since $\tau_{yz} = -\frac{\partial \phi}{\partial x}$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0 \quad (7)$$

Similarly

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = \left(\frac{\nu}{1+\nu}\right)\frac{P}{I} - \frac{d^2 f}{dy^2} \quad (8)$$

integrating the equ (8) we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left(\frac{\nu}{1+\nu}\right)\frac{Py}{I} - \left(\frac{df}{dy}\right) + C \quad (9)$$

C is constant of integration.

Consider the rotation of an element of area in the plane of a cross-section of the cantilever given by

or

rotation in x-y plane → $$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \quad (10)$$

about z axis.

ω_z is the rotation of element in x-y plane about z axis. the local twist at a point (x, y) of a c/s is defined as

Twist per unit length

$$\frac{\partial \omega_z}{\partial z} = \frac{1}{2}\left(\frac{\partial^2 u_y}{\partial z \partial x} - \frac{\partial^2 u_x}{\partial z \partial y}\right)$$

$$= \frac{1}{2}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right)$$

$$= \frac{1}{2G}\left(\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{zx}}{\partial y}\right) \quad \text{— (11)}$$

$G = \frac{\tau}{\gamma}$

$\gamma_{yz} = \frac{\partial u_y}{\partial z}$

$\gamma_{zx} = \frac{\partial u_x}{\partial z}$

Substitute eqn (6) in (11)

$$\frac{\partial \omega_z}{\partial z} = -\frac{1}{2G}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{df}{dy}\right) \quad \text{— (12)}$$

Using eqn (9) and eqn (12) we get

$$-2G\frac{\partial \omega_z}{\partial z} = \left(\frac{\nu}{1+\nu}\right)\frac{Py}{I} + C \quad \text{— (13)}$$

Term represents twist θ of c/s; represents bending; C represents torsion

θ = C @ centre 'O' @ y = 0

The mean value of the local twist over the section or at the centroid of the section is equal to the constant 'C'. If the x axis is an axis of symmetry, bending by a force 'P' in this axis will result in a symmetrical pattern of rotation 'ω_z' of elements of the c/s, with a mean value of zero over the whole c/s. The mean value of $\frac{\partial \omega_z}{\partial z}$ will thus also be zero and thus C will also be zero in eqn (13). If the c/s is not symmetrical then again defining bending without torsion. (only bending), the value of C will be zero.

Hence eqn (9) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left(\frac{\nu}{1+\nu}\right)\frac{Py}{I} - \frac{df}{dy} \quad \text{— (14)}$$

Substituting eqn (6) in eqn (4) we get

$$\frac{\partial \phi}{\partial y}\left(\frac{dy}{ds}\right) + \frac{\partial \phi}{\partial x}\left(\frac{dx}{ds}\right) = \frac{d\phi}{ds} = \left[\frac{Px^2}{2I} - f(y)\right]\frac{dy}{ds} \quad \text{— (15)}$$

From equ (15) the value of the function ϕ along the boundary of the c/s can be calculated if the function f(y) is chosen. Equ (14) together with equ (15) determines the stress function 'ϕ'. Generally the function f(y) is chosen in such a manner that the right hand of equ (14) becomes zero. Then ϕ becomes constant along the boundary.

Case 1:- Bending of a bar of circular cross-section

Let the boundary of the c/s of the bar is given by

$$x^2+y^2=r^2 \qquad \text{--- (16)}$$

The right hand of boundary condition equ (15) becomes zero

if $f(y) = \frac{P}{2I}x^2 = \frac{P(r^2-y^2)}{2I}$ --- (17) $\quad \frac{d\phi}{ds}=0$ because ϕ is constant on the boundary

Substituting equ (17) in equ (14) we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = \left(\frac{1+2\nu}{1+\nu}\right)\frac{Py}{I}$$

or

$$\nabla^2\phi = \left(\frac{1+2\nu}{1+\nu}\right)\frac{Py}{I} \qquad \text{--- (18)}$$

$$\frac{\partial f}{\partial y} = \frac{P(-2y)}{2I} = -\frac{Py}{I}$$

$$\left(\frac{\nu}{1+\nu}+1\right) = \left(\frac{1+2\nu}{1+\nu}\right)$$

The boundary condition and equ (18) are satisfied by taking

$$\phi = m(x^2+y^2-r^2)y \qquad \text{--- (19)}$$

where m is a constant factor

Equ (18) is satisfied if we take

$$m = \frac{(1+2\nu)}{8(1+\nu)}\left(\frac{P}{I}\right)$$

Equ (19) then becomes

$$\phi = \frac{(1+2\nu)}{8(1+\nu)}\left(\frac{P}{I}\right)(x^2+y^2-r^2)y \qquad \text{--- (20)}$$

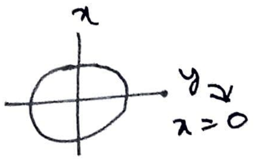
The stress components from eqn (6) becomes

$$\tau_{xz} = \frac{(3+2\nu)}{8(1+\nu)}\left(\frac{P}{I}\right)\left(r^2 - x^2 - \frac{(1-2\nu)}{(3+2\nu)}y^2\right)$$

$$\tau_{yz} = \frac{-(1+2\nu)}{4(1+\nu)}\left(\frac{Pxy}{I}\right)$$ — (21)

$$\tau_{zx} = \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + f(y)$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x}$$



$$f(y) = \frac{P(r^2 - y^2)}{2I}$$

Along the horizontal diameter of the c/s,
$x = 0$ (on y axis)

From eqn (21) we get

$$\tau_{xz} = \frac{(3+2\nu)}{8(1+\nu)}\left(\frac{P}{I}\right)\left(r^2 - \frac{(1-2\nu)}{(3+2\nu)}y^2\right)$$ — (22)

$$\tau_{yz} = 0$$

$$(\tau_{xz})_{y=0} = (\tau_{xz})_{max} = \frac{(3+2\nu)}{8(1+\nu)}\left(\frac{P}{I}\right)$$ — (23)

$$(\tau_{xz})_{y=\pm r} = \frac{(1+2\nu)}{4(1+\nu)}\frac{Pr^2}{I}$$ — (24)

From the elementary beam theory, based on the assumptions that the shearing stress τ_{xz} is uniformly distributed along the horizontal diameter of the c/s is:

$$\tau_{xz} = \frac{4}{3}\left(\frac{P}{A}\right)$$ — (25)

A is area of c/s of the bar

Take $\nu = 0.3$

$$I = \frac{\pi d^4}{64}$$

$$(\tau_{xz})_{max} \text{ @ } y=0 = 1.38\frac{P}{A}$$

$$(\tau_{xz})_{y=\pm r} = 1.23\frac{P}{A}$$

Case 2: Bending of a Prismatic bar of elliptic cross-section

let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ——— (26)

be the boundary of the C/s. The right hand side of eqn (15) will vanish if we take

$$f(y) = -\frac{P}{2I}\left(\frac{a^2}{b^2}y^2 - a^2\right) \quad \text{——— (27)}$$

substituting eqn (27) in eqn (14) we get

$$\nabla^2\phi = \frac{Py}{I}\left(\frac{a^2}{b^2} + \left(\frac{\nu}{1+\nu}\right)\right) \quad \text{——— (28)}$$

The boundary condition and eqn (28) are satisfied by taking

$$\phi = \frac{(1+\nu)a^2 + \nu b^2}{2(1+\nu)(3a^2+b^2)}\left(\frac{P}{I}\right)\left(x^2 + \frac{a^2}{b^2}y^2 - a^2\right)y \quad \text{——— (29)}$$

The stress components from eqn (6) becomes

$$\tau_{xz} = \frac{2(1+\nu)a^2 + b^2}{(1+\nu)(3a^2+b^2)}\left(\frac{P}{2I}\right)\left[a^2 - x^2 - \frac{(1-2\nu)a^2}{2(1+\nu)a^2+b^2}y^2\right] \quad \text{——— (30)}$$

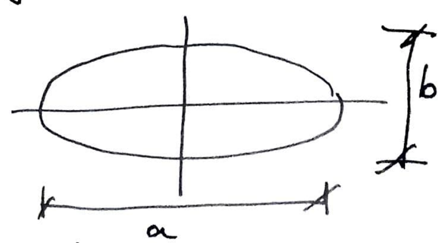
$$\tau_{yz} = \frac{-(1+\nu)a^2 + \nu b^2}{(1+\nu)(3a^2+b^2)}\left(\frac{Pxy}{I}\right) \quad \text{——— (31)}$$

$$(\tau_{xz})_{x=0} = \frac{2(1+\nu)a^2+b^2}{(1+\nu)(3a^2+b^2)}\left(\frac{P}{2I}\right)\left[a^2 - \frac{(1-2\nu)a^2}{2(1+\nu)a^2+b^2}y^2\right] \quad \text{——— (32)}$$

$$(\tau_{yz})_{x=0} = 0 \quad \text{——— (33)}$$

if $b \ll a$ then neglecting the $\frac{b^2}{a^2}$ term we get

$$(\tau_{xz})_{max} = \frac{Pa^2}{3I} = \frac{4P}{3A} \quad \text{——— (34)}$$



which coincides with the solution of elementary beam theory.

If $b >> a$ then

$$(\tau_{xz})_{max} = \left(\frac{2}{1+\nu}\right)\frac{P}{A}$$

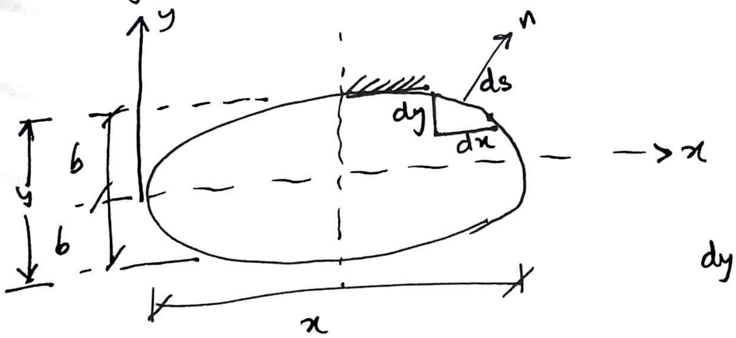
also $$(\tau_{xz})_{y=\pm b} = \left(\frac{4\nu}{1+\nu}\right)\frac{P}{A}$$ — (35)

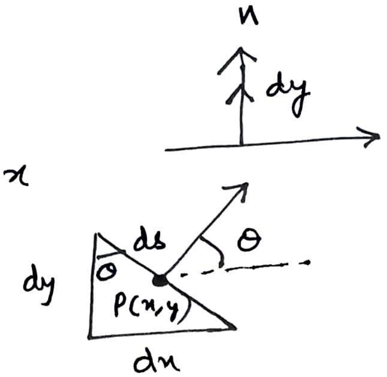
Case 3:- Bending of a Prismatic bar of Rectangular cross-section

Consider a rectangular bar of width '2b' and thickness '2a'.
the equation for the boundary line is

$$(x^2 - a^2)(y^2 - b^2) = 0 \quad \text{——— (36)}$$

If we substitute $f(y) = \frac{Pa^2}{2I}$ in equ (15) then the right hand side of equ (15) becomes zero along the sides $x = \pm a$ of the rectangle. Along the vertical sides $y = \pm b$ the derivative $\frac{dy}{ds}$ is zero. thus the right hand side of equ (15) is zero along the boundary lines and we can take $\phi = 0$ at the boundary.





$\cos\theta = \frac{dy}{ds}$

$\theta = 90°$ when $y = \pm b$

ϕ = constant along the boundary ——— (37)

Equ (14) becomes $\nabla^2\phi = \left(\frac{\nu}{1+\nu}\right)\frac{Py}{I}$

substitute $\frac{df}{dy} = 0$ in equ (14)

From equ (6) the shearing stresses can be resolved into the following two systems

$$\tau'_{xz} = \frac{P}{2I}(a^2 - x^2), \quad \tau'_{yz} = 0$$

$$\tau''_{xz} = \frac{\partial\phi}{\partial y}, \quad \tau''_{yz} = -\frac{\partial\phi}{\partial x}$$ — (38)

Substitute ~~[illegible]~~ $f(y) = \frac{Pa^2}{2I}$ in eqn (6)

$\frac{d\phi}{dy} = 0 \qquad -\frac{d\phi}{dx} = 0$ because $\phi = 0$

Single primed stresses represents the parabolic stress distributions given by the elementary beam theory.

Double primed stresses represent necessary corrections to the elementary solutions.

The boundary conditions are satisfied by taking the stress function ϕ in the form of double fourier series.

$$\phi = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{2m+1,n} \cos \frac{(2m+1)\pi x}{2a} \cdot \sin \frac{n\pi y}{b} \quad \text{—— (39)}$$

by substituting in eqn (37) the

Calculate the fourier coefficients ~~[illegible]~~ by substituting in eqn (37) the eqn (39) evaluate ϕ from which the shear stresses are calculated by substituting in ~~[illegible]~~

eqn (38).

GR22 2022-23 M.Tech MTECH STE 110, Section: A GR22D5002 Advanced Solid Mechanics Sessional Marks

S.No	Roll No	MID-I Marks	MID-II Marks	Tutorial Marks	Assessment Marks	Sessional Marks
1	22241D2001	25	21	5	5	
2	22241D2002	20	21	5	5	
3	22241D2003	14	7	5	5	
4	22241D2004	29	29	5	5	
5	22241D2005	18	18	5	5	
6	22241D2006	21	25	5	5	
7	22241D2007	25	24	5	5	
8	22241D2008	19	16	5	5	
9	22241D2009	20	17	5	5	
10	22241D2010	16	21	5	5	
11	22241D2011	15	11	5	5	
12	22241D2012	22	22	5	5	
13	22241D2013	21	17	5	5	
14	22241D2014	21	14	5	5	
15	22241D2015	27	20	5	5	
16	22241D2016	30	27	5	5	
17	22241D2017	15	09	5	5	
18	22241D2018	28	27	5	5	
19	22241D2019	14	07	5	5	

Faculty Signature

THEORETICAL CONCEPTS OF PLASTICITY

- Dr V Srinivasa Reddy

Unit 5 contents : Concepts of plasticity, Plastic Deformation, Strain Hardening, Idealized Stress- Strain curve, Yield Criterions, Plastic Stress-Strain Relations.

What causes the failure?

It is known from the results of material testing that when bars of ductile materials are subjected to uniform tension, the stress-strain curves show a linear range within which the materials behave in an elastic manner and a definite yield zone where the materials undergo permanent deformation. In the case of the so-called brittle materials, there is no yield zone. However, a brittle material, under suitable conditions, can be brought to a plastic state before fracture occurs.

It was stated that the state of stress at any point can be characterized by the six rectangular stress components—three normal stresses and three shear stresses. Similarly it was shown that the state of strain at a point can be characterized by the six rectangular strain components

When failure occurs, the question that arises is: what causes the failure? Is it a particular state of stress, or a particular state of strain or some other quantity associated with stress and strain? Further, the cause of failure of a ductile material need not be the same as that for a brittle material.

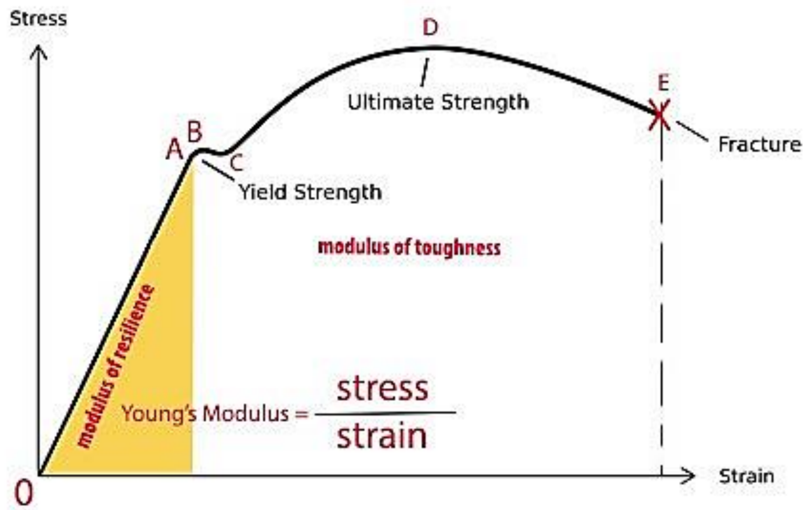
Consider, for example, a uniform rod made of a ductile material subject to tension. When yielding occurs, (i) The principal stress s at a point will have reached a definite value, usually denoted by σ_y; (ii) The maximum shearing stress at the point will have reached a value equal to $\tau = 1/2\sigma_y$; (iii) The principal extension will have become $\varepsilon = \sigma_y/E$; (iv) The octahedral shearing stress will have attained a value equal $\sqrt{2}$ /3) σ_y; and so on.

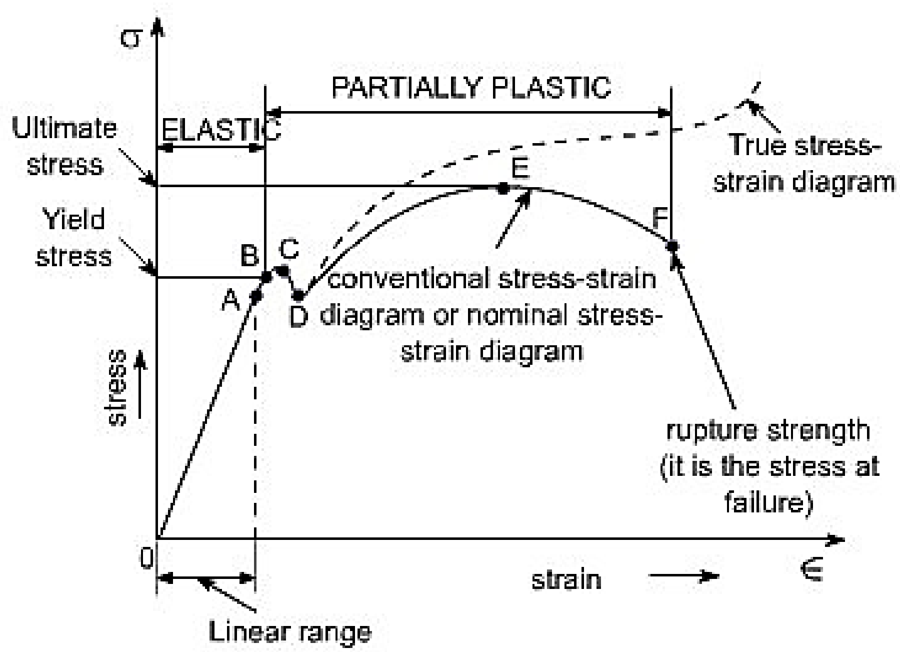
Any one of the above or some other factors might have caused the yielding. Further, as pointed out earlier, the factor that causes a ductile material to yield might be quite different from the factor that causes fracture in a brittle material under the same loading conditions. Consequently, there will be many criteria or theories of failure. It is necessary to remember that failure may mean fracture or yielding. Whatever may be the theory adopted, the information regarding it will have to be obtained from a simple test, like that of a uniaxial tension or a pure torsion test. This is so because the state of stress or strain which causes the failure of the material concerned can easily be calculated. The critical value obtained from this test will have to be applied for the stress or strain at a point in a general machine or a structural member so as not to initiate failure at that point. There are six main theories of failure. Another theory, called Mohr's theory, is a graphical approach.

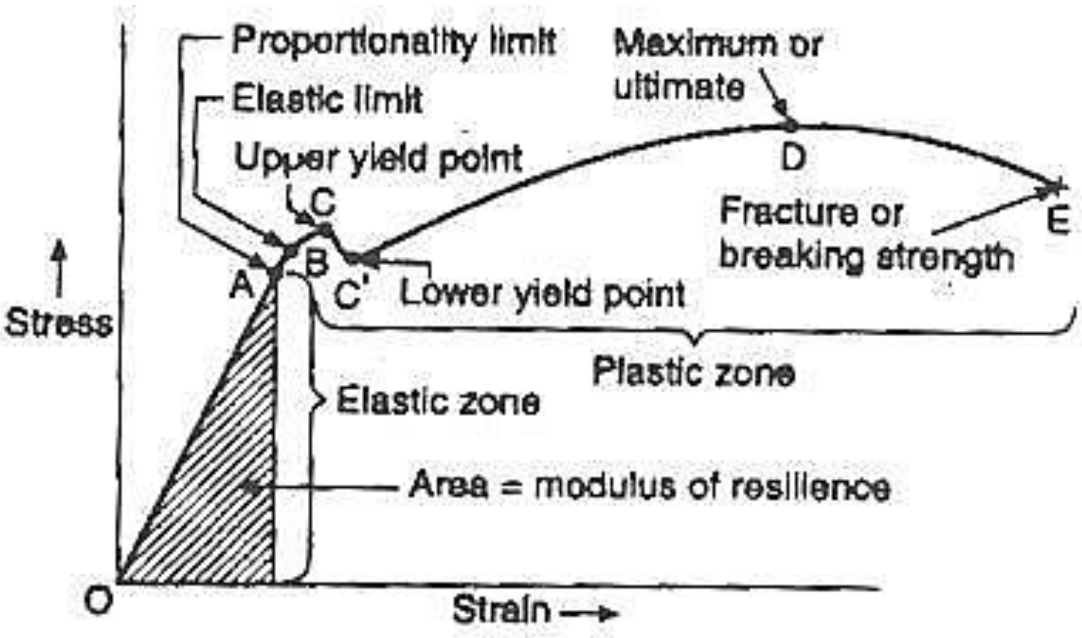
- It is concerned with materials which initially deform elastically, but which deform plastically upon reaching a yield stress.
- In metals and other crystalline materials, the occurrence of plastic deformations at the micro-scale level is due to the motion of dislocations and the migration of grain boundaries on the micro-level.
- There are two broad groups of metal plasticity problem which are of interest to the engineer and analyst.
- The first involves relatively small plastic strains, often of the same order as the elastic strains which occur. Analysis of problems involving small plastic strains allows one to design structures optimally, so that they will not fail when in service, but at the same time are not stronger than they really need to be. In this sense, plasticity is seen as a material failure.

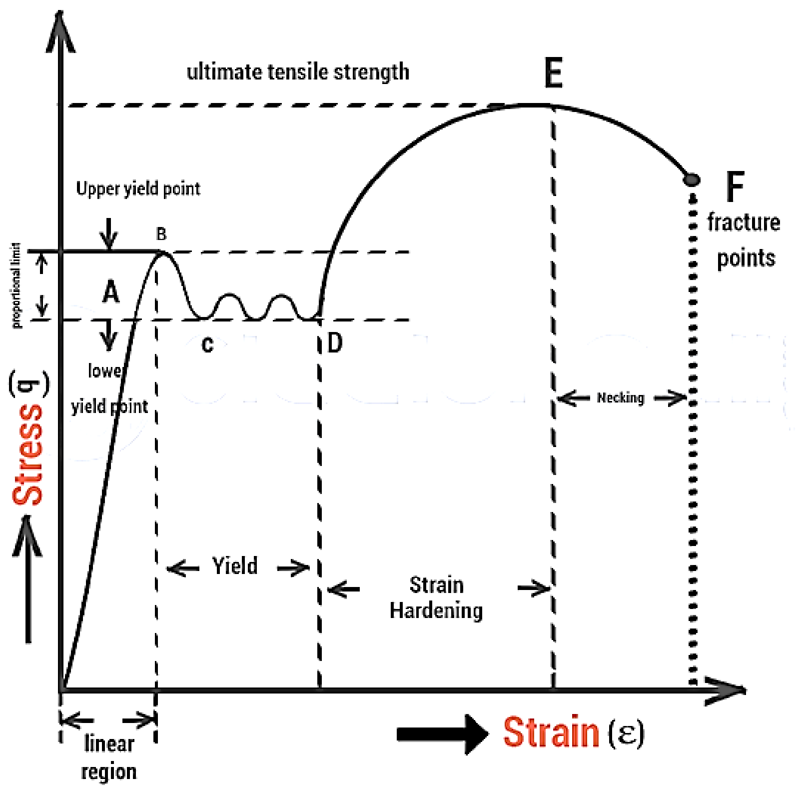
- The second type of problem involves very large strains and deformations, so large that the elastic strains can be disregarded. In these latter-type problems, a simplified model known as perfect plasticity is usually employed.
- Plastic deformations are normally rate independent, that is, the stresses induced are independent of the rate of deformation (or rate of loading).
- Plastic deformation is a non-reversible process where Hooke's law is no longer valid.
- One aspect of plasticity in the viewpoint of structural design is that it is concerned with predicting the maximum load, which can be applied to a body without causing excessive yielding.

Idealized Stress-strain curve









Why is there a dip in the stress strain curve for mild steel after the ultimate point?

Nominal stress – Strain OR Conventional Stress – Strain diagrams:

Stresses are usually computed on the basis of the original area of the specimen; such stresses are often referred to as conventional or nominal stresses.

True stress – Strain Diagram:

Since when a material is subjected to a uniaxial load, some contraction or expansion always takes place. Thus, dividing the applied force by the corresponding actual area of the specimen at the same instant gives the so called true stress.

- The maximum load which the specimen can withstand without failure is called the load at the ultimate strength.

Beyond point E, the cross-sectional area of the specimen begins to reduce rapidly over a relatively small length of bar and the bar is said to form a neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at point F.

In a stress/strain diagram the increase in stress (pressure or load) is assumed to continue at a set rate. Strain (deflection of the material under the stress) increases in a linear relationship until the stress reaches the yield strength of the material and it "gives up". This is the end of "elastic" deflection, where the material would return to its unstressed form when the stress is removed. Beyond that point the strain is "plastic" deflection where the material will remain mostly in the deflected (bent) position. The materials strength to resist the applied load decrease and for the same load material stretches so strain increases without increase in the stress. It loses its strength as there is significant reduction in its cross sectional area.

Why the lower yield point stress value of mild steel is consider as a strength of material instead of upper yield point stress?

As you increase the applied load beyond elastic limit (point B), material starts elongate plastically i.e. it does not regain its original shape after removing the load. Mild steel has dislocations (Dislocations are defects present in crystal areas where atoms are out of position (irregular alignment)) pinned by carbon particles. So as you move further, the energy required to unpin these dislocations increases till Point C which is ‘Upper Yield Point’. As soon as dislocations get free, the stress induced drops to a lower value at Point C’ known as ‘Lower Yield Point’. **When the upper yield point is achieved, dislocations get free causing the stress lower down. This phenomenon is momentary** i.e. UYP is unstable. The lower yield point is more stable as it is the effect of this phenomenon. Hence, we take the Lower Yield Point (point C’) into consideration while designing the components.

- Basically there are three types of failure in case of mechanical component i.e

1) Failure due to elastic deformation
2) Failure due to plastic deformation
3) Failure due to fracture

When component deforms elastically it's dimensions changes and it fails. And this failure is known as failure due to elastic deformation

When component undergoes plastic deformation it's dimension changes permanently and failure takes place this is known as failure due to plastic deformation.

For ductile metals elastic failure is criteria of failure because ductile metals undergo elastic deformation before failure. And elastic deformation starts at lower yield point.

- Plastic deformation is a state in which a material doesn't, take back its original shape or stay deformed. Materials have some elasticity in it so when a stress is applied on it (suppose a tensile stress) it changes its shape know as strain . So in elastic deformation it regains its shape after the applied stress is removed like a rubber but above a certain limit plastic deformation happen and the material stays in deformed state even after removing the source of stress.

What is strain hardening region in stress strain curve? Why it is called so?

When a metal is stressed beyond its elastic limit it enters the plastic region (The region in which residual strain remains upon unloading). When the load is increased further (a kind of rearrangement occurs at atom level and the mobility of the dislocation decreases), 'dislocation density' increases that in turn makes the metal harder and stronger through the resulting plastic deformation.

It means, it's more difficult to deform the metal as the strain increases and hence it's called "strain hardening". This tends to increase the strength of the metal and decrease its ductility.

When you are conducting a tensile test on a material, after the elastic limit the material starts getting plastically deformed. During the plastic deformation, because of the process of dislocations interactions within the material, the tensile strength increases as the material is getting deformed. This increase in the tensile strength of the material continues till it reaches a maximum in the stress ~strain curve.

This increase in the tensile strength of the material is due to strain hardening which is due to the increased dislocations interactions during the deformation of the tensile test. This is called Strain -hardening.

After reaching the maximum, instability sets in due to some inhomogeneity in the material, and the tensile specimen under deformation starts necking (reduction in the cross section of the tensile specimen). This necking continues until the specimen breaks at the end of the tensile test.

It is called hardening because stress rate increases with respect to strain so it means that the material becomes stiffer and stiffer as strain increases thus is called strain hardening. It is the region between yield limit and Ultimate strength. The various dislocations present move become tangled or intertwine with other dislocations giving rise to a situation where further movement of dislocations becomes tough. This leads to hardening of the material and resists further deformation.

It is also called cold working as if this process is done in low temperatures, it would prevent the atoms from coming back to their positions. At higher temperatures, the atoms acquire enough kinetic energy to be able to move easily. Thus, the material strengthening gained might be lost or becomes lesser at higher temperatures.

It is going to be concave up. second derivative of stress with respect to strain is positive.

slope increase = hardening if slope decreases it is called softening.

At strain hardening region, with the increasing stresses(pressure), stacking up of atoms happens. This provides resistance to the dislocation travel thereby decreasing the deformation and increasing the strength of material. In laymen we can say strength is directly proportional to strain rate.

In the same way the region between ultimate tensile strength to breaking point is called strain softening region.

On the application of load on given material, after yield point is reached, recrystallization is not possible. Atoms get dislocated. (Length of the test specimen increases and width decreases, phenomena of necking occurs. As atom to atom distance decreases due to above reason, it offers higher and higher resistance so we need to apply gradually more load/force to further deform the specimen. This phenomena is known as strain hardening (increase in strength due to strain occurred as a result of load applied initially)

Criteria for yielding or Theories of failure or yield criteria

Yield point under simplified condition of uniaxial tension is widely known and documented. But such simplified conditions [1 – Pure uniaxial tension 2 – Pure shear] are rare in reality.

In many situations complex and multiaxial stresses are present and, in this situation, it is necessary to know when a material will yield.

Mathematically and empirically, the relationships between the yield point under uniaxial tensile test and yield strength under complex situations have been found out. These relationships are known as yield criteria. Thus yield criterion is defined as mathematical and empirically derived relationship between yield strength under uniaxial tensile load and yielding under multiaxial complex stress situation.

<u>What is the meaning about yield criterion?</u>

In the case the stress is un-axial and that stress will cause yielding so this stress can readily be determined. But what if there are several stress acting at a point in different direction The criteria for deciding which combination of multi-axial stress will cause yielding are called criteria.

A yield criterion, often expressed as yield surface, or yield locus, is an hypothesis concerning the limit of elasticity under any combination of stresses.

<u>True elastic limit</u>

The lowest stress at which dislocations move. This definition is rarely used, since dislocations move at very low stresses, and detecting such movement is very difficult.

<u>Proportionality limit</u>

Up to this amount of stress, stress is proportional to strain (Hooke's law), so the stress-strain graph is a straight line, and the gradient will be equal to the elastic modulus of the material.

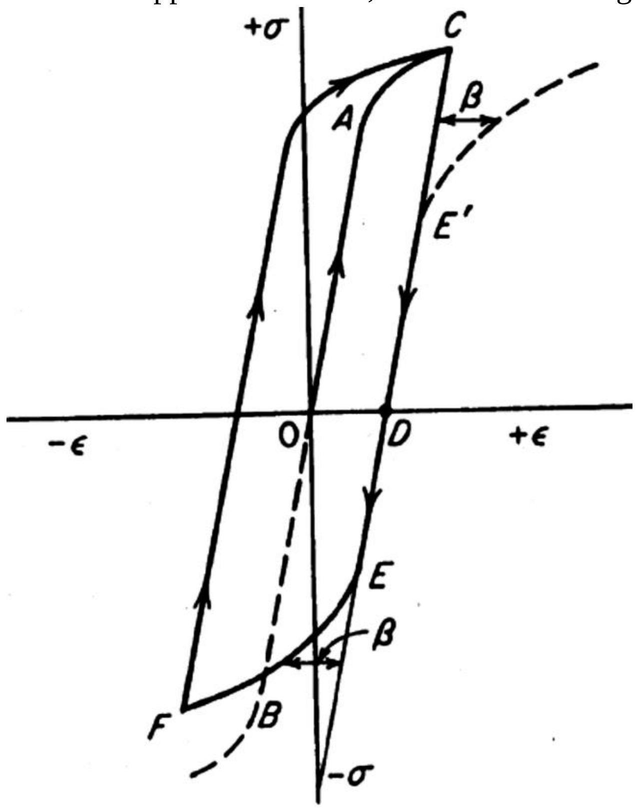
<u>Elastic limit (yield strength)</u>

Beyond the elastic limit, permanent deformation will occur. The lowest stress at which permanent deformation occurs can be measured. This requires a manual load-unload procedure, and the accuracy is critically dependent on equipment and operator skill. For elastomers, such as rubber, the elastic limit is much larger than the proportionality limit. Also, precise strain measurements have shown that plastic strain begins at low stresses.

<u>Bauschinger effect</u>

For most ductile metals that are isotropic, the following assumptions are invoked: There is no Bauschinger effect, thus the yield strengths in tension and compression are equivalent.

The lowering of yield stress for a material when deformation in one direction is followed by deformation in the opposite direction, is called Bauschinger effect.



General Theory of Plasticity defines -

1. Yield criteria : predicts material yield under multi-axial state of stress
2. Flow rule : relation between plastic strain increment and stress increment. A flow rule which relates increments of plastic deformation to the stress components
3. Hardening rule: Evolution of yield surface with strain

Theories of Failure or Yield criteria

Some Yield criteria developed over the years are:

1. Maximum Principal Stress Criterion:- used for brittle materials
2. Maximum Principal Strain Criterion:- sometimes used for brittle materials
3. Strain energy density criterion:- ellipse in the principal stress plane
4. Maximum shear stress criterion (a.k.a Tresca):- popularly used for ductile materials
5. Von Mises or Distortional energy criterion:- most popular for ductile materials

General Terminology in Plasticity

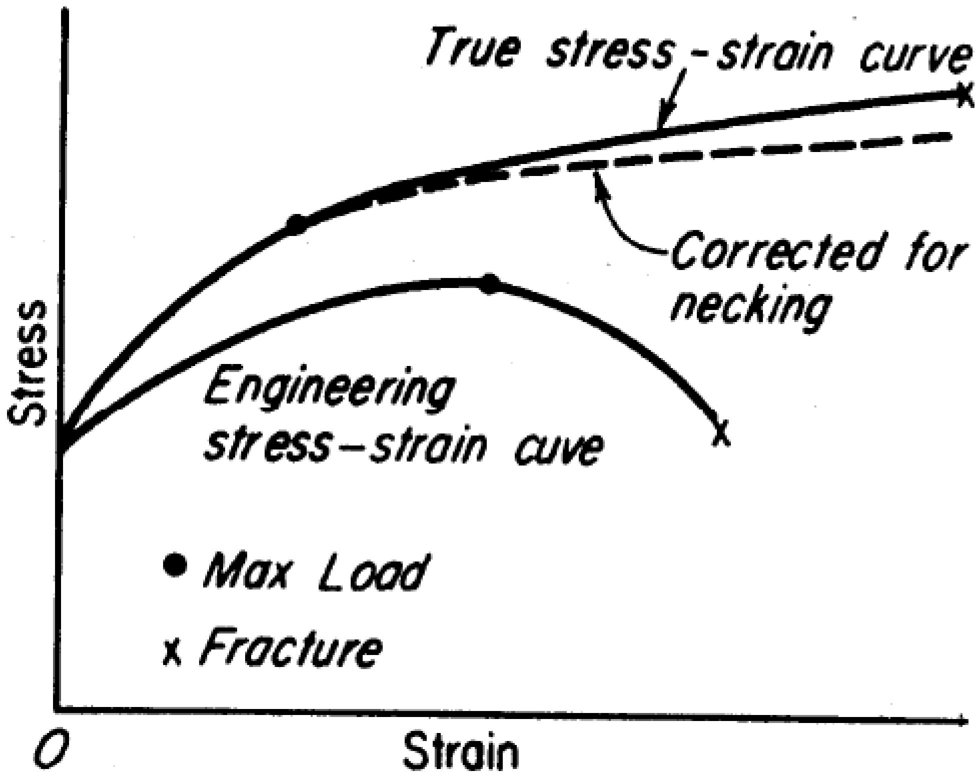
Isotropic – Isotropic materials have elastic properties that are independent of direction. Most common structural materials are isotropic.

Anisotropic – Materials whose properties depend upon direction. An important class of anisotropic materials is fiber-reinforced composites.

Homogeneous – A material is homogeneous if it has the same composition at every point in the body. A homogeneous material may or may not be isotropic.

Effective stress and effective strain:

Effective stress is defined as that stress which when reaches critical value, yielding can commence. True Stress-True Strain Curve Also known as the flow curve.



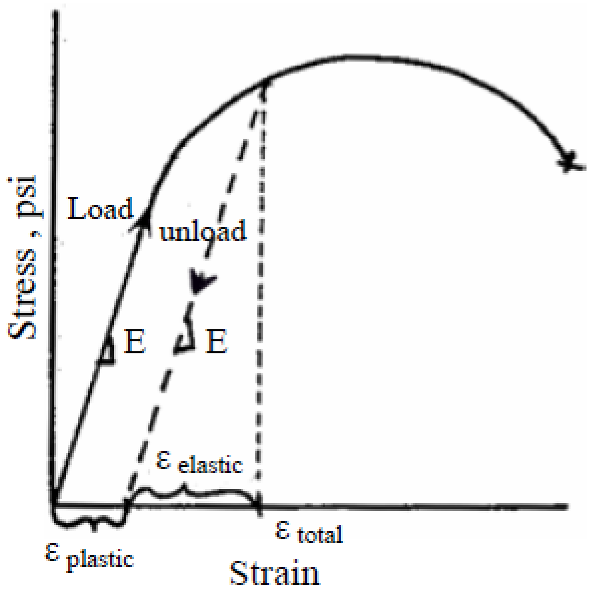
Plastic Deformation

After a material has reached its elastic limit, or yielded, further straining will result in permanent deformation. After yielding not all of the strain will be recovered when the load is removed. Plastic deformation is defined as permanent, non-recoverable deformation. Plastic deformation is not linear with applied stress. Recall if a material experiences only elastic deformation, when the stress is removed the elastic strain will be recovered. If a material is loaded beyond its yield point it experiences both elastic and plastic strain. After yielding the rate of straining is no longer linear as the applied stress increases. When the stress is removed, only the elastic strain is recovered; the plastic strain is permanent.

Elastic deformation occurs as the interatomic bonds stretch, but the atoms retain their original nearest neighbors and they "spring back" to their original positions when the load is removed. Clearly in order to have permanent deformation there must be permanent movement in the interatomic structure of the material. Although some of the atoms move away from their original nearest neighbors not all of the interatomic bonds are broken (this is evident because we can achieve permanent deformation without fracture of the material). The mechanism for permanent deformation is called slip. Slip occurs when planes of densely packed atoms slide over one another: individual bonds are broken and reformed with new atoms in a step-wise fashion until the desired deformation is achieved.

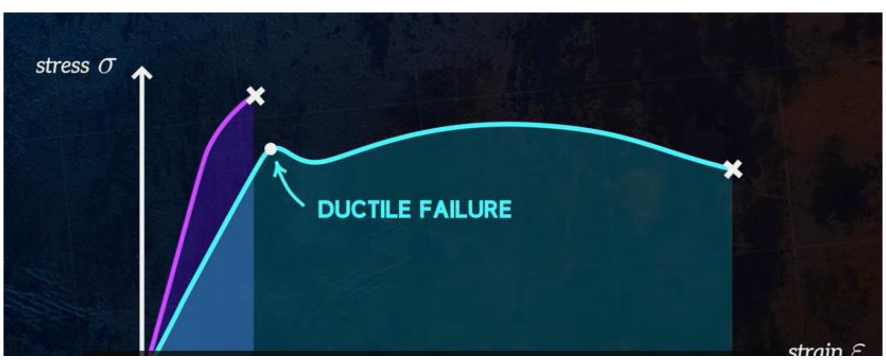
total strain = elastic strain + plastic strain

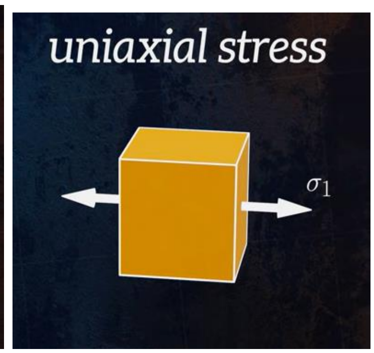
The recovery (or "unload") curve that is produced when the load is removed from a specimen is parallel to E. The amount of strain recovered during the unloading process is the elastic strain; the amount of strain that remains in the specimen after unloading is the plastic strain.



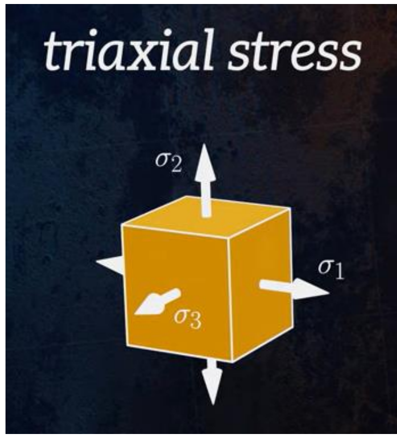
Yield criterions or theories of failure

In the picture you can see this bracket which is holding some weight as you keep increasing the weight you will know at what maximum weight this bracket may fail. How can you predict at what load this object may fail. Which means how can we predict failure. To what level stresses in the object need to reach for it will fail. Let us define what failure is. For the ductile materials failure is considered to occur at the start of plastic deformation. For brittle materials failure happens at fracture. Failure points can be easily identified or determined using some simple tests such as tensile test etc under uniaxial state of stress. This state of stress is very simple and ideally do not exist. So failure in ductile materials occurs when the normal stress in the object reaches yield strength of the material whereas in brittle materials if the normal strength reaches ultimate strength of the material failure occurs.





Let us consider the case of uniaxial stress in which predicting failure is very easy. But in case of multiaxial or triaxial state of stresses it is not so easy to predict failure.



In fact in case of triaxial state of stress, there is no proper universally accepted method to determine the reasons for failure. Instead, we need to predict the failure by using one of the failure theories which will work relatively better under certain circumstances based on experiments. Because each body may fail in different ways, failure theories which may apply for ductile materials may not be applicable to brittle materials and vice-versa. So how does a failure theory actually help us in predicting failure. These theories help us to predict the failure by comparing the stress state of the body with its material properties like yield or ultimate strength which can be easily determined using a uniaxial test.

The stress state at a point can be described using three principal stresses so most failure theories are defined as the function of the principal stresses and the material strength.

$$f(\sigma_1, \sigma_2, \sigma_3) = \sigma_y, \sigma_u$$

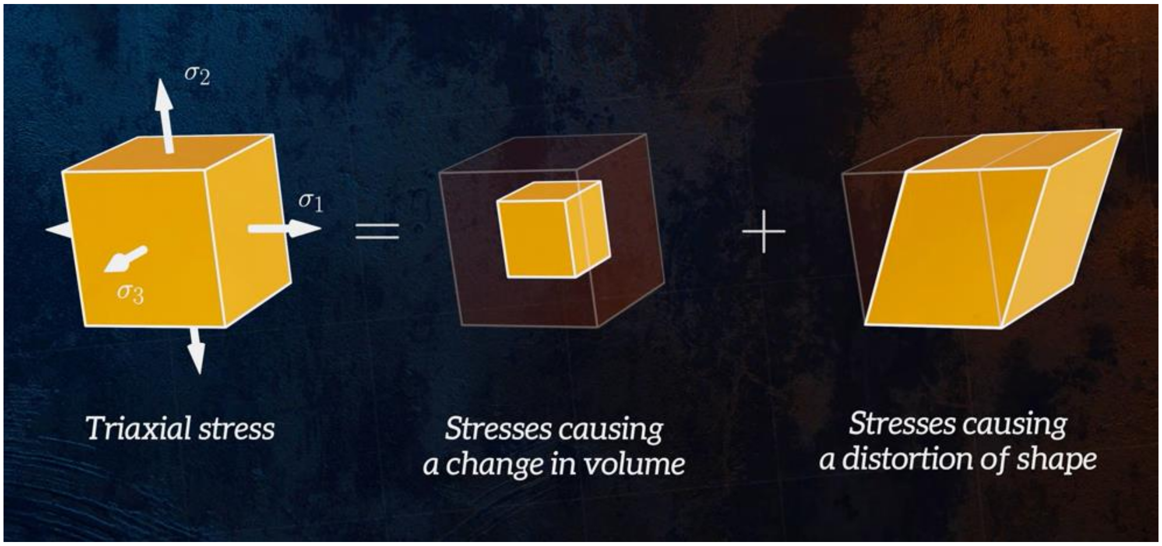
The simplest failure theory is the one in which failure occurs when the maximum or minimum principal stresses reach the yield or ultimate strength of the material.

$\sigma_1 = \sigma_y, \sigma_u$ or $\sigma_3 = \sigma_y, \sigma_u$

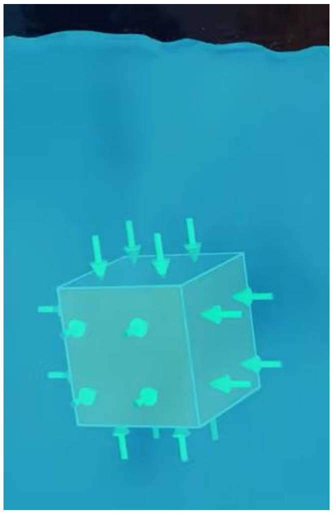
$\sigma_1 = \sigma_y, \sigma_u$ or $\sigma_3 = -\sigma_y, -\sigma_u$

This is called maximum principal stress theory or Rankine theory. It is a simple theory but not a good failure theory particularly for ductile materials.

Let us look at some better failure theories for ductile materials. Any good failure theory needs to be validated with the experimental observations. There is one key observation that the failure thoeries for ductile materials need to understand that the hydrostatic stresses do not cause yielding in ductile materials.



A triaxial state of stress can be decomposed into stresses which can cause the change in volume and stresses which can cause the shape distortion.

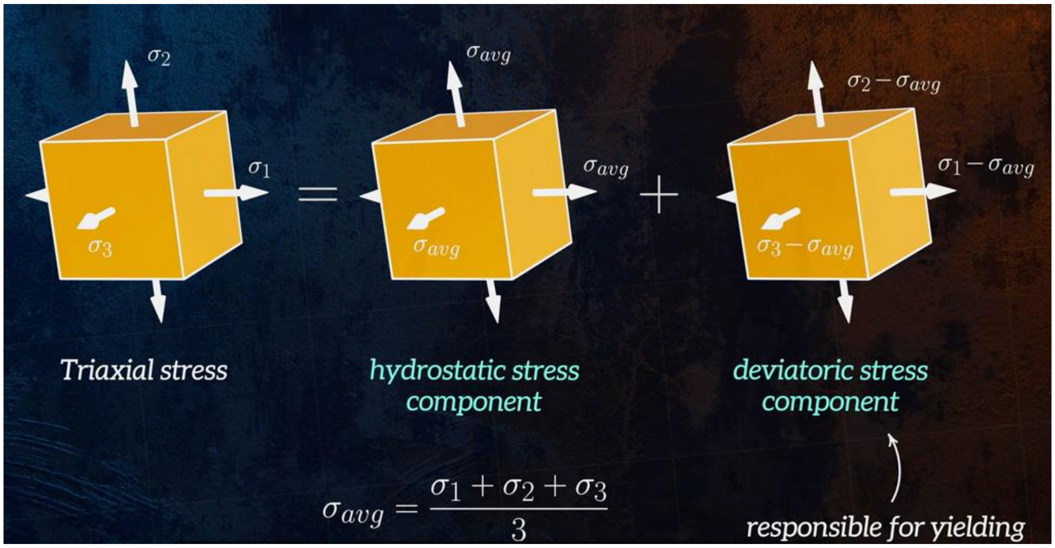


Stresses which can cause the change in volume are called hydrostatic stresses (a kind of stresses developed on an object when it is immersed in the liquid). For the hydrostatic stress configuration, the three principal stresses will be equal and there are no shear stresses.

$\sigma_1 = \sigma_2 = \sigma_3$

For the triaxial state of stresses, hydrostatic component can be calculates as the average of the three principal stresses.

The mechanism that causes the yielding of ductile materials is the shear deformation. Since in the state of hydrostatic stresses there is no shear stresses and even if this component is very large but still will not contribute to yielding so yielding is caused by the stresses which causes shape distortion. Stresses which causes shape distortion are responsible for yielding. These are called deviatoric stresses. This deviatoric component can be calculated by subtracting the hydrostatic component from the each of the principal stresses.



The hydrostatic and deviatoric components of state of triaxial stress can be expressed in matrix form. Here the stress state is described using the principal stresses.

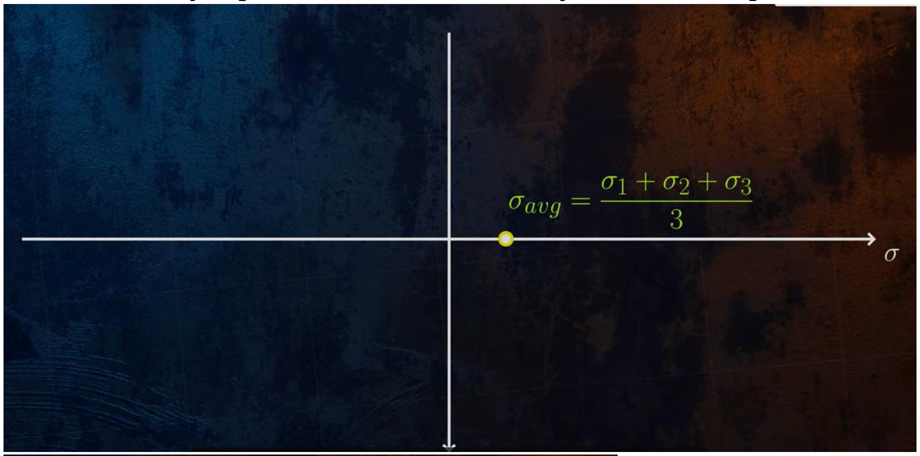
$$\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_{avg} & 0 & 0 \\ 0 & \sigma_{avg} & 0 \\ 0 & 0 & \sigma_{avg} \end{bmatrix} + \begin{bmatrix} \sigma_1 - \sigma_{avg} & 0 & 0 \\ 0 & \sigma_2 - \sigma_{avg} & 0 \\ 0 & 0 & \sigma_3 - \sigma_{avg} \end{bmatrix}$$

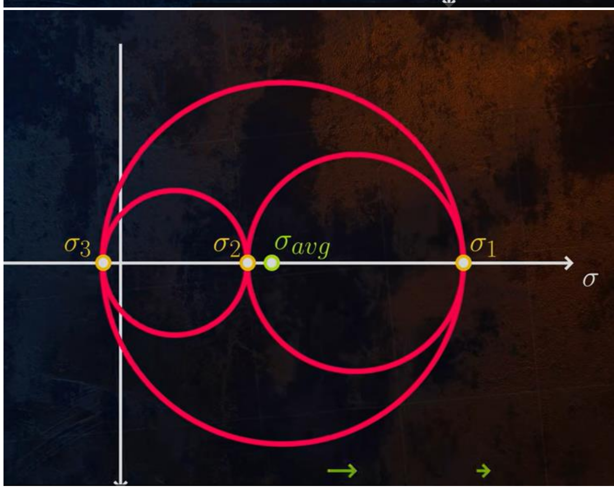
If you need to express the stress state in any other orientation of the stress element, then state of stress can be expressed as follows

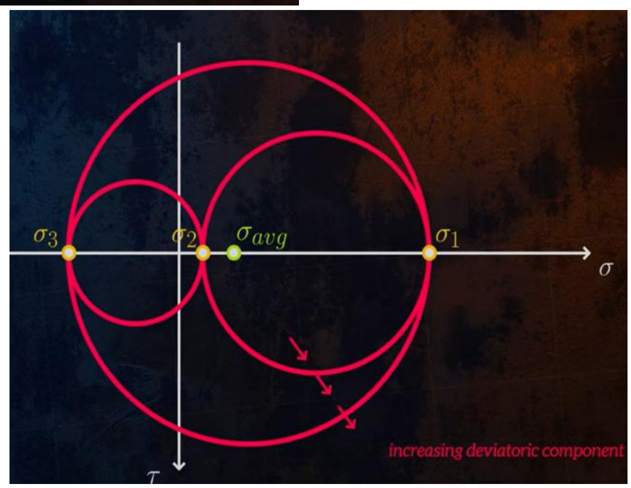
$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_{avg} & 0 & 0 \\ 0 & \sigma_{avg} & 0 \\ 0 & 0 & \sigma_{avg} \end{bmatrix} + \begin{bmatrix} \sigma_x - \sigma_{avg} & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma_{avg} & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \sigma_{avg} \end{bmatrix}$$

Mohr's circle can be used to understand the state of stress in terms of principal stresses.

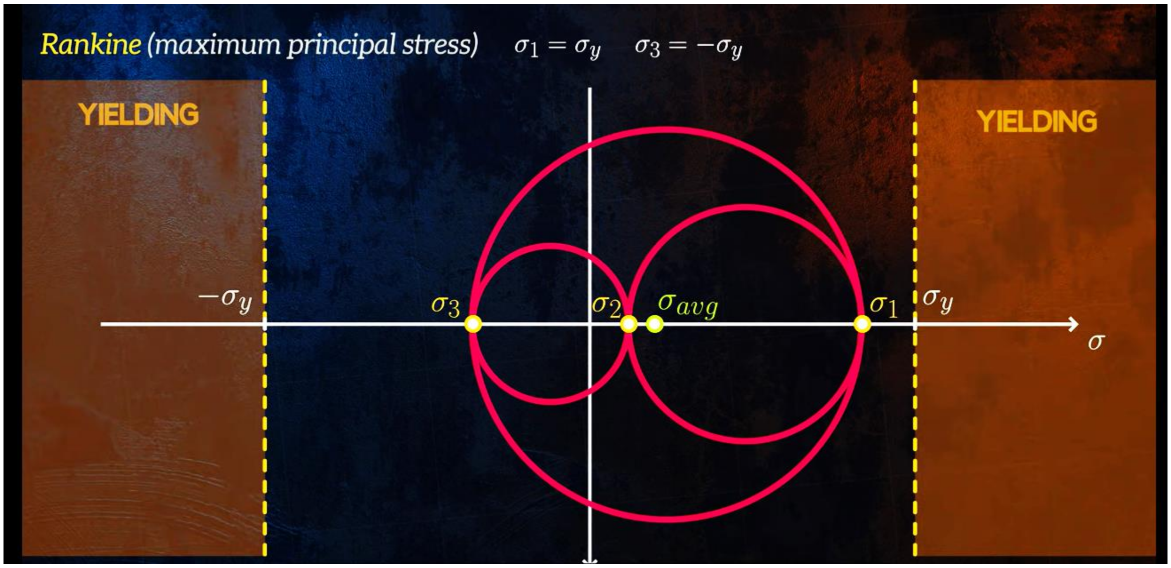
For hydrostatic stress component there are no shear stresses. So the mohr's circle will be reduced to single point equal to the average of the three principal stresses. Shifting the mohr's circle horizontally represents the increase in hydrostatic component.







Increasing the radius of the mohr's circle without changing the average stress represents the increase in the deviatoric component. Since the failure of the ductile materials depends on the deviatoric component, a good failure theory for ductile materials should produce the same results regardless of where mohr's circle is located on the horizontal axis. This explains why the principal stress theory is not the good theory for ductile materials because it is not consistent with the observation that the yielding is independent of the hydrostatic stress.



Two failed theories which are consistent with the observations are Tresca and Von Mises failure criteria. So there two are most commonly used failure theories for ductile materials.

Tresca failure criterion (Maximum shear stress theory)

It states that the yielding occurs when the maximum shear stress is equal to the shear stress at yielding in a uniaxial tensile test. Mathematically defines as,

$$\tau_{max} = \tau_y$$

So graphically using mohr's circle, it can be represented as

This theory is consistent with observation that the hydrostatic stresses do not effect the yielding means it is insignificant where the mohr's circle is located on the horizontal axis. It is common to express this theory as a function of principal stresses instead of as a function of shear stresses. You can observe that in triaxial stress state the maximum shear stress is equal to the radius of the outer circle which is the difference between the maximum and minimum principal stresses divided by 2

$$\tau_{max}=\tau_y$$

$$(\sigma_1-\sigma_3)/2=\tau_y$$

Mohr's cicle for a uniaxial tensile test at yielding looks like as follow

The intermediate (σ_2) and minimum principal (σ_3) stresses are equal to zero and maximum principal stress (σ_1) will be equal to the yield strength of the material.

$$\sigma_2=\sigma_3=0$$

$$\sigma_1=\sigma_y$$

$$(\sigma_1-\sigma_3)/2=\tau_y$$

$$(\sigma_y-0)/2=\tau_y$$

$$\sigma_y/2=\tau_y$$

Shear stress at yielding is equal to half of the yield strength of the ductile material.

So we can rewrite the equations as follows

$$(\sigma_1-\sigma_3)/2=\tau_y$$

$$\sigma_y/2=\tau_y$$

so from the above equations

$$(\sigma_1-\sigma_3)/2= \sigma_y/2 \;\; => (\sigma_1-\sigma_3)= \sigma_y$$

The above is the Tresca yield criterion.

<u>Von Mises failure criterion</u>

(Maximum distortion energy theory)

It is sometimes it is called as Maxwell- Huber-Hencky-von Mises theory.

It states that the yielding occurs when the maximum distortion energy in a material is equal to the distortion energy at the yielding in a uniaxial tensile test.

$$DE_{max}=DE_y$$

What is distortion energy?

It is essentially the portion of strain energy in a stressed element corresponding to the effect of the deviatoric stresses.

In triaxial state of stress, the maximum distortion energy per unit volume can be calculated from the principal stresses using the equation

$DE_{max}= ((1+\upsilon)/6E)[(\sigma_1-\sigma_2)^2 +(\sigma_2-\sigma_3)^2 +(\sigma_3-\sigma_1)^2]$

We know that at yielding during the tensile test the maximum principal stress is equal to yield strength of the material and the other two principal stresses are equal to zeros.

$\sigma_1=\sigma_y;\ \sigma_2=\sigma_3=0$

So distortion energy at yielding in a tensile test (DE_y)

$DE_y=((1+\upsilon)/6E)[(\sigma_1-0)^2 +(0-0)^2 +(0-\sigma_1)^2] = ((1+\upsilon)/6E)[(\sigma_y)^2 +(\sigma_y)^2] =((1+\upsilon)/3E)\sigma_y^2$

So $DE_{max}=DE_y$

$((1+\upsilon)/6E)[(\sigma_1-\sigma_2)^2 +(\sigma_2-\sigma_3)^2 +(\sigma_3-\sigma_1)^2] = ((1+\upsilon)/3E)\sigma_y^2$

$(1/2)[(\sigma_1-\sigma_2)^2 +(\sigma_2-\sigma_3)^2 +(\sigma_3-\sigma_1)^2] = \sigma_y^2$

$\sqrt{(1/2)[(\sigma_1-\sigma_2)^2 +(\sigma_2-\sigma_3)^2 +(\sigma_3-\sigma_1)^2]} = \sigma_y$

This the yield criterion of von Mises theory

Again this theory considers the difference between principal stresses and so is independent of the hydrostatic stresses.

$$\underbrace{\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]}}_{\text{equivalent von Mises stress } \sigma_{eq}} = \sigma_y$$

If the σ_{eq} is larger than the yield strength of the material, yielding is predicted to have occurred.

When comparing failure theories it can be useful to plot their yield surfaces.

<u>What is Yield Surface?</u>

It is the representation of failure theory in the principal stress space.

Let us take the plane stress case where one of the 3 principal stresses is zero.

Conventionally the 3 principal stresses are ordered in such a way that σ_1 is greater than or equal to σ_2 which is greater than or equal to σ_3. In reality we cannot determine the order of principal stresses so we consider σ_A , σ_B and σ_C as 3 principal stresses.

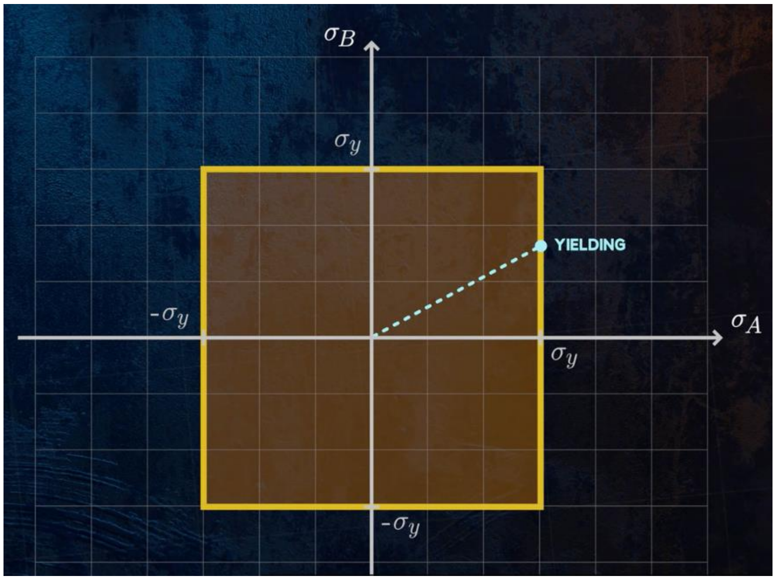
$\sigma 1 \geq \sigma 2 \geq \sigma 3$

Since $\sigma 3=0$, the two axes of the yield surface graph corresponds to two non-zero principal stresses σ_A and σ_B.

<u>Yield surface of Rankine theory</u>

The yield surface for maximum principal stress theory is quite easy to plot because it says that yielding begins when either of the principal stresses is equal to yield strength of the material.

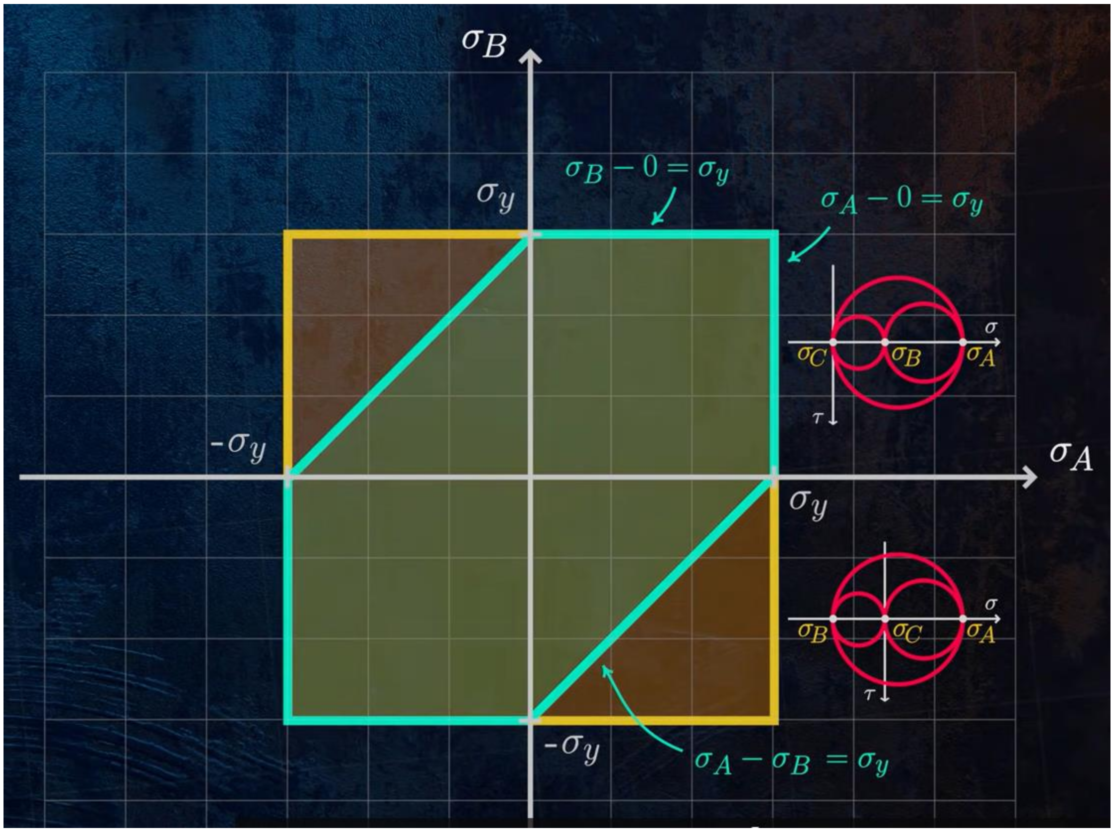
$|\sigma 1|= \sigma y$; $|\sigma 2|= \sigma y$



Yielding is expected to occur when the state of stress reaches this thick line.

Yield surface of Tresca theory

Theory states that yielding occurs when the difference between the maximum and minimum principal stresses is equal to the yield strength of the material. Taking the difference between maximum and minimum principal stresses is not so simple because plane stress itself is a 3 dimensional case of stress state.



($\sigma 1$-$\sigma 3$)= σy is the Tresca yield criterion.

The top right quadrant of the graph σA and σB are positive and σC which is zero is the minimum principle stress. Then the yield surface looks like this. In bottom right quadrant σB is negative and σA is positive which means that σB is the minimum principal stress. Then the yield surface looks like this. Repeating this process for the other 2 quadrants completes the Tresca yield surface.

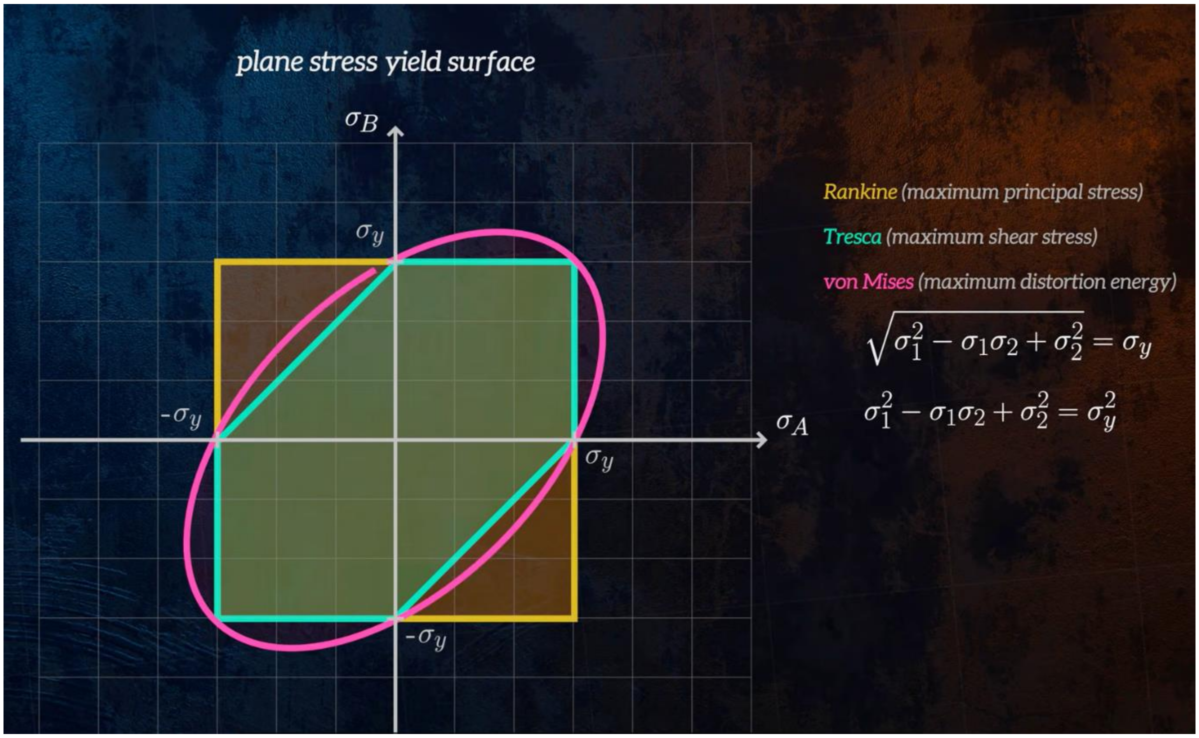
Yield surface of von Mises theory

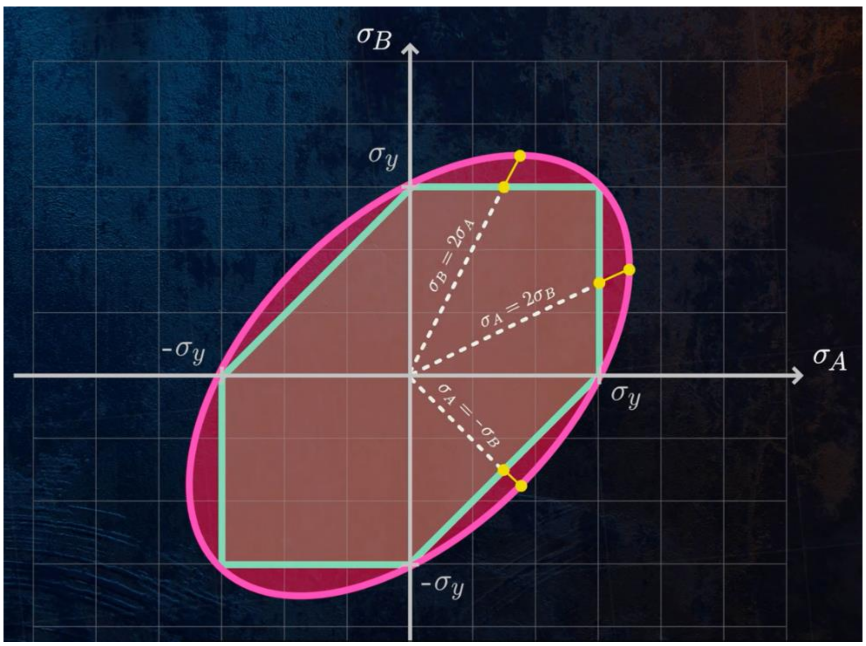
$\sqrt{[\sigma 1^2 - \sigma 1\sigma 2 + \sigma 2^2]} = \sigma y$ is the yield criterion of von Mises theory.

Von Mises theory for the plane stress conditions can be expressed in the form of above equation in terms of principal stresses. When we square both sides of the equation it forms the equation of ellipse, which gives us the von Mises yield surface.

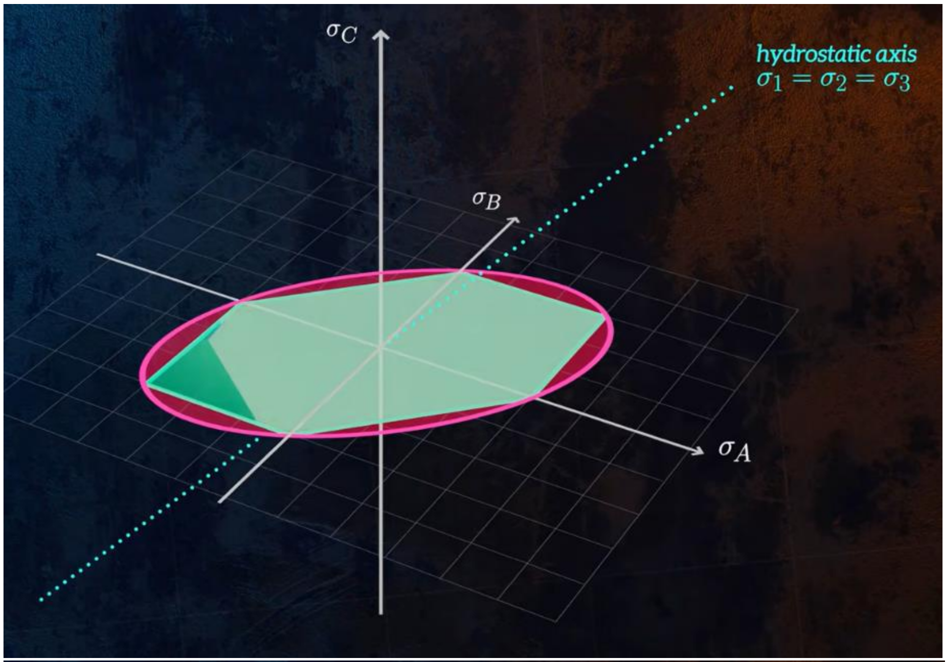
It is clear that maximum principal stress theory has large areas where its use is potentially unsafe. Both tresca and von mises theories agree with experimental observations although von mises is slightly better. Tresca yield surface lies entirely inside the von mises yield surface meaning that Tresca is more conservative (more traditional approach) and easier to apply

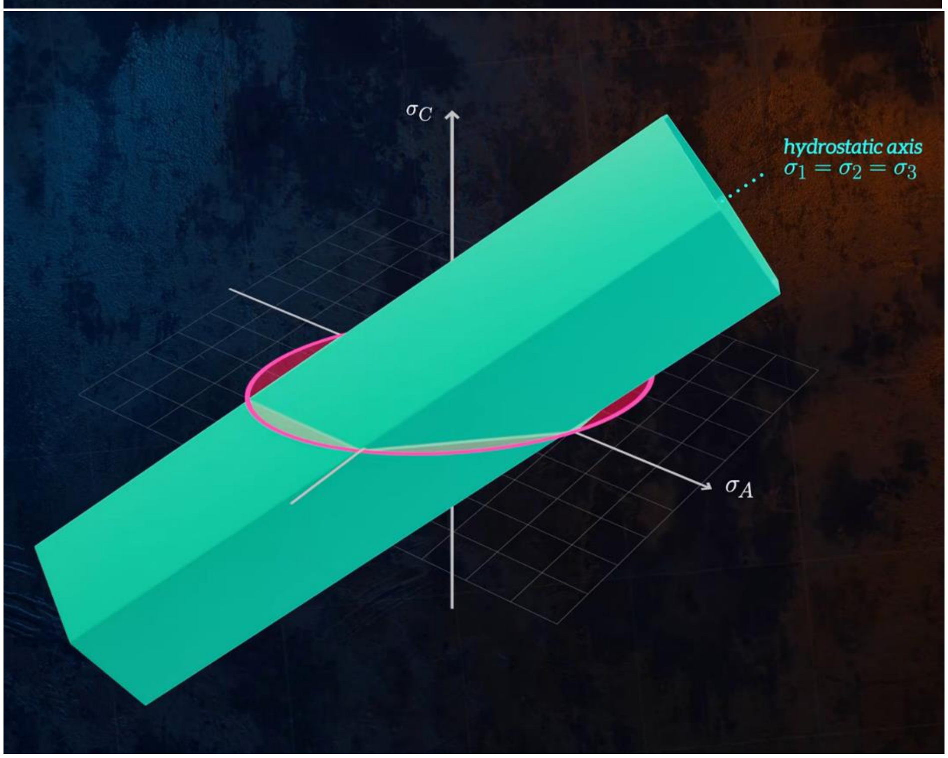
The maximum difference between the two theories can be calculated as 15.5%

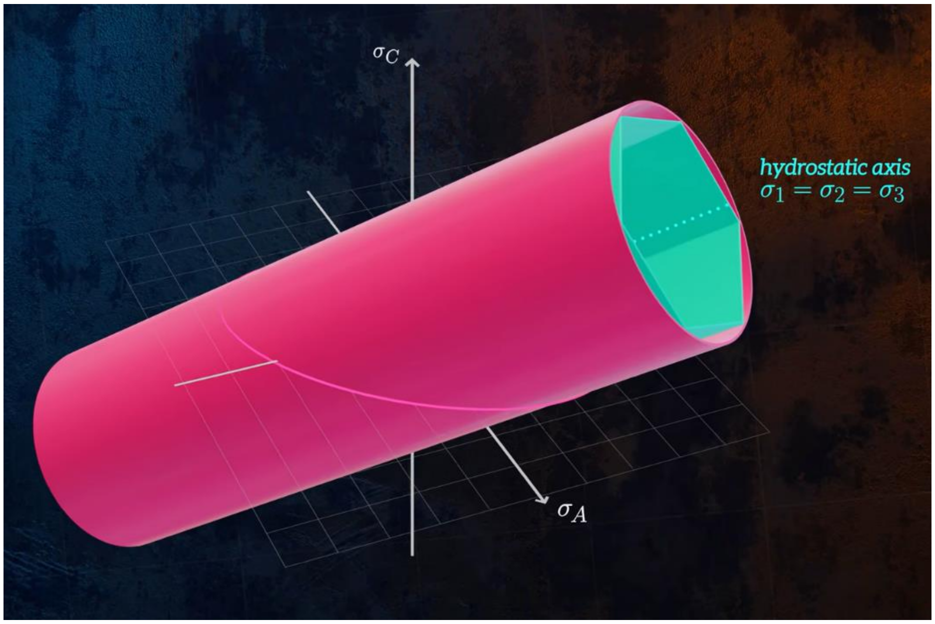




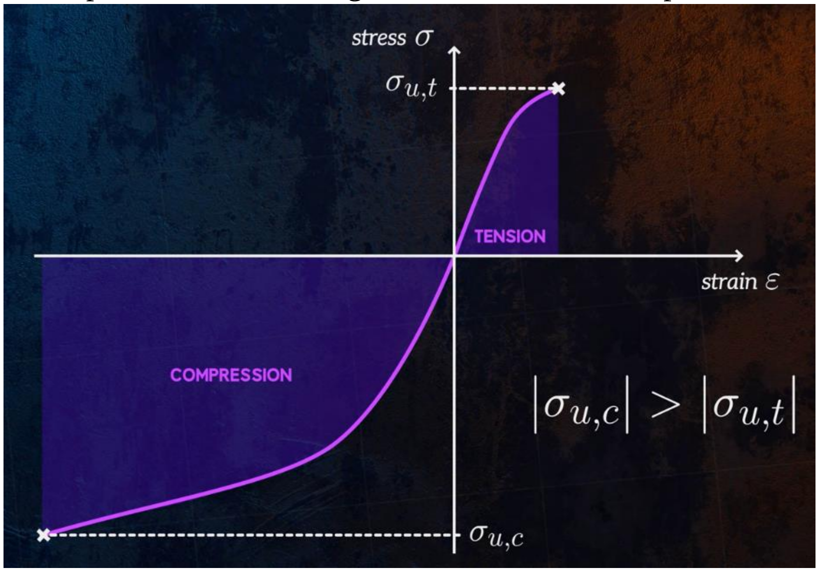
For a general three dimensional case of stress state, σ_3 can never be non-zero. Tresca and von mises yield surfaces are not affected by the hydrostatic stresses so to obtain the yield surfaces in 3 dimensional case we just need to extent the plane stress case yield surfaces along the hydrostatic axis.


σ_C
hydrostatic axis
$\sigma_1 = \sigma_2 = \sigma_3$
σ_B
σ_A


σ_C
hydrostatic axis
$\sigma_1 = \sigma_2 = \sigma_3$
σ_A

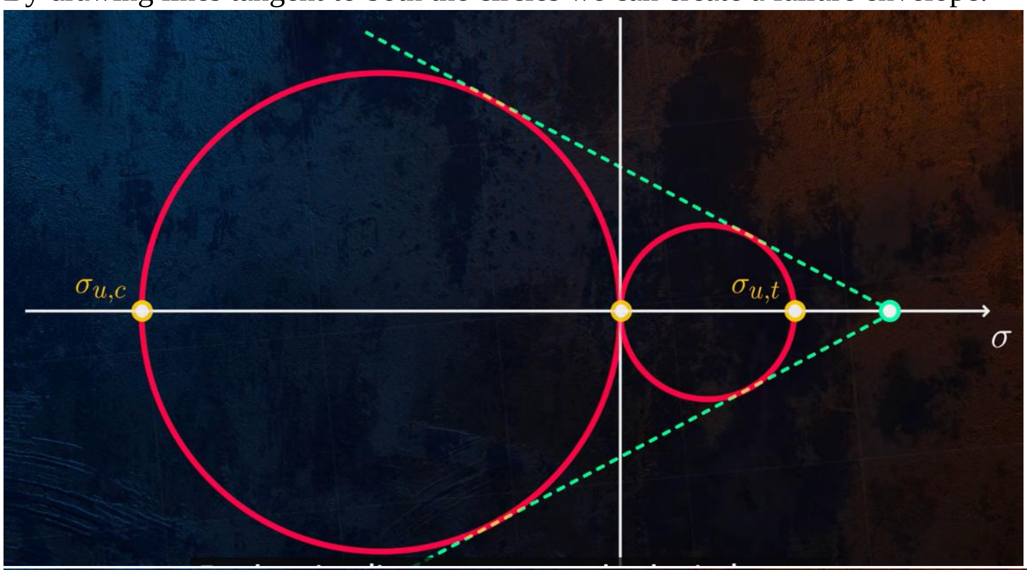


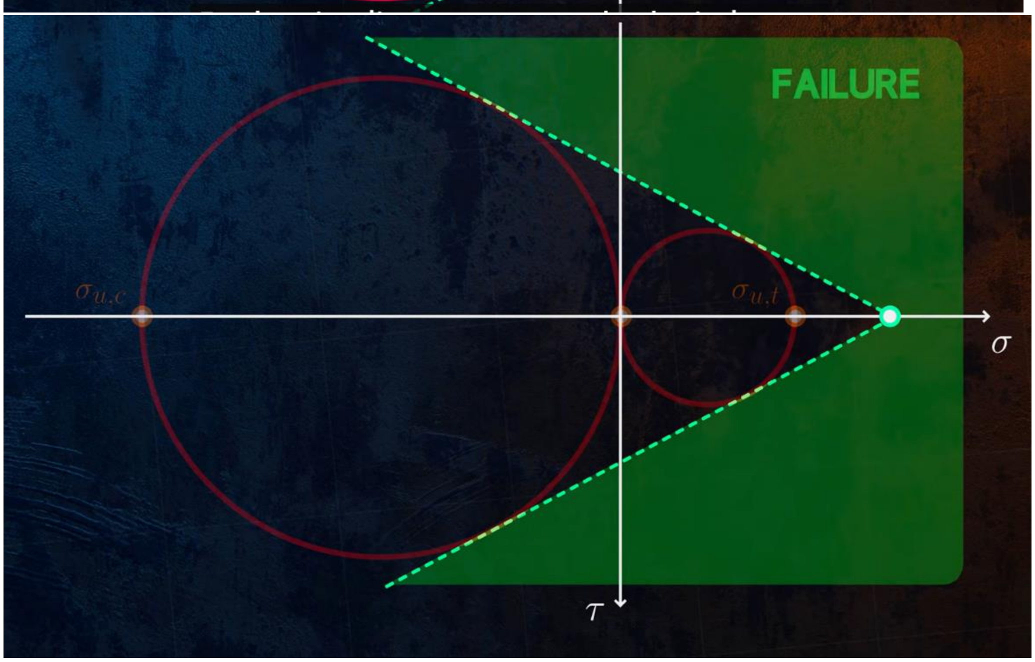
Failure of brittle materials is different from the ductile materials. For brittle materials failure is considered by fracture rather than the yielding. In brittle materials unlike the ductile materials the compressive strengths will be larger than the tensile strength. This needs to be considered in the failure theory for brittle materials meaning that to assess the failure in brittle materials we need determine the two separate ultimate strengths for tension and compression.



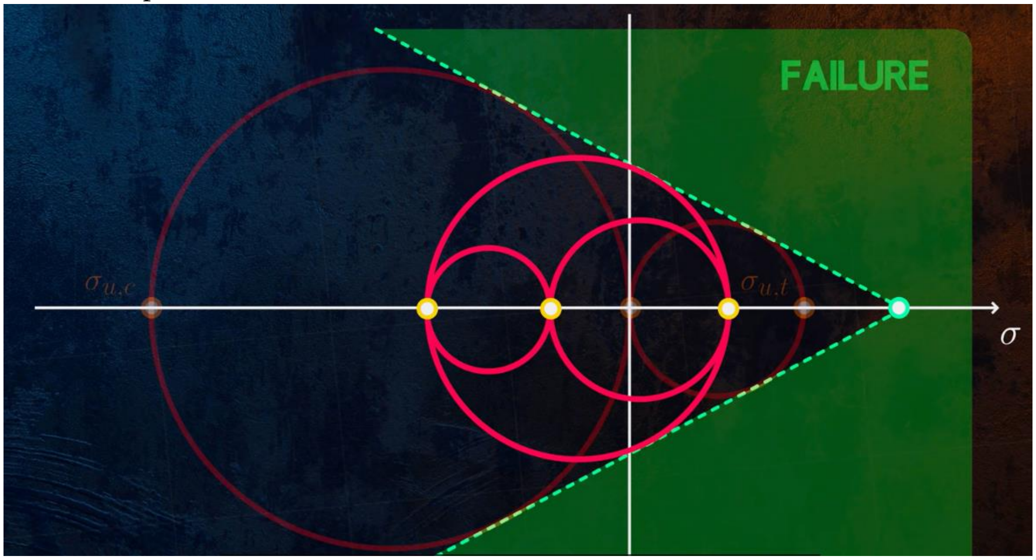
Coulomb-Mohr's theory is the failure theory often used to use for brittle materials. Unlike the failure theories of ductile materials where the hydrostatic stresses are not significant, in Coulomb-Mohr's theory, failure is sensitive to hydrostatic stress and requires both compressive and tensile ultimate strengths. The easiest way to define this theory is to make use of mohr's circle. We start by drawing the mohr's circles corresponding to failure in the uniaxial tensile and compressive tests.

By drawing lines tangent to both the circles we can create a failure envelope.

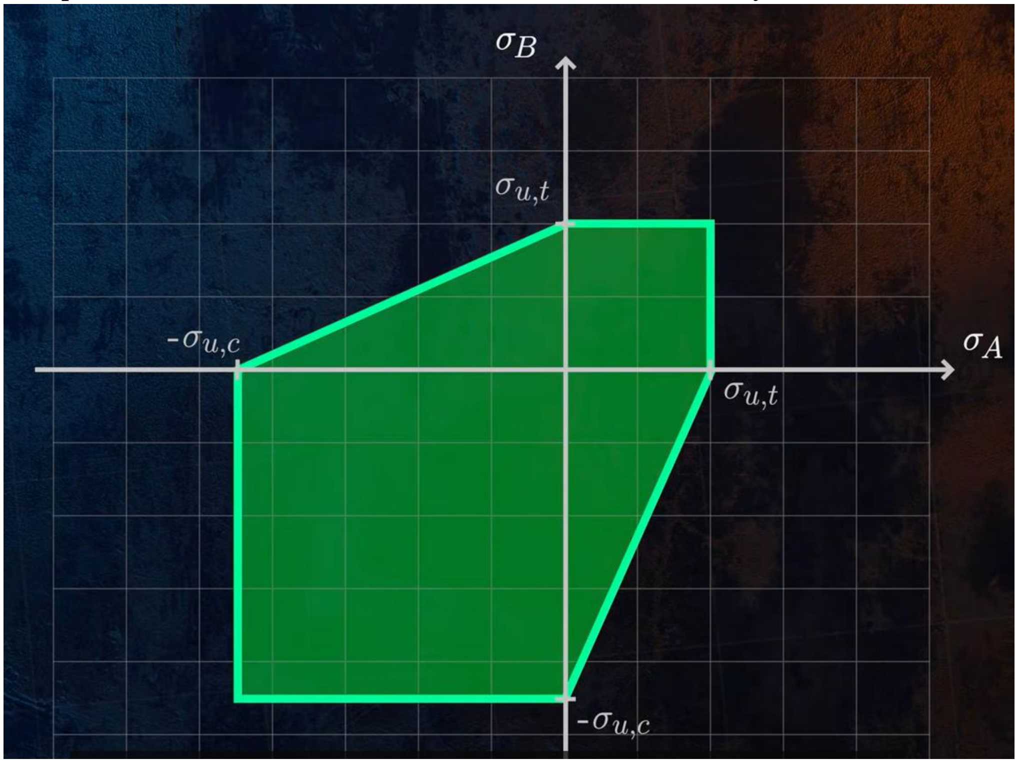




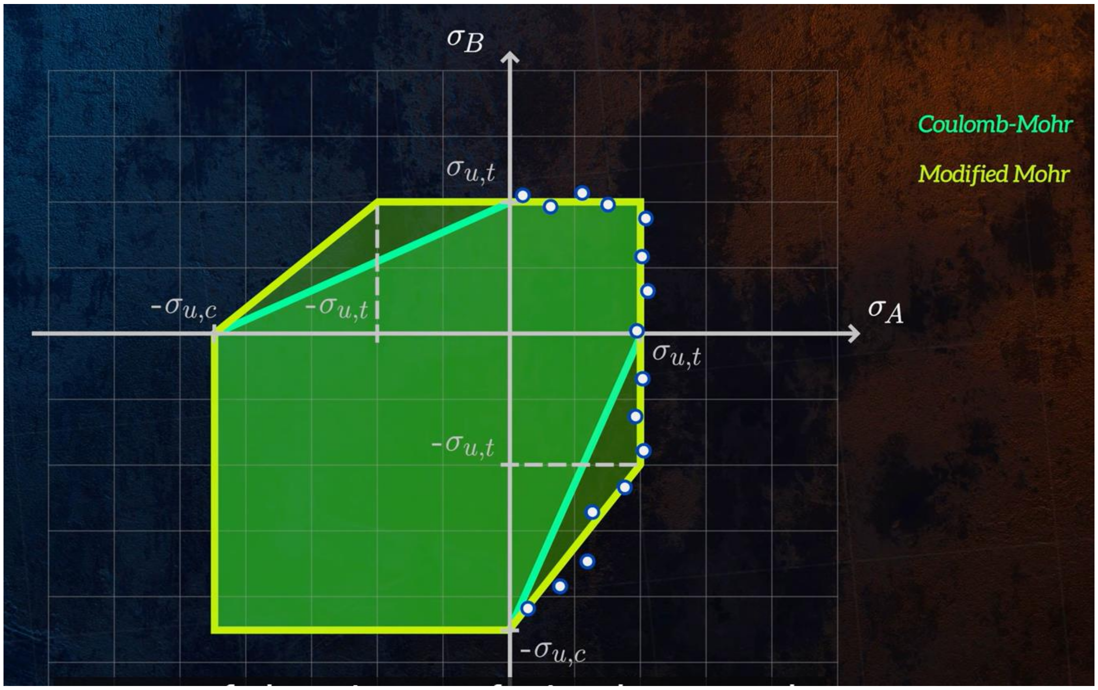
Coulomb-Mohr's theory states that a material will fail for a stress state with a mohr's circle that reaches this envelope.



The plane stress failure surface for Coulomb-Mohr's theory looks like below



Since the Coulomb-Mohr's theory don't fit well accurately with experimental observations especially in bottom right quadrant, a modified Mohr's theory is proposed which fits better with experimental data.

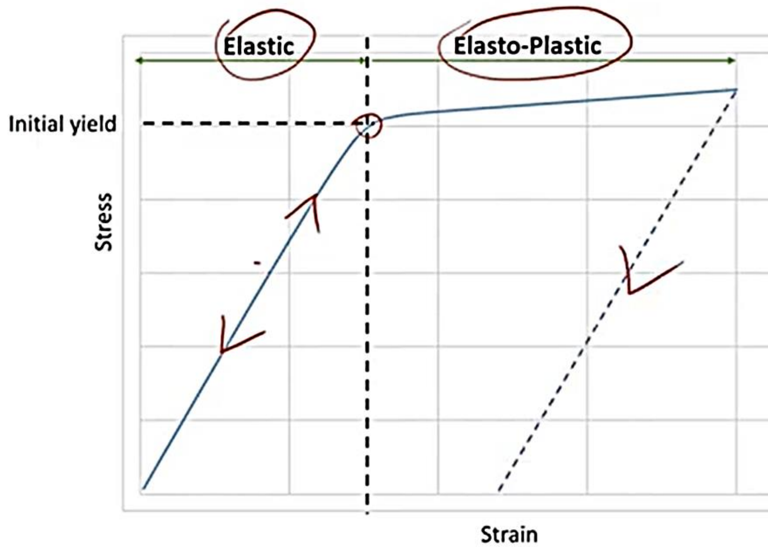


**************The End of PART 1******************
PART 2 follows………..

Following PART 1…………

Why study plasticity?

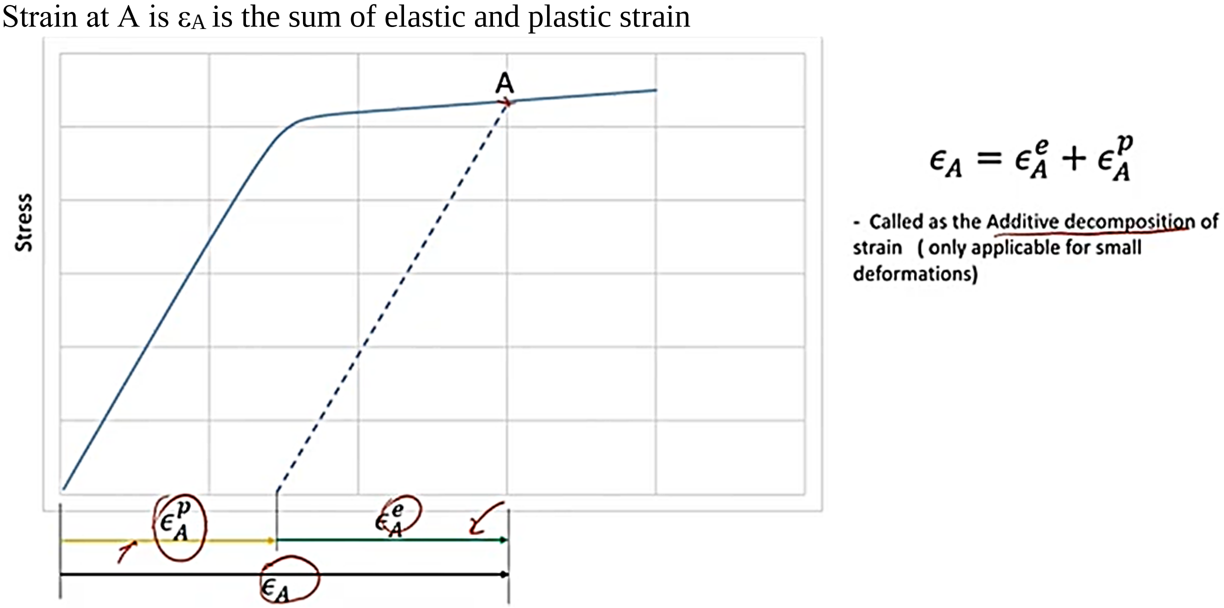
- Many materials fail by fatigue which is governed by the plastic deformation.
- Plasticity is used to design better fatigue resistant materials and structures
- Plastic deformation in metals is due to the shear in metals where as in soil and rock things are different depends on pressure or load acting
- Sliding or slip at molecular level causes the plastic deformation



In elastic regime, the loading and unloading follow the same path. In elasto-plastic zone, the loading and unloading paths are different.

Elastic and plastic strains

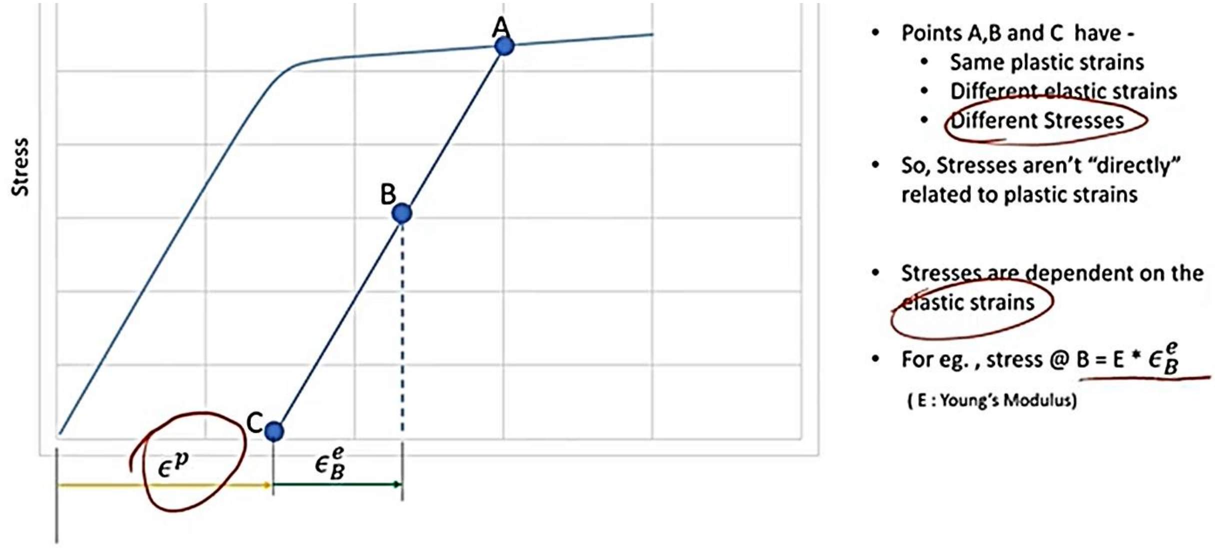
Strain at A is ε_A is the sum of elastic and plastic strain



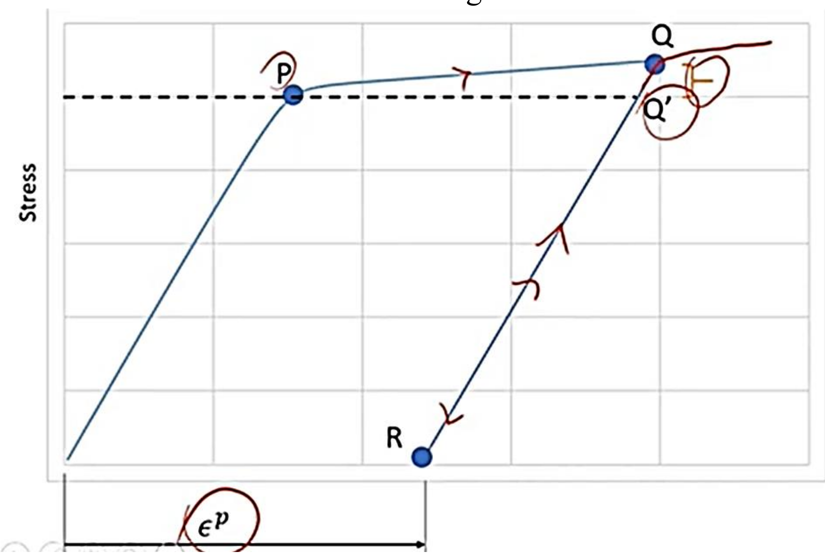
$\varepsilon_A{}^P$ – plastic strain

$\varepsilon_A{}^E$ – elastic strain (recovered strain) or elastic portion of the strain

There is only one stress nothing called plastic stress.



Plastic strains do not induce any stress. Stresses are related to elastic strains. What does plastic strain contribute to…it contributes to strength.



Let us load the material till Q and unload it to R with P as initial yield. Let us reload the material from R towards Q, one may expect the material to yield at Q^1 which is an initial yield but yields at Q. The material has gained some strength corresponding to QQ^1 which could be related to the plastic strain. When the materials gains strength it is called strain hardening when it loses strength in plastic strain it is known as strain softening.

<u>How to model the plasticity?</u>

There are three elements for Plasticity modelling-

1. **Yield condition**

Means at what combination of stresses does the material yield?

It is represented by the yield surface.

If Stress state is on the yield surface- Elasto-plastic regime
If inside yield surface- Elastic regime
It will never go beyond the yield surface.

2. **Flow rule**

Gives a mathematical description about how the material will flow beyond initial yield
Roughly relation between the plastic strain and stress

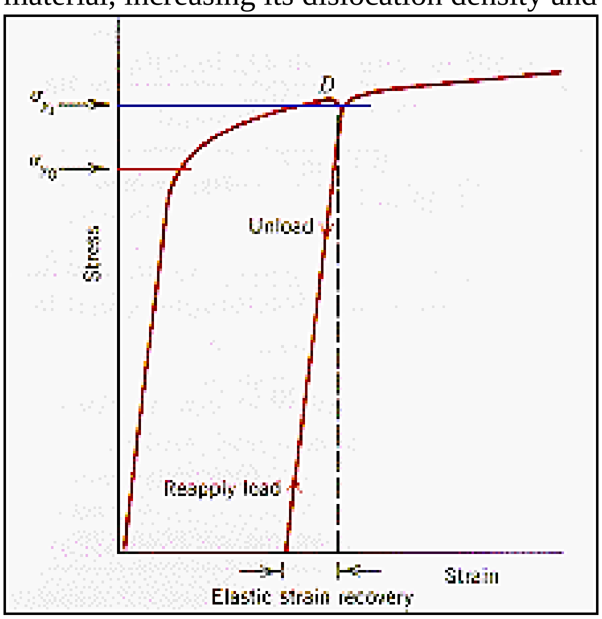
3. **Hardening rule**

Gives the description of evolution of yield surface with plastic strain
Basic models in plasticity : 1) Isotropic hardening and 2) Kinematic hardening
For ID the threshold of the plasticity or yield surface is the point
For 3D problem, it becomes a surface called yield surface. Once the material

Stress-strain relations in plastic deformation is called plastic stress-strain curve.

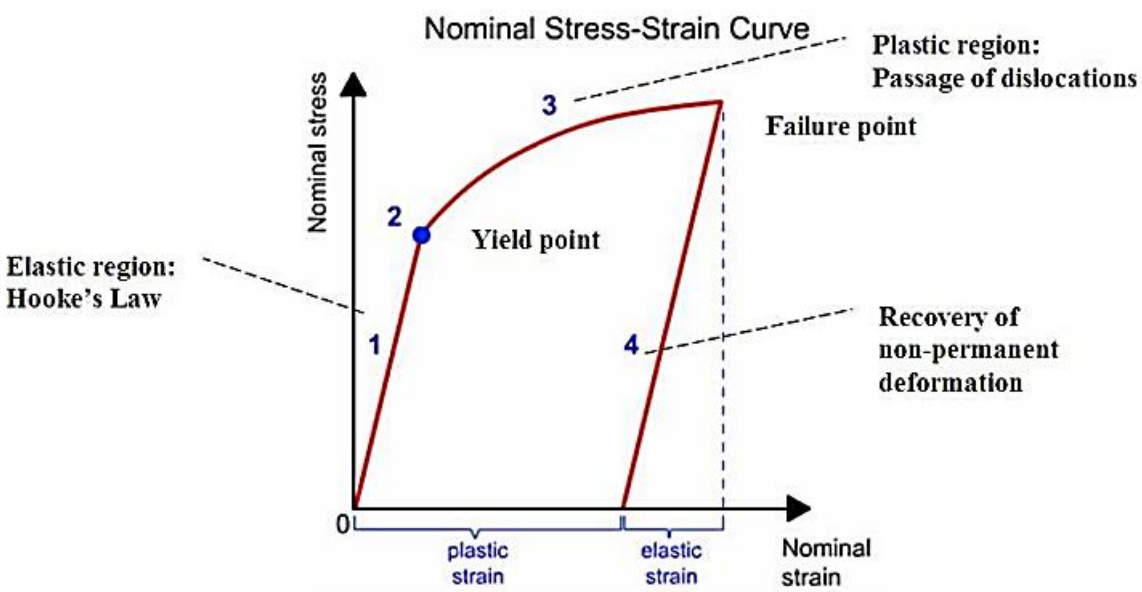
<u>What is Plastic Deformation?</u>

When a material experiences an applied stress its dimensions will change. For low values of the stress the material exhibits an elastic strain. The stress-strain curve shown in the diagram indicates that this elastic behavior continues until the applied stress becomes larger than the yield stress, s_{y0} (red line), of the material. At this point the material starts to show plastic deformation. If the deformation is continued to the point D on the diagram and the stress is then reduced to zero, the sample recovers the elastic component of the strain but retains the plastic deformation strain component. Reapplying the stress yields an initial elastic response with the same slope (elastic modulus) as the initial loading, however, the yield stress marking the transfer to plastic deformation has increased to s_{y1} (blue line). The plastic deformation strain-hardened (work-hardened) the material, increasing its dislocation density and increasing the yield stress.



	Stress, σ	Total Strain, ε	Elastic Strain, ε_e	Plastic Strain, ε_p
Yield Point:	S_{ty}	$S_{ty}/E + 0.002$	S_{ty}/E	0.002
Ultimate Point:	S_{tu}	$S_{tu}/E + \varepsilon_f$	S_{tu}/E	ε_f

Note that when determining the strain at the yield point, a plastic strain of 0.002 was assumed.



What is Elasticity?

Objects deform when pushed, pulled, and twisted. **Elasticity** is the measure of the amount that the object can return to its original shape after these external forces and pressures stop. This is what allows springs to store elastic potential energy.

What is Plasticity?

The opposite of elasticity is plasticity; when something is stretched, and it stays stretched, the material is said to be plastic. When energy goes into changing the shape of some material and it stays changed, that is said to be *plastic deformation*. When the material goes back to its original form, that's *elastic deformation.*

Plastic flow takes place a stress point reaches the boundary of the elastic

What are the Tresca and von Mises theories of yield criterion or failure

1. Tresca criterion (Maximum shearing stress theory)

Yielding will occur when the maximum shear stress reaches the values of the maximum shear stress occurring at yielding under uniaxial tension (or compression) test.

The maximum shear stress in multi-axial stress = the maximum shear stress in simple tension

$$\max\left\{\frac{\sigma_1-\sigma_2}{2},\frac{\sigma_1-\sigma_3}{2},\frac{\sigma_2-\sigma_3}{2}\right\}=\frac{\sigma_0}{2}$$

2. The von-Mises yield criterion (Octahedral shearing stress theory) or distortion energy criterion.

Yielding begin when the octahedral shear stress reaches the octahedral shear stress at yield in simple tension.

$$\tau_{oct} = \tau_{oct,o}$$

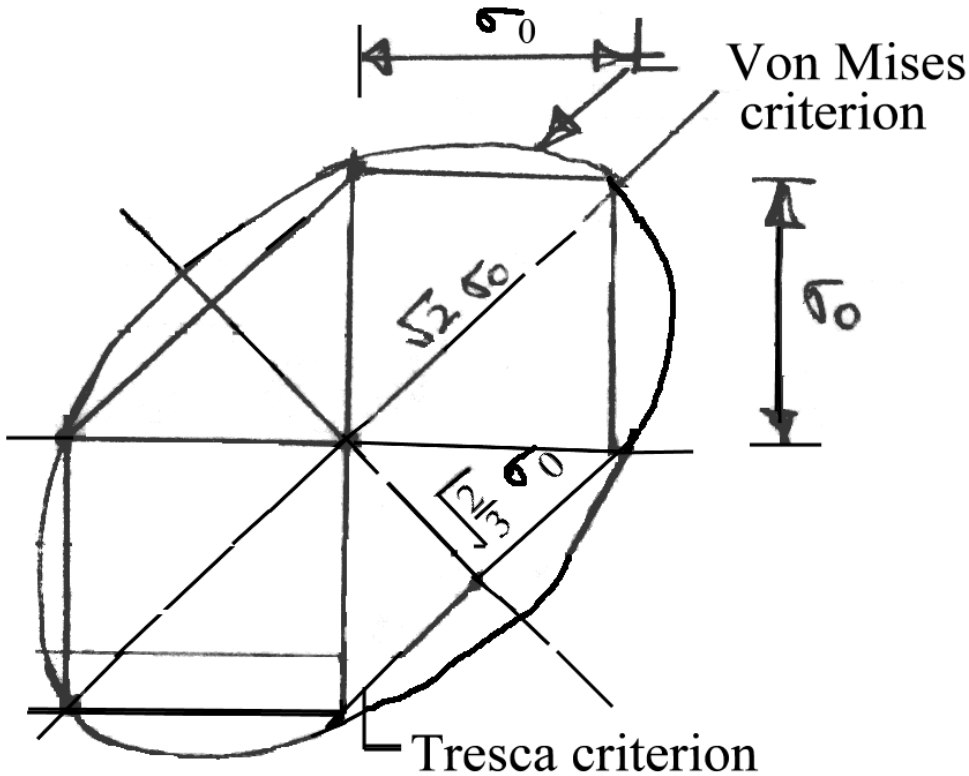
$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6(\tau_{xy}{}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

$$\tau_{oct,o} = \frac{\sqrt{2}}{3}\sigma_0$$

<u>List the advantages of Von Mises criterion along with the Limitations of Tresca</u>

1. It overcomes major deficiency of Tresca criterion. Von Mises criterion implies that yielding is not dependent on any particular normal stress but instead, depends on all three principal shearing stresses.
2. Von Mises criterion conforms the experimental data better than Tresca and therefore more realistic.
3. Since it involves squared terms, the result is independent of sign of individual stresses. This is an important since it is not necessary to know which is the largest and the smallest principal stress in order to use this criterion.
4. Tresca criterion ignores the effect of intermediate principal stress and this is a major drawback of this.
5. Von Mises criterion take into consideration the intermediate principal stress and hence move realistic.
6. The predications offered by Von Mises criterion conforms empirical data.
7. The application of Von Mises yield criterion holds good for both ductile and brittle materials.
8. Tresca criterion do not yield good results for brittle materials.
9. The yield stress predicted by Von Mises criterion is 15. 5% greater than the yield stress predicted by Tresca criterion.
10. Tresca criterion is preferred in analysis for simplicity.
11. Von Mises criterion is preferred where more accuracy is desired.
12. Von Mises criterion is represented by a right circular cylinder whereas the Tresca criterion is represented by a regular hexagonal prism.

Draw the yield surfaces for Von Mises and Tresca criterion

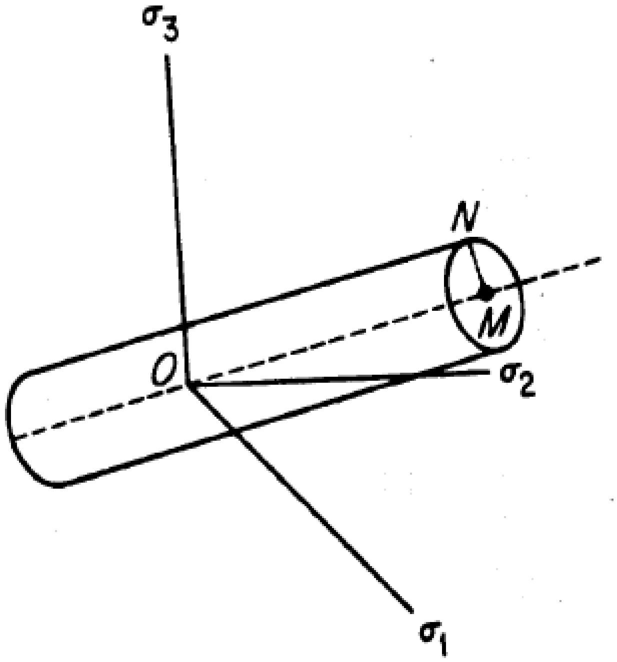


What is Yield Surface?

Yield surface is described in three dimensional space of stresses, and encompasses (holds within) the elastic region of material behavior. The states of stress of material inside the yield surface are elastic, when the stress reaches this surface it reaches the yield point. Then the material behaviour becomes plastic, because the stress cannot cross this surface.

Useful means of describing yield surface include expressing it in the terms of principal stresses ($\sigma_1, \sigma_2, \sigma_3$), or using stress invariants (I_1, I_2, I_3).

The yield criteria can be represented geometrically by a cylinder oriented at equal angles to the σ_1, σ_2, & σ_3 axes.



1. A state of stress which gives a point inside the cylinder represents elastic behavior.
2. Yielding begins when the state of stress reaches the surface of the cylinder.
3. MN, the cylinder radius is the deviatoric stress.
4. The cylinder axis, OM, which makes equal angles with the principal axes represents the hydrostatic component of the stress tensor.

There are several different yield surfaces known in engineering, and those most popular are listed below.

1. Tresca - Guest yield surface
2. Huber - Mises - Hencky, also known as Prandtl - Reuss yield surface
3. Mohr - Coulomb yield surface
4. Drucker - Prager yield surface
5. Brestler - Pister criterion
6. Willam - Warnke criterion

Plastic Stress-strain relations

1. In elastic regime, the stress-strain relations are uniquely determined by the Hooke's law.
2. In plastic deformation, the strains also depend on the history of loading. It is necessary to determine the differentials or increments of plastic strains throughout the loading path and then obtain the total strain by integration.
3. Plastic strains are independent of the loading path.

For Example

• A rod, 50 mm long, is extended to 60 mm and then compressed back to 50 mm.

On the basis of total deformation:

$$\varepsilon = \int_{50}^{60} \frac{dL}{L} + \int_{60}^{50} \frac{dL}{L} = 0$$

On an incremental basis:

$$\varepsilon = \int_{50}^{60}\frac{dL}{L} + \int_{60}^{50} -\frac{dL}{L} = 2\ln 1.2 = 0.365$$

Two general categories of plastic stress-strain relationships.
• Incremental or flow theories relate stresses to plastic strain increments.
• Deformation or total strain theories relate the stresses to total plastic strains.

Explain the Flow rule

1. Stress vs. strain relationship in plasticity called the flow rule.
2. As pressure is applied the material resists the deformation. So greater and greater force is needed to continue the deformation up to a point when the material begins to lose coherence (no longer elastic) and the deformation becomes permanent and the resistance to deformation decreases, so less force is required. The behavior of the material past that maximum point is then described by the "plastic flow rule". As if applying pressure to a plastic.
3. Flow rule is roughly the relation between "plastic strain" (not the total strain) and stress, it gives a description of how a material flows beyond initial yield.

What is strain hardening?

If you plastically deform a solid, then unload it, and then try to re-load it so as to induce further plastic flow, its resistance to plastic flow will have increased i.e. its yield point/elastic limit increases (meaning plastic flow begins at a higher stress than in the preceding cycle- so we say the resistance to plastic flow increases]. This is known as 'strain hardening'

How to model strain hardening?

There are different ways of modelling strain hardening for a finite element material model. Discussed below are the two simplest approaches:

1. Isotropic hardening.
2. Kinematic hardening.

For isotropic hardening, if you plastically deform a solid, then unload it, then try to reload it again, you will find that its yield stress (or elastic limit) would have increased compared to what it was in the first cycle.

Again, when the solid is unloaded and reloaded, yield stress (or elastic limit) further increases. [as long as it is reloaded past its previously reached maximum stress]. This continues until a stage (or a cycle) is reached that the solid deforms elastically throughout [that is, if the cycles of load are always to the same level, then after just one cycle your specimen on subsequent cycles will just be loading and unloading along the elastic line of the stress strain curve].This is isotropic hardening.

Essentially, isotropic hardening just means if you load something in tension past yield, when you unload it, then load it in compression, it will not yield in compression until it reaches the level past yield that you reached when loading it in tension. In other words if the yield stress in tension increases due to hardening the compression yield stress grows the same amount even though you might not have been loading the speciment in compression. It is a type of hardening used in mathematical models for finite element analysis to describe plasticity. though it is not absolutely correct for real materials.

Isotropic hardening is not useful in situations where components are subjected to cyclic loading.[real metals exhibit some isotropic hardening and some kinematic hardening.

Isotropic hardening does not account for Bauschinger effect and predicts that after a few cycles, the material (solid) just hardens until it responds elastically.

To fix this, alternative laws i.e. kinematic hardening laws have been introduced. As per these hardening laws, the material softens in compression and thus can correctly model cyclic behaviour and Bauschinger effect.

What is hardening rule mean?

What is Isotropic Hardening and Kinematic hardening?

A hardening rule, which prescribes the work hardening of the material and the change in yield condition with the progression of plastic deformation.

Most materials exhibit some degree of hardening as an accompaniment to plastic straining. In general this means that the shape and size of the yield surface changes during plastic loading. This change may be rather arbitrary and extremely difficult to describe accurately. Therefore, hardening is often described by a combination of two specific types of hardening, namely isotropic hardening and kinematic hardening

Isotropic hardening is irreversible; once the material has experienced a certain degree of hardening the yield limit is shifted permanently. Isotropic hardening rule states that the yield surface expands proportionally in all directions when yield stress is exceeded.

Kinematic hardening rule states that the yield surface does not exceed, but translates in the direction of the stress rising and stays in the same area and shape. (yield surface remains the same shape but expands with increasing stress)

What is Flow plasticity mean?

Principle of Normality and Plastic Potential

1. Flow plasticity is a solid mechanics theory that is used to describe the plastic behavior of materials.
2. Flow plasticity theories are characterized by the assumption that a flow rule exists that can be used to determine the amount of plastic deformation in the material.
3. In flow plasticity theories it is assumed that the total strain in a body can be decomposed additively (or multiplicatively) into an elastic part and a plastic part. The elastic part of the strain can be computed from a linear elastic or hyperelastic constitutive model. However, determination of the plastic part of the strain requires a flow rule and a hardening model.
4. Typical flow plasticity theories for unidirectional loading (for small deformation perfect plasticity or hardening plasticity) are developed on the basis of the following requirements:

1. The material has a linear elastic range.
2. The material has an elastic limit defined as the stress at which plastic deformation first takes place
3. Beyond the elastic limit the stress state always remains on the yield surface

4. The total strain is a linear combination of the elastic and plastic parts
5. The plastic part cannot be recovered while the elastic part is fully recoverable.

Briefly explain various theories of failure

1. Maximum Principal Stress Theory (Rankine)

According to this theory, the maximum principal stress in the material determines failure regardless of what the other two principal stresses are, so long as they are algebraically smaller.

This theory is not much supported by experimental results

The maximum principal stress theory, because of its simplicity, is considered to be reasonably satisfactory for brittle materials which do not fail by yielding.

Criterion:

if $\sigma 1 > \sigma 2 > \sigma 3$ are the principal stresses at a point and σy the yield stress or tensile elastic limit for the material under a uniaxial test, then failure occurs when $\sigma 1 \geq \sigma y$

2. **Maximum Shearing Stress Theory**

If $\sigma 1 > \sigma 2 > \sigma 3$ are the three principal stresses at a point, failure occurs when

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_y}{2}$$

where $\sigma_y/2$ is the shear stress at yield point in a uniaxial test.

For ductile load carrying members where large shears occur and which are subject to unequal triaxial tensions, the maximum shearing stress theory is used because of its simplicity.

3. **Maximum Elastic Strain Theory**

According to this theory, failure occurs at a point in a body when the maximum strain at that point exceeds the value of the maximum strain in a uniaxial test of the material at yield point.

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right] \geq \frac{\sigma_y}{E}$$

ε_1 is the principal strain.

This is not supported by experiments. While the maximum strain theory is an improvement over the maximum stress theory, it is not a good theory for ductile materials.

For materials which fail by brittle fracture, one may prefer the maximum strain theory to the maximum stress theory.

4. **Octahedral Shearing Stress Theory**

According to this theory, the critical quantity is the shearing stress on the octahedral plane. The plane which is equally inclined to all the three principal axes Ox, Oy and Oz is called the octahedral plane. The normal to this plane has direction cosines nx, ny and nz = 1/ 3 . The tangential stress on this plane is the octahedral shearing stress.

$$\tau_{\text{oct}} = \frac{1}{3}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]^{1/2}$$

$$= \frac{\sqrt{2}}{3}\left(I_1^2 - 3I_2\right)^{1/2}$$

$$\tau_{oct} = \frac{1}{3}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]^{1/2} \geq \frac{\sqrt{2}}{3}\sigma_y$$

This theory is supported quite well by experimental evidences.
This theory is equivalent to the maximum distortion energy theory

5. **Maximum Elastic Energy Theory (Beltrami and Haigh)**

According to this theory, failure at any point in a body subject to a state of stress begins only when the energy per unit volume absorbed at the point is equal to the energy absorbed per unit volume by the material when subjected to the elastic limit under a uniaxial state of stress.

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \geq \sigma_y^2$$

This theory does not have much significance since it is possible for a material to absorb considerable amount of energy without failure or permanent deformation when it is subjected to hydrostatic pressure.

6. **Energy of Distortion Theory (Huber, von Mises and Hencky)**

According to this theory, it is not the total energy which is the criterion for failure; in fact the energy absorbed during the distortion of an element is responsible for failure.

$$\tau_{oct} = \frac{1}{3}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]^{1/2} \geq \frac{\sqrt{2}}{3}\sigma_y$$

Therefore, the octahedral shearing stress theory and the distortion energy theory are identical.

<u>What is significance of the theories of failure?</u>

The mode of failure of a member and the factor that is responsible for failure depend on a large number of factors such as the nature and properties of the material, type of loading, shape and temperature of the member, etc. We have observed, for example, that the mode of failure of a ductile material differs from that of a brittle material. While yielding or permanent deformation is the characteristic feature of ductile materials, fracture without permanent deformation is the characteristic feature of brittle materials. Further, if the loading conditions are suitably altered, a brittle material may be made to yield before failure. Even ductile materials fail in a different manner when subjected to repeated loadings (such as fatigue) than when subjected to static loadings. All these factors indicate that any rational procedure of design of a member requires the determination of the mode of failure (either yielding or fracture), and the factor (such as stress, strain and energy) associated with it. If tests could be performed on the actual member subjecting it to all the possible conditions of loading that the member would be subjected to during operation, then one could determine the maximum loading condition that does not cause failure. But this may not be possible except in very simple cases. Consequently, in complex loading conditions, one has to identify the factor associated with the failure of a member and take precautions to see that this factor does not exceed the maximum allowable value. This information is obtained by performing a suitable test (uniform tension or torsion) on the material in the laboratory.
In discussing the various theories of failure, we have expressed the critical value associated with each theory in terms of the yield point stress σy obtained from a uniaxial tensile stress. This was done

since it is easy to perform a uniaxial tensile stress and obtain the yield point stress value. It is equally easy to perform a pure torsion test on a round specimen and obtain the value of the maximum shear stress τy at the point of yielding. Consequently, one can also express the critical value associated with each theory of failure in terms of the yield point shear stress τy. In a sense, using σy or τy is equivalent because during a uniaxial tension, the maximum shear stress τ at a point is equal to 1/2σ; and in the case of pure shear, the normal stresses on a 45° element are σ and $-\sigma$, where σ is numerically equivalent to τ.

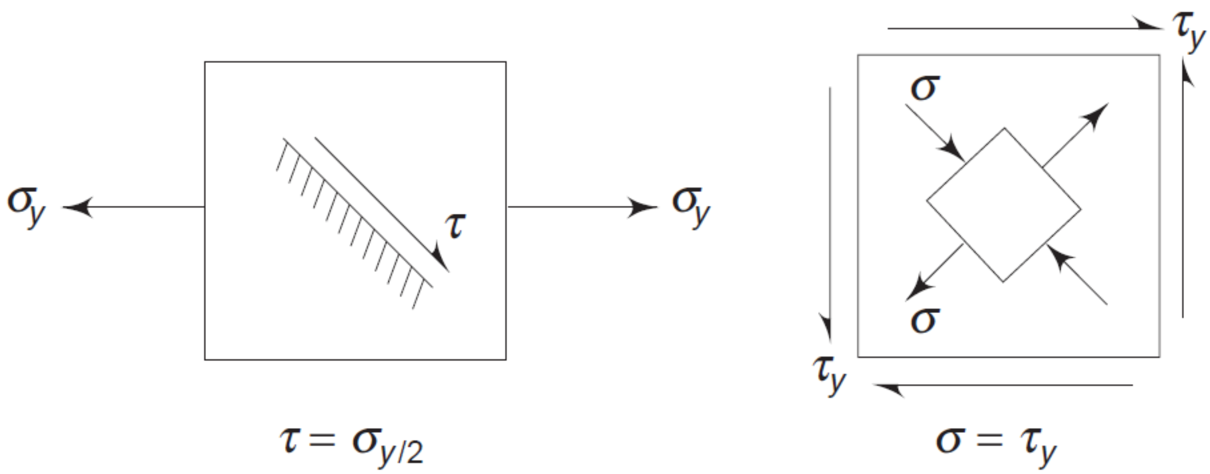


Figure: Uniaxial and pure shear state of stress

If one uses the yield point shear stress *ty* obtained from a pure torsion test, then the critical value associated with each theory of failure is as follows:

1. Maximum Normal Stress Theory

According to this theory, failure occurs when the normal stress s at any point in the stressed member reaches a value $\sigma \geq \tau y$. This is because, in a pure torsion test when yielding occurs, the maximum normal stress σ is numerically equivalent to τy.

2. Maximum Shear Stress Theory

According to this theory, failure occurs when the shear stress τ at a point in the member reaches a value $\tau \geq \tau y$

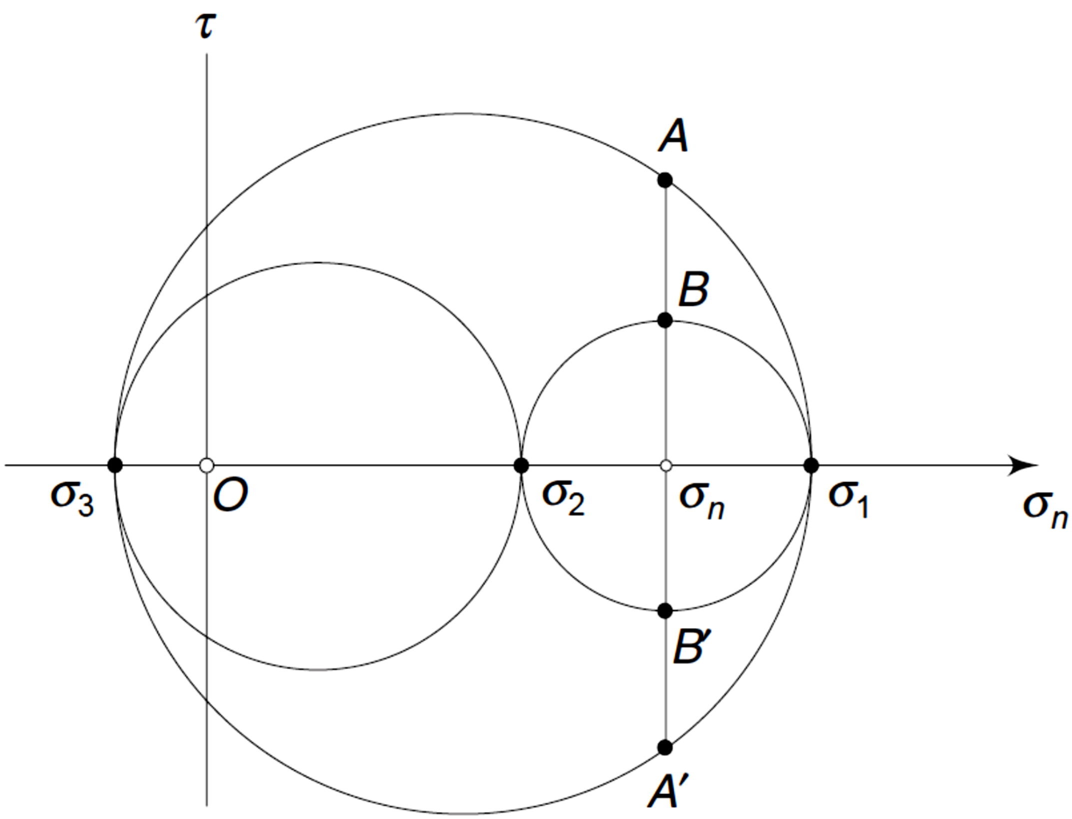
Failure theory	*Tension*	*Shear*	*Relationship*
Max. normal stress	σ_y	$\sigma_y = \tau_y$	$\tau_y = \sigma_y$
Max. shear stress	$\tau = \frac{1}{2}\sigma_y$	τ_y	$\tau_y = 0.5\,\sigma_y$
Max. strain $\left(\nu = \frac{1}{4}\right)$	$\varepsilon = \frac{1}{E}\sigma_y$	$\varepsilon = \frac{5}{4}\frac{\tau_y}{E}$	$\tau_y = 0.8\,\sigma_y$
Octahedral shear	$\tau_{\text{oct}} = \frac{\sqrt{2}}{3}\sigma_y$	$\tau_{\text{oct}} = \sqrt{\frac{2}{3}}\,\tau_y$	$\tau_y = 0.577\,\sigma_y$
Max. energy $\left(\nu = \frac{1}{4}\right)$,	$U = \frac{1}{2E}\sigma_y^2$	$U = \frac{5}{4}\frac{1}{E}\tau_y^2$	$\tau_y = 0.632\,\sigma_y$
Distortion energy	$U^* = \frac{1+\nu}{3}\frac{\sigma_y^2}{E}$	$U^* = (1+\nu)\frac{\tau_y^2}{E}$	$\tau_y = 0.577\,\sigma_y$

In designing a member to carry a given load without failure, usually a factor of safety *N* is used. The purpose is to design the member in such a way that it can carry *N* times the actual working load without failure. It has been observed that one can associate different factors for failure according to the particular theory of failure adopted.

Explain the Mohr's Theory of Failure

All the theories of failure had one common feature, that is the criterion of failure is unaltered by a reversal of sign of the stress. While the yield point stress *sy* for a ductile material is more or less the same in tension and compression, this is not true for a brittle material. In such a case, according to the maximum shear stress theory, we would get two different values for the critical shear stress. Mohr's theory is an attempt to extend the maximum shear stress theory (also known as the stress-difference theory) so as to avoid this objection.

To explain the basis of Mohr's theory, consider Mohr's circles, for a general state of stress.



$\sigma1$, $\sigma2$ and $\sigma3$ are the principal stresses at the point. Consider the line *ABB'A'* . The points lying on *BA* and *B'A'* represent a series of planes on which the normal stresses have the same magnitude σn but different shear stresses. The maximum shear stress associated with this normal stress value is τ, represented by point *A* or *A'*. The fundamental assumption is that if failure is associated with a given normal stress value, then the plane having this normal stress and a maximum shear stress accompanying it, will be the critical plane. Hence, the critical point for the normal stress σn will be the point *A*. From Mohr's circle diagram, the planes having maximum shear stresses for given normal stresses, have their representative points on the outer circle. Consequently, as far as failure is concerned, the critical circle is the outermost circle in Mohr's circle diagram, with diameter ($\sigma1$ – $\sigma3$).

Now, on a given material, we conduct three experiments in the laboratory, relating to simple tension, pure shear and simple compression. In each case, the test is conducted until failure occurs. In simple tension, $\sigma 1 = \sigma yt$, $\sigma2 = \sigma3 = 0$. The outermost circle in the circle diagram (there is only one circle) corresponding to this state is shown as *T* in Fig. below.

The plane on which failure occurs will have its representative point on this outer circle. For pure shear, $\tau ys = \sigma1 = -\sigma3$ and $\sigma2 = 0$. The outermost circle for this state is indicated by *S*. In simple compression, $\sigma1 = \sigma2 = 0$ and $\sigma3 = -\sigma yc$. In general, for a brittle material, σyc will be greater than σyt numerically. The outermost circle in the circle diagram for this case is represented by *C*.

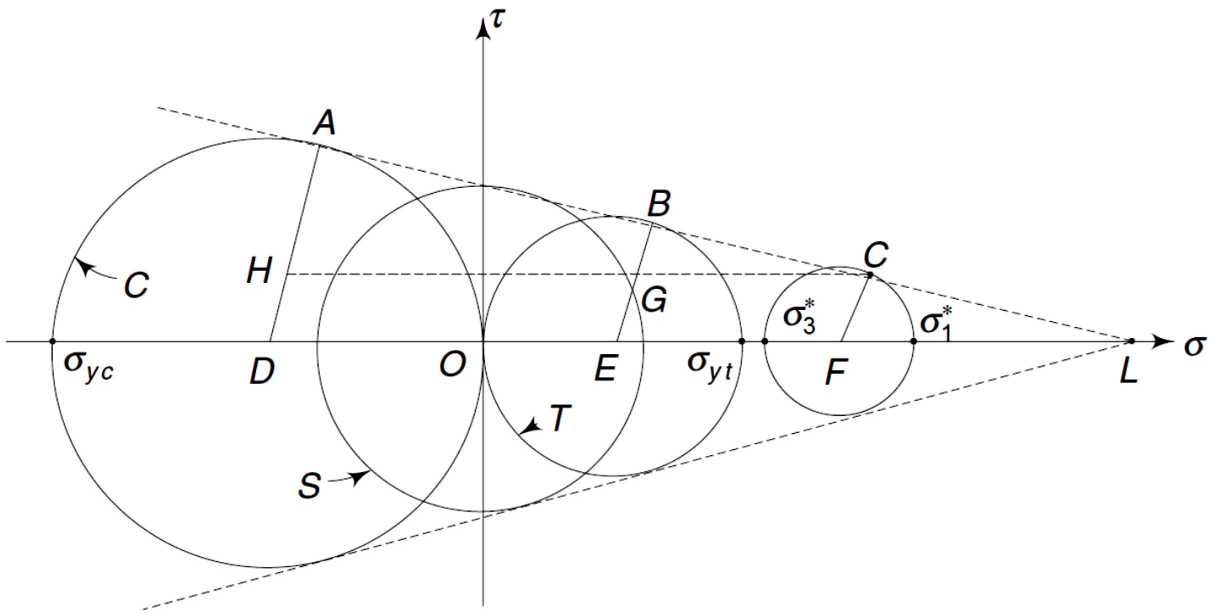
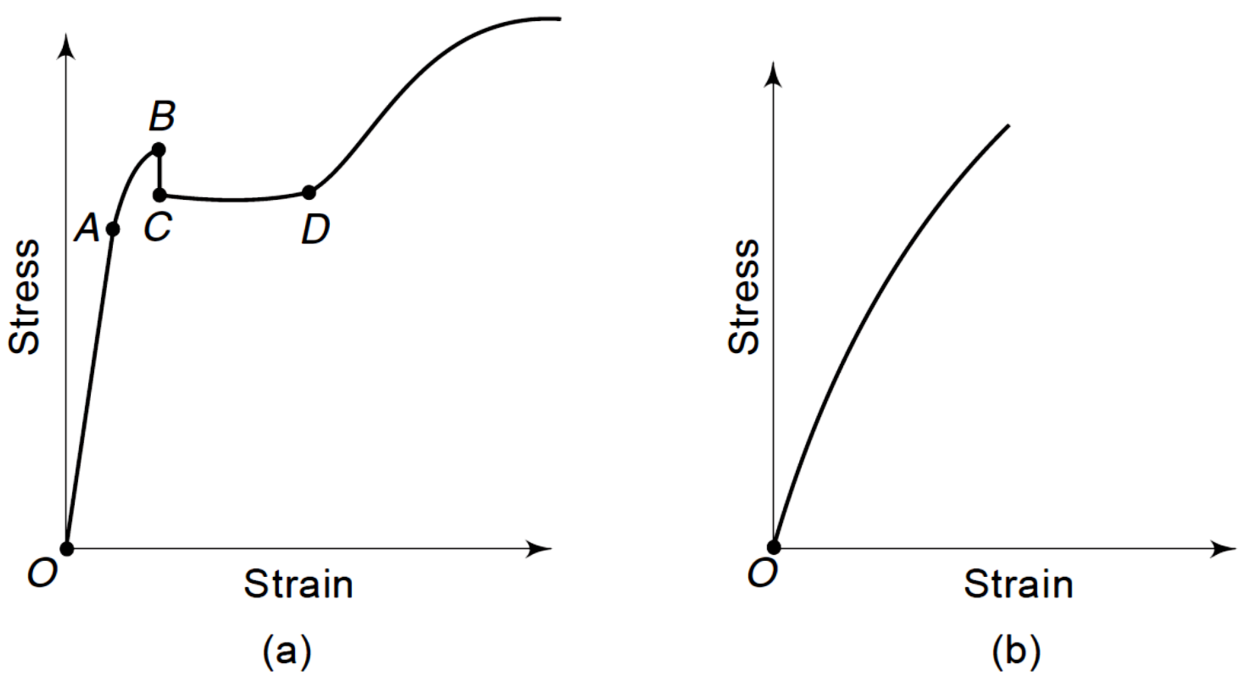


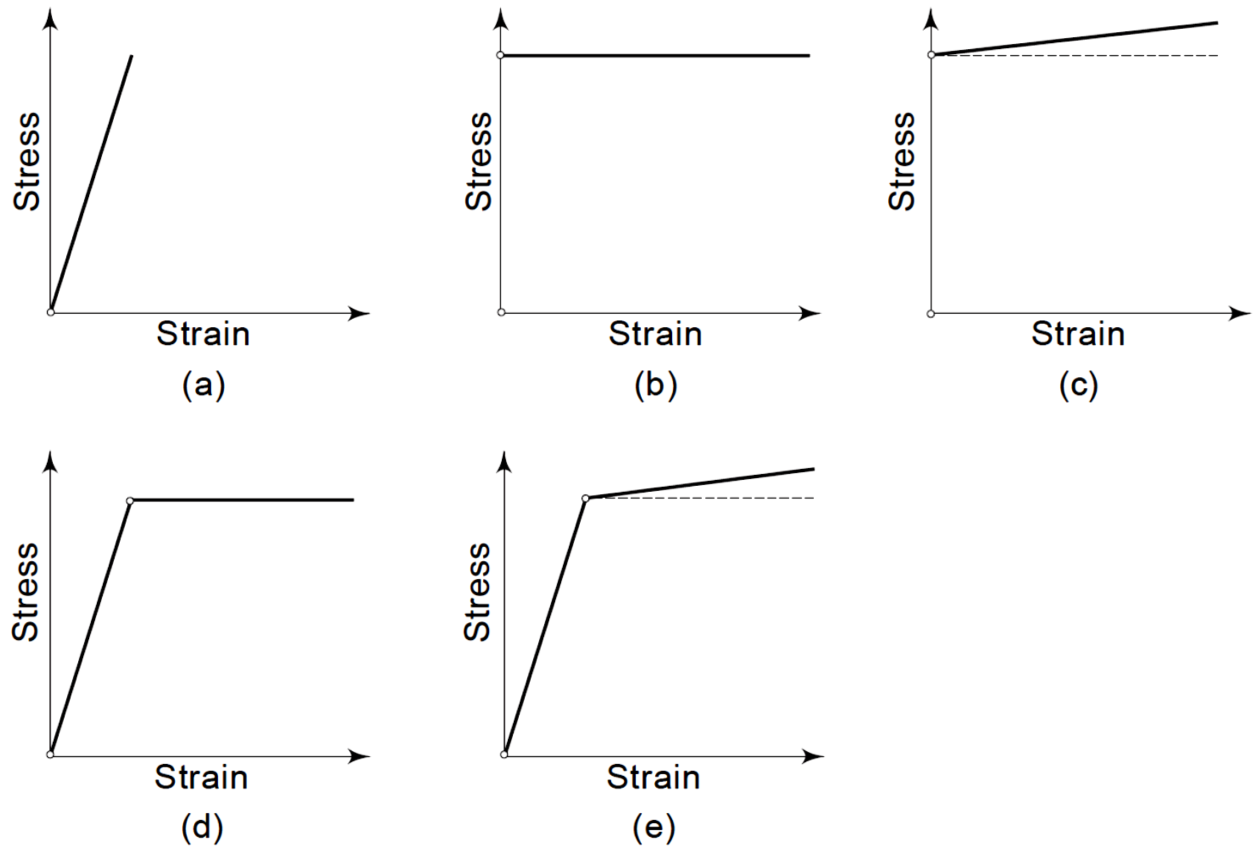
Diagram representing Mohr's failure theory

In addition to the three simple tests, we can perform many more tests (like combined tension and torsion) until failure occurs in each case, and correspondingly for each state of stress, we can construct the outermost circle. For all these circles, we can draw an envelope. The point of contact of the outermost circle for a given state with this envelope determines the combination of σ and τ, causing failure. Obviously, a large number of tests will have to be performed on a single material to determine the envelope for it.

Stress–strain diagram for (a) Ductile material (b) Brittle material



In order to develop stress-strain relations during plastic deformation, the actual stress-strain diagrams are replaced by less complicated ones. These are shown in Fig. below.



Ideal stress-strain diagram for a material that is (a) Linearly elastic (b) Rigid perfectly plastic (c) Rigid-linear work hardening (d) Linearly elastic-perfectly plastic (e) Linearly elastic-linear work hardening

In these, Fig. (a) represents a linearly elastic material, while Fig. (b) represents a material which is rigid (i.e. has no deformation) for stresses below σy and yields without limit when the stress level reaches the value σy. Such a material is called a rigid perfectly plastic material. Figure (c) shows the behaviour of a material which is rigid for stresses below σy and for stress levels above σy a linear work hardening characteristics is exhibited. A material exhibiting this characteristic behaviour is designated as rigid linear work hardening. Figure (d) and (e) represent respectively linearly elastic - perfectly plastic and linearly elastic–linear work hardening.

<u>What is Stress Space And Strain Space?</u>

The state of stress at a point can be represented by the six rectangular stress components τij (i, j = 1, 2, 3). One can imagine a six-dimensional space called the stress space, in which the state of stress can be represented by a point. Similarly, the state of strain at a point can be represented by a point in a six- dimensional strain space. In particular, a state of plastic strain ε_p can be so represented. A history of loading can be represented by a path in the stress space and the corresponding deformation or strain history as a path in the strain space.

Stress–Strain Relations (Plastic Flow)
Or
Plastic Stress-strain relations (Prandtl–Reuss Equations)

When a stress point reaches this boundary of yield surface, plastic deformation takes place. In this context, one can speak of only the change in the plastic strain rather than the total plastic strain because the latter is the sum total of all plastic strains that have taken place during the previous strain history of the specimen. Consequently, the stress–strain relations for plastic flow relate the strain increments. Another way of explaining this is to realise that the process of plastic flow is irreversible; that most of the deformation work is transformed into heat and that the stresses in the final state depend on the strain path.

Consequently, the equations governing plastic deformation cannot, in principle, be finite relations concerning stress and strain components as in the case of Hooke's law, but must be differential relations.

The following assumptions are made:

(i) The body is isotropic

(ii) The volumetric strain is an elastic strain and is proportional to the mean pressure

(iii) The total strain increments are made up of the elastic strain increments and plastic strain increments

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p} \quad \text{--------(1)}$$

(iv) The elastic strain increments are related to stress components through Hooke's law

$$d\varepsilon_{xx}^{e} = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$d\varepsilon_{yy}^{e} = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$d\varepsilon_{zz}^{e} = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$d\varepsilon_{xy}^{e} = d\gamma_{xy}^{e} = \frac{1}{G}\tau_{xy}$$

$$d\varepsilon_{yz}^{e} = d\gamma_{yz}^{e} = \frac{1}{G}\tau_{yz}$$

$$d\varepsilon_{zx}^{e} = d\gamma_{zx}^{e} = \frac{1}{G}\tau_{zx} \quad \text{---------------------(2)}$$

(v) The deviatoric components of the plastic strain increments are proportional to the components of the deviatoric state of stress

$$d\varepsilon_{ij}^{p} = d\lambda\, s_{ij} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3)}$$

Equations (1), (2) and (3) constitute the Prandtl–Reuss equations.

***************THE END*************

THEORY OF ELASTICITY AND PLASTICITY

HAND BOOK

Dr. V S Reddy

- **Explain St.Venant's Theory using a suitable example of torsional problem.**

  Or

- **Using saint venant semi inverse method for the problem of Torsion of straight bars derive the solution.**

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = 0$$

$$\gamma_{yz} = \theta\left(\frac{\partial \psi}{\partial y} + x\right)$$

$$\gamma_{zx} = \theta\left(\frac{\partial \psi}{\partial x} - y\right)$$

$$u_x = -\theta yz \qquad u_y = \theta xz \qquad u_z = \theta\psi(x, y)$$

$$u_x = -r\theta z \sin\beta \qquad u_y = r\theta z \cos\beta$$

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{yz} = G\theta\left(\frac{\partial \psi}{\partial y} + x\right)$$

$$\tau_{zx} = G\theta\left(\frac{\partial \psi}{\partial x} - y\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$$

$$G\theta\left(\frac{\partial \psi}{\partial x} - y\right)n_x + G\theta\left(\frac{\partial \psi}{\partial y} + x\right)n_y = 0$$

$$\iint_R \tau_{zx}\, dx\, dy = \iint_R \tau_{yz}\, dx\, dy = 0$$

$$T = \iint_R (\tau_{yz} x - \tau_{zx} y)\, dx\, dy$$

$$= G\theta \iint_R \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right) dx\, dy$$

Writing J for the integral

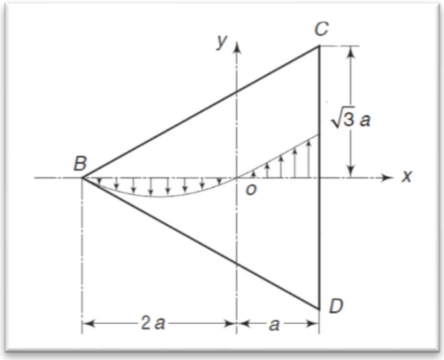
$$J = \iint_R \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right) dx\, dy$$

we have $\quad T = GJ\theta$

- **Establish the torsional moment carrying capacity of an equilateral triangle cross sectional bar.**

Consider the warping function

$$\psi = A(y^3 - 3x^2 y)$$



$$x - a = 0 \quad \text{on } CD$$
$$x - \sqrt{3}\,y + 2a = 0 \quad \text{on } BC$$
$$x + \sqrt{3}\,y + 2a = 0 \quad \text{on } BD$$

$$J = \frac{3}{5} I_p$$

$$\theta = \frac{T}{GJ} = \frac{5}{3}\frac{T}{GI_P}$$

The stress components are

$$\tau_{yz} = G\theta\left(\frac{\partial \psi}{\partial y} + x\right)$$
$$= G\theta\left(3Ay^2 - 3Ax^2 + x\right)$$
$$= \frac{G\theta}{2a}\left(x^2 - y^2 + 2ax\right)$$

and

$$\tau_{zx} = G\theta\left(\frac{\partial \psi}{\partial y} - y\right)$$
$$= \frac{G\theta\, y}{a}(x - a)$$

$$\tau_{max} = \frac{3G\theta a}{2}$$

- **Explain Theories of Failure and give the governing equations. Also explain the limitations of those theories.**

Or

- **Explain the different theories failure and write yield criterion for each.**

Limitations:

- Out of the four theories, only the maximum normal stress theory predicts failure for brittle materials.
- The rest of the three theories are applicable for ductile materials. Out of these three, the distortion energy theory provides most accurate results in majority of the stress conditions. The strain energy theory needs the value of Poisson's ratio of the part material, which is often not readily available. The maximum shear stress theory is conservative.
- For simple unidirectional normal stresses all theories are equivalent, which means all theories will give the same result.

Theories of failure

- Max. principal stress theory – Rankine
- Max. principal strain theory – St. Venants
- Max. strain energy – Beltrami
- Distortional energy – von Mises
- Max. shear stress theory – Tresca
- Octahedral shear stress theory

- **<u>Max. principal stress theory or normal stress theory (</u>(Rankine's theory)**

1. According to this theory, the maximum principal stress in the material determines failure the other two principal stresses are algebraically smaller.
2. This theory is not much supported by experimental results.
3. A pure state of hydrostatic pressure $[\sigma_1 = \sigma_2 = \sigma_3 = -p\ (p > 0)]$ cannot produce permanent deformation in compact crystalline or amorphous solid materials but produces only a small elastic contraction. This contradicts the maximum principal stress theory.
4. The maximum principal stress theory cannot be a good criterion for failure.

If the principal stresses σ_1, σ_2 and σ_3 are arranged such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at the point will be

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \qquad (1.63a)$$

In the *xy* plane, the maximum shear stress will be

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) \qquad (1.63b)$$

Thus, if $\sigma 1 > \sigma 2 > \sigma 3$ are the principal stresses at a point and σy the yield stress or tensile elastic limit for the material under a uniaxial test, then failure occurs when

$$\sigma_1 \geq \sigma_y$$

- **<u>Max. shear stress theory (Tresca or Guest's Theory)</u>**

1. Assuming $\sigma_1 > \sigma_2 > \sigma_3$, yielding, according to this theory, occurs when the maximum shearing stress reaches a critical value.
2. The maximum shearing stress theory is accepted to be fairly well justified for ductile materials.
3. However, as remarked earlier, for ductile load carrying members where large shears occur and which are subject to unequal triaxial tensions, the maximum shearing stress theory is used because of its simplicity.
4. If $\sigma_1 > \sigma_2 > \sigma_3$ are the three principal stresses at a point, failure occurs when

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_y}{2}$$

where $\sigma_y/2$ is the shear stress at yield point in a uniaxial test.

- **<u>Max. principal strain theory</u> <u>(Saint Venant's Theory)</u>**

1) According to this theory, failure occurs at a point in a body when the maximum strain at that point exceeds the value of the maximum strain in a uniaxial test of the material at yield point.
2) Thus, if σ_1, σ_2 and σ_3 are the principal stresses at a point, failure occurs when

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \geq \frac{\sigma_y}{E}$$

3) We have observed that a material subjected to triaxial compression does not suffer failure, thus contradicting this theory. Also, in a block subjected to a biaxial tension, as shown in fig
   the principal strain ε_1 is

   $\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$ and is smaller than σ_1/E because of σ_2.
4) Therefore, according to this theory, σ_1 can be increased more than σ_y without causing failure, whereas, if σ_2 were compressive, the magnitude of σ_1 to cause failure would be less than σ_y. However, this is not supported by experiments.
5) While the maximum strain theory is an improvement over the maximum stress theory, it is not a good theory for ductile materials.
6) For materials which fail by brittle fracture, one may prefer the maximum strain theory to the maximum stress theory.

- **<u>Distortional energy theory (von-Mises theory) or (von Mises-Hencky's theory)</u>**

1) According to this theory, it is not the total energy which is the criterion for failure; in fact the energy absorbed during the distortion of an element is responsible for failure.
2) The energy of distortion can be obtained by subtracting the energy of volumetric expansion from the total energy. It was known that any given state of stress can be uniquely resolved into an isotropic state and a pure shear (or deviatoric) state. σ_1, σ_2 and σ_3 are the principal stresses at a point.

The expression for the energy of distortion.

$$U^* = \frac{1}{6G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right)$$

or

$$U^* = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

In a uniaxial test, the energy of distortion is equal to $\frac{1}{6G}\sigma_y^2$.

Hence, according to the distortion energy theory, failure occurs at that point where σ_1, σ_2 and σ_3 are such that

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2\sigma_y^2 \qquad (4.13)$$

But we notice that the expression for the octahedral shearing stress from Eq. (1.22) is

$$\tau_{\text{oct}} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

Hence, the distortion energy theory states that failure occurs when

$$9\tau_{\text{oct}}^2 = \geq 2\sigma_y^2$$

or

$$\tau_{\text{oct}} = \geq \frac{\sqrt{2}}{3}\sigma_y$$

Therefore, the octahedral shearing stress theory and the distortion energy theory are identical.

- **Maximum Strain energy theory (Beltrami and Heigh's Thoery)**

1) According to this theory, failure at any point in a body subject to a state of stress begins only when the energy per unit volume absorbed at the point is equal to the energy absorbed per unit volume by the material when subjected to the elastic limit under a uniaxial state of stress.
2) The energy *U* per unit volume is

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

In a uniaxial test, the energy stored per unit volume at yield point or elastic limit is $1/2E\,\sigma_y^2$. Hence, failure occurs when

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \geq \sigma_y^2$$

3) This theory does not have much significance since it is possible for a material to absorb considerable amount of energy without failure or permanent deformation when it is subjected to hydrostatic pressure.

- **Octahedral Shearing Stress Theory**

1) According to this theory, the critical quantity is the shearing stress on the octahedral plane. The plane which is equally inclined to all the three principal axes *Ox*, *Oy* and *Oz* is called the octahedral plane. The normal to this plane has direction cosines n_x, n_y and $n_z = 1/\sqrt{3}$. The tangential stress on this plane is the octahedral shearing stress.
2) The normal and shearing stresses on these planes are called the octahedral normal stress and octahedral shearing stress respectively. If σ_1, σ_2 and σ_3 are the principal stresses at a point, then

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}I_1 \quad (1.43)$$

and

$$\tau_{oct}^2 = \frac{1}{9}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad (1.44a)$$

or

$$9\tau_{oct}^2 = 2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \quad (1.44b)$$

or

$$\tau_{oct} = \frac{\sqrt{2}}{3}(I_1^2 - 3I_2)^{1/2} \quad (1.44c)$$

It is important to remember that the octahedral planes are defined with respect to the principal axes and not with reference to an arbitrary frame of reference.

In a uniaxial test, at yield point, the octahedral stress $(\sqrt{2}/3)\ \sigma_y = 0.47\sigma_y$. Hence, according to the present theory, failure occurs at a point where the values of principal stresses are such that

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \geq \frac{\sqrt{2}}{3}\sigma_y \qquad (4.4a)$$

or

$$\left(I_1^2 - 3I_2\right) \geq \sigma_y^2 \qquad (4.4b)$$

3) This theory is supported quite well by experimental evidences. This theory is equivalent to the maximum distortion energy theory.

- **Explain saint venant's Semi inverse method. Apply the same to an elliptical cross section and obtain shear stress and displacements in the cross section.**

Or

Derive the equations for twisting moment and shear stresses in straight bars of non-circular cross sections. Hence evaluate the same for an elliptical cross section.

Saint venant's Semi inverse method:

Or

Equations for twisting moment and shear stresses in straight bars of non-circular cross sections

Or

Derive using St. Venants semi inverse method the stress function for Torsion of non circular shafts and obtain Twisting moment in term of this stress function. Hence apply this to an elliptic c/s and obtain distribution of shear stresses in a c/s.

$$u_x = -r\theta z \sin\beta$$
$$u_y = r\theta z \cos\beta$$

$$u_x = -\theta yz$$
$$u_y = \theta xz$$
$$u_z = \theta\psi(x, y)$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = 0$$
$$\gamma_{yz} = \theta\left(\frac{\partial\psi}{\partial y} + x\right)$$
$$\gamma_{zx} = \theta\left(\frac{\partial\psi}{\partial x} - y\right)$$

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)$$

$$\tau_{zx} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right)$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi = 0$$

$$G\theta\left(\frac{\partial\psi}{\partial x} - y\right)n_x + G\theta\left(\frac{\partial\psi}{\partial y} + x\right)n_y = 0$$

$$\iint_R \tau_{zx}\, dx\, dy = \iint_R \tau_{yz}\, dx\, dy = 0$$

$$T = \iint_R (\tau_{yz} x - \tau_{zx} y)\, dx\, dy$$

$$= G\theta \iint_R \left(x^2 + y^2 + x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x}\right) dx\, dy$$

Writing J for the integral

$$J = \iint_R \left(x^2 + y^2 + x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x}\right) dx\, dy$$

we have $T = GJ\theta$

Elliptical cross section:

$$\psi = Axy$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi = 0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{a^2}{b^2} = \frac{1-A}{1+A}$$

$$A = \frac{b^2 - a^2}{b^2 + a^2}$$

$$\psi = \frac{b^2 - a^2}{b^2 + a^2}\, xy$$

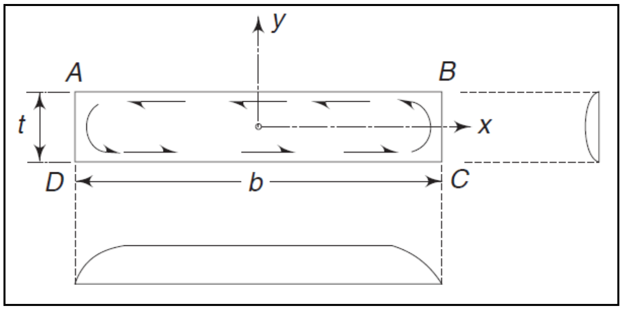
$$T = GJ\theta = G\theta \frac{\pi a^3 b^3}{a^2 + b^2}$$

$$\theta = \frac{T}{G}\frac{a^2 + b^2}{\pi a^3 b^3}$$

$$\tau_{yz} = \frac{2Tx}{\pi a^3 b} \qquad \tau_{zx} = \frac{2Ty}{\pi a b^3}$$

$$\tau_{max} = \frac{2T}{\pi a^3 b^3}(a^4 b^2)^{1/2} = \frac{2T}{\pi a b^2}$$

- **How is membrane analogy applied to a problem of torsion in non-circular shafts, evaluate shear stress in a narrow rectangular section and apply the same to twist in rolled profiled steel sections.**



$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$$

$$\phi = G\theta\left(\frac{t^2}{4} - y^2\right)$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = 0$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = -2G\theta\, y$$

$$(\tau_{zx})_{\max} = \pm\, G\theta\, t$$

$$T = 2\iint \phi\, dx\, dy$$

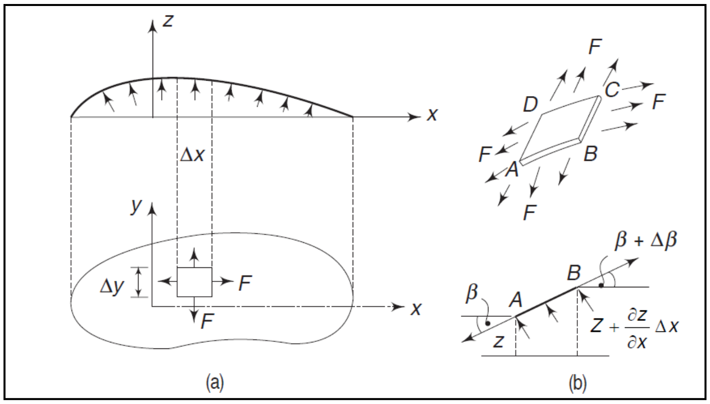
$$T = \frac{1}{3} bt^3\, G\theta$$

$$\theta = \frac{1}{G}\frac{3T}{bt^3}, \qquad \tau_{zx} = -\frac{6T}{bt^3} y, \qquad (\tau_{zx})_{\max} = \pm\frac{3T}{bt^2}$$

- **Explain soap film method or membrane analogy method**

  Or

- **Explain membrane analogy for a obtaining behaviour of non circular shafts under torsion.**

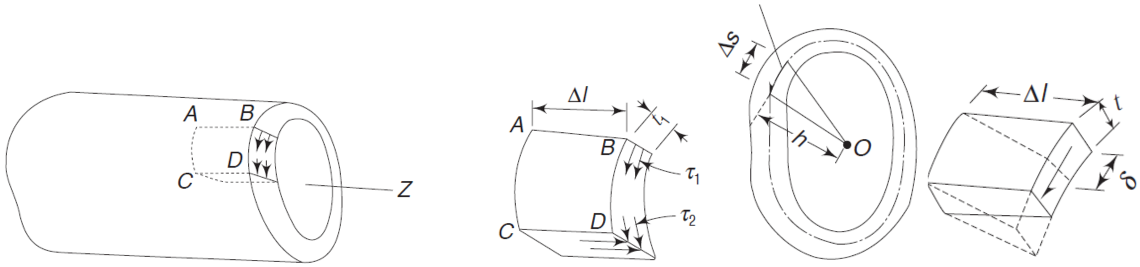


$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{F}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = -2G\theta$$

Force p acting upward on the membrane element ABCD. F be the uniform tension per unit length of the membrane.

Short notes on Torsion of thin tubes



$$T = \Sigma\, 2q\, \Delta A = 2qA$$

Generally known as the Bredt–Batho formula.

The total elastic strain energy is therefore

$$U = \frac{T^2\, \Delta l}{8A^2 G} \oint \frac{ds}{t}$$

Hence, the twist or the rotation per unit length ($\Delta l = 1$) is

$$\theta = \frac{\partial U}{\partial T} = \frac{T}{4A^2 G} \oint \frac{ds}{t}$$

Explain about Yield criteria

Plastic yielding of the material subjected to any external forces is of considerable importance in the field of plasticity. For predicting the onset of yielding in ductile material, there are at present two generally accepted criteria,

1) Von Mises' or Distortion-energy criterion
2) Tresca or Maximum shear stress criterion

- **<u>Distortional energy theory (von-Mises theory) or (von Mises-Hencky's theory)</u>**

3) According to this theory, it is not the total energy which is the criterion for failure; in fact the energy absorbed during the distortion of an element is responsible for failure.
4) The energy of distortion can be obtained by subtracting the energy of volumetric expansion from the total energy. It was known that any given state of stress can be uniquely resolved into an isotropic state and a pure shear (or deviatoric) state. σ_1, σ_2 and σ_3 are the principal stresses at a point.

The expression for the energy of distortion.

$$U^* = \frac{1}{6G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right)$$

or

$$U^* = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

In a uniaxial test, the energy of distortion is equal to $\frac{1}{6G}\sigma_y^2$.

Hence, according to the distortion energy theory, failure occurs at that point where σ_1, σ_2 and σ_3 are such that

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2\sigma_y^2 \tag{4.13}$$

But we notice that the expression for the octahedral shearing stress from Eq. (1.22) is

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

Hence, the distortion energy theory states that failure occurs when

$$9\tau_{oct}^2 = \geq 2\sigma_y^2$$

or

$$\tau_{oct} = \geq \frac{\sqrt{2}}{3}\sigma_y$$

Therefore, the octahedral shearing stress theory and the distortion energy theory are identical.

- **<u>Max. shear stress theory (Tresca or Guest's Theory)</u>**

5. Assuming $\sigma_1 > \sigma_2 > \sigma_3$, yielding, according to this theory, occurs when the maximum shearing stress reaches a critical value.
6. The maximum shearing stress theory is accepted to be fairly well justified for ductile materials.

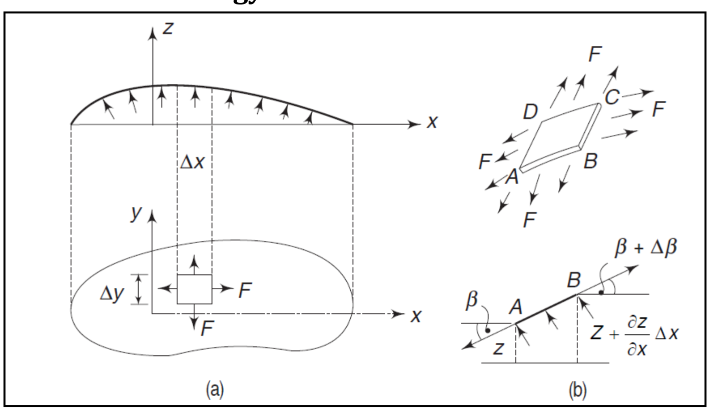
7. However, as remarked earlier, for ductile load carrying members where large shears occur and which are subject to unequal triaxial tensions, the maximum shearing stress theory is used because of its simplicity.
8. If $\sigma_1 > \sigma_2 > \sigma_3$ are the three principal stresses at a point, failure occurs when

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_y}{2}$$

where $\sigma_y/2$ is the shear stress at yield point in a uniaxial test.

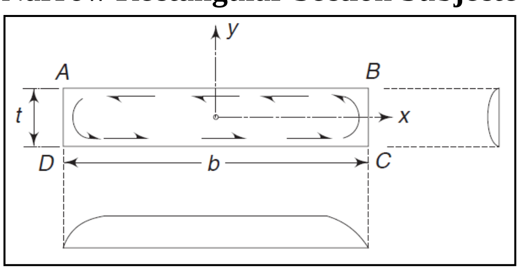
- **Explain membrane analogy .Apply this analogy to the problem of bar of narrow rectangular section subjected to torsion and calculate the stresses.**
  or
  Explain membrane analogy for torsion of prismatic shafts. Hence obtain solution to the problem of torsion. Hence obtain solution to the problem of a bar with narrow rectangular cross section.

1) Membrane Analogy



$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{F}$$

2) Narrow Rectangular Section Subjected To Torsion



$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$$

$$\phi = G\theta\left(\frac{t^2}{4} - y^2\right)$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = 0$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = -2G\theta\, y$$

$$(\tau_{zx})_{\max} = \pm\, G\theta\, t$$

$$T = 2\iint \phi\, dx\, dy$$

$$T = \frac{1}{3} bt^3\ G\theta$$

$$\theta = \frac{1}{G}\frac{3T}{bt^3}, \qquad \tau_{zx} = -\frac{6T}{bt^3} y, \qquad (\tau_{zx})_{\max} = \pm\frac{3T}{bt^2}$$

⌖ **Write the assumptions of plasticity.**

In formulating a basic plasticity theory the following assumptions are usually made:
(1) the response is independent of rate effects
(2) the material is incompressible in the plastic range
(3) there is no Bauschinger effect
(4) the yield stress is independent of hydrostatic pressure
(5) the material is isotropic

⌖ **Explain Saint Venant's semi inverse method for evaluation of torsion in prismatic shafts. Hence calculate torsional moment and shear stresses in terms of stress function.**

Saint Venant's semi inverse method for evaluation of torsion in prismatic shafts:

(Already answered- Check)

Calculate torsional moment and shear stresses in terms of stress function:

(Prandtl's torsion stress function)

$$\frac{\partial \tau_{zx}}{\partial z} = 0, \quad \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

$$\gamma_{yz} = -\frac{1}{G}\frac{\partial \phi}{\partial x}, \quad \text{and} \quad \gamma_{zx} = \frac{1}{G}\frac{\partial \phi}{\partial y}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = a \text{ constant } F$$

$$\frac{d\phi}{ds} = 0$$

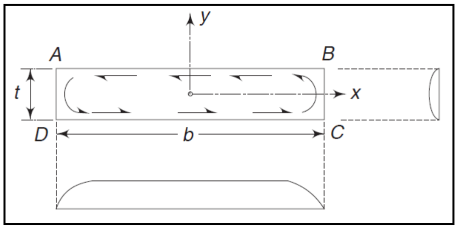
$$T = \iint_R (x\tau_{zy} - y\tau_{zx})\ dx\, dy$$

$$T = 2\iint \phi\ dx\, dy$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = -2G\theta$$

- **Calculate shear stresses and twisting moment in a narrow rectangular section. Obtain the same for a rolled profile section.**

Shear stresses and twisting moment in a narrow rectangular section:



$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$$

$$\phi = G\theta\left(\frac{t^2}{4} - y^2\right)$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = 0$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = -2G\theta\, y$$

$$(\tau_{zx})_{\max} = \pm\, G\theta\, t$$

$$T = 2\iint \phi\, dx\, dy$$

$$T = \frac{1}{3} bt^3\, G\theta$$

$$\theta = \frac{1}{G}\frac{3T}{bt^3}, \qquad \tau_{zx} = -\frac{6T}{bt^3} y, \qquad (\tau_{zx})_{\max} = \pm\frac{3T}{bt^2}$$

Rolled profile section:

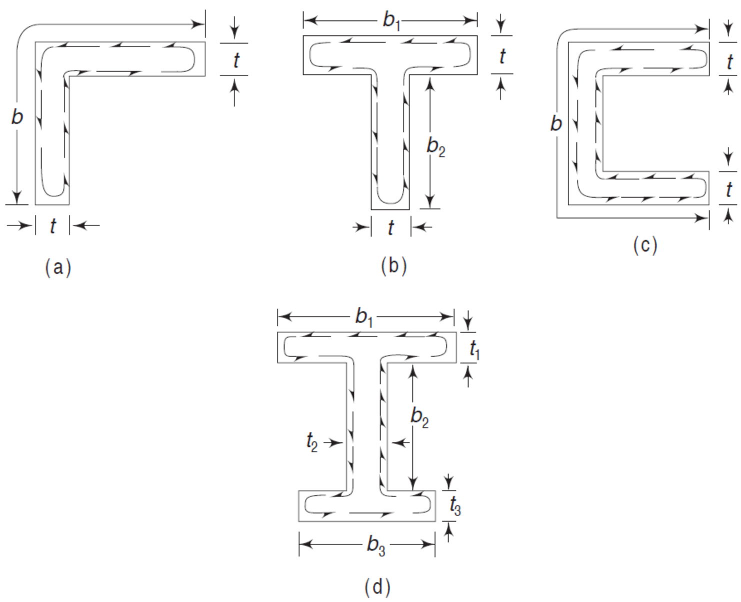


Fig. 7.18 *Torsion of rolled sections*

$$\phi = G\theta\left(\frac{t_i^2}{4} - y^2\right) \qquad (i = 1, 2 \text{ or } 3)$$

$$(\tau_{zx})_{\max} = \pm\, G\theta\, t$$

$$T = 2\iint \phi\, dx\, dy$$

$$T = \frac{1}{3} bt^3 \, G\theta$$

$$T = \frac{1}{3} G\theta(b_1 t_1^3 + b_2 t_2^3 + b_3 t_3^3)$$

$$\theta = \frac{1}{G}\frac{3T}{bt^3}, \qquad \tau_{zx} = -\frac{6T}{bt^3} y, \qquad (\tau_{zx})_{\max} = \pm\frac{3T}{bt^2}$$

- **If a cantilever beam is subjected to point load at the free end calculate shear stresses if the cross section is circular.**

  Or

- **Evaluate shear stresses in a rectangular section of a cantilever beam loaded at the free end.**

  Or

- **Evaluate shear stresses in a cantilever bar with a point load at the force end. Obtain stresses variation in the cross section if the bar is circular in section.**

⌖ Define warping.

The theory of torsion presented here concerns **torques (**the term torque is usually used instead of moment in the context of twisting shafts) which twists the members but which *do not induce any warping*, that is, cross sections which are perpendicular to the axis of the member remain so after twisting.

On the basis of the solution of circular shafts, we assume that the cross-sections rotate about an axis; the twist per unit length being θ. A section at distance z from the fixed end will, therefore, rotate through θz. A point $P(x, y)$ in this section will undergo a displacement $r\theta z$, as shown in Fig. 7.3. The components of this displacement are

$$u_x = -r\theta z \sin \beta$$
$$u_y = r\theta z \cos \beta$$

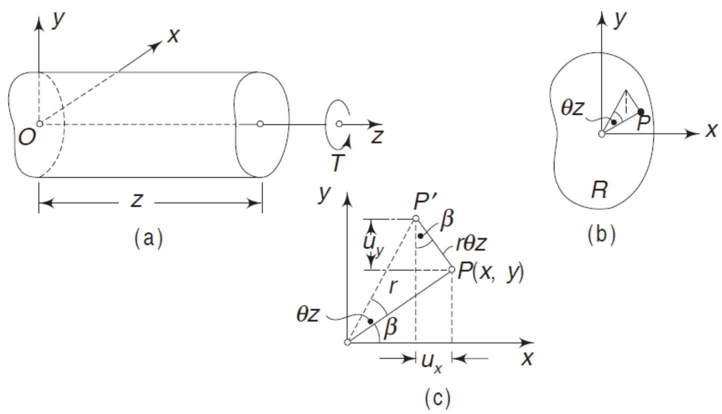
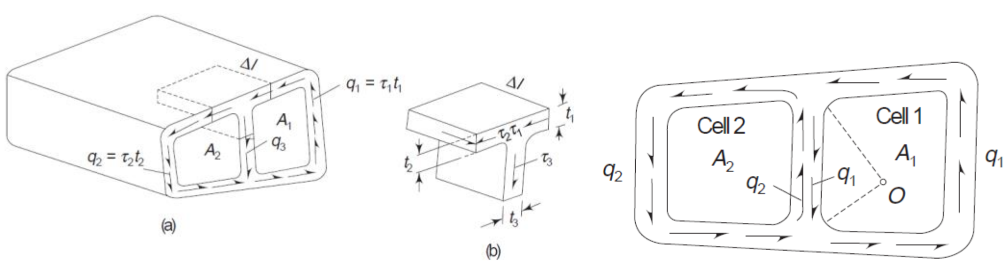


Fig. 7.3 *Prismatic bar under torsion and geometry of deformation*

In addition to these x and y displacements, the point P may undergo a displacement uz in z direction. This is called warping.

⌖ Torsion of hollow shaft (or) hollow sections or thin-walled multiple-cell closed sections



$$q_1 = q_2 + q_3$$

$$T_1 = 2q_1 A_1$$

$$T_2 = 2q_2 (A_2 + A_1^*) - 2q_2 A_1^*$$

$$T = T_1 + T_2 = 2q_1 A_1 + 2q_2 A_2$$

$$2G\theta = \frac{1}{A}\oint \frac{q\,ds}{t}$$

Then, for cell 1

$$2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2)$$

For cell 2

$$2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1)$$

solve for q_1, q_2 and θ.

APPLIED ELASTICITY & PLASTICITY

1. **BASIC EQUATIONS OF ELASTICITY**
   Introduction, The State of Stress at a Point, The State of Strain at a Point, Basic Equations of Elasticity, Methods of Solution of Elasticity Problems, Plane Stress, Plane Strain, Spherical Co-ordinates, Computer Program for Principal Stresses and Principal Planes.

2. **TWO-DIMENSIONAL PROBLEMS IN CARTESIAN CO-ORDINATES**
   Introduction, Airy's Stress Function – Polynomials : Bending of a cantilever loaded at the end ; Bending of a beam by uniform load, Direct method for determining Airy polynomial : Cantilever having Udl and concentrated load of the free end; Simply supported rectangular beam under a triangular load, Fourier Series, Complex Potentials, Cauchy Integral Method , Fourier Transform Method, Real Potential Methods.

3. **TWO-DIMENSIONAL PROBLEMS IN POLAR CO-ORDINATES**
   Basic equations, Biharmonic equation, Solution of Biharmonic Equation for Axial Symmetry, General Solution of Biharmonic Equation, Saint Venant's Principle, Thick Cylinder, Rotating Disc on cylinder, Stress-concentration due to a Circular Hole in a Stressed Plate (Kirsch Problem), Saint Venant's Principle, Bending of a Curved Bar by a Force at the End.

4. **TORSION OF PRISMATIC BARS**
   Introduction, St. Venant's Theory, Torsion of Hollow Cross-sections, Torsion of thin-walled tubes, Torsion of Hollow Bars, Analogous Methods, Torsion of Bars of Variable Diameter.

5. **BENDING OF PRISMATIC BASE**
   Introduction, Simple Bending, Unsymmetrical Bending, Shear Centre, Solution of Bending of Bars by Harmonic Functions, Solution of Bending Problems by Soap-Film Method.

6. **BENDING OF PLATES**
   Introduction, Cylindrical Bending of Rectangular Plates, Slope and Curvatures, Lagrange Equilibrium Equation, Determination of Bending and Twisting Moments on any plane, Membrane Analogy for Bending of a Plate, Symmetrical Bending of a Circular Plate, Navier's Solution for simply supported Rectangular Plates, Combined Bending and Stretching of Rectangular Plates.

7. **THIN SHELLS**
   Introduction, The Equilibrium Equations, Membrane Theory of Shells, Geometry of Shells of Revolution.

8. **NUMERICAL AND ENERGY METHODS**
   Rayleigh's Method, Rayleigh – Ritz Method, Finite Difference Method, Finite Element Method.

9. **HERTZ'S CONTACT STRESSES**
Introduction, Pressure between Two-Bodies in contact, Pressure between two-Spherical Bodies in contact, Contact Pressure between two parallel cylinders, Stresses along the load axis, Stresses for two Bodies in line contact Exercises.

10. **STRESS CONCENTRATION PROBLEMS**
Introduction, Stress-Concentration Factor, Fatigue Stress-Concentration Factors.

http://books.google.co.in/books?id=KzunZOFUWnoC&lpg=PP1&ots=PrfjDf51Uj&dq=advanced%20mechanics%20of%20solids%20by%20ls%20srinath&pg=PP1#v=onepage&q&f=false

Unit 1

BASIC EQUATIONS OF ELASTICITY

Structure

1.1. Introduction

Elasticity: All structural materials possess to a certain extent the property of *elasticity*, i.e., if external forces, producing *deformation* of a structure, do not exceed a certain limit, the deformation disappears with the removal of the forces. Throughout this book it will be assumed that the bodies undergoing the action of external forces are *perfectly elastic*, i.e., that they resume their initial form completely after removal of forces.

The molecular structure of elastic bodies will not be considered here. It will be assumed that the matter of an elastic body is homogeneous and continuously distributed over its volume so that the smallest element cut from the body possesses the same specific physical properties as the body. To simplify the discussion it will also be assumed that the body is *isotropic*, i.e., that the elastic properties are the same in all directions.

Structural materials usually do not satisfy the above assumptions. Such an important material as steel, for instance, when studied with a microscope, is seen to consist of crystals of various kinds and various orientations. The material is very far from being homogeneous, but experience shows that solutions of the theory of elasticity based on the assumptions of homogeneity and isotropy can be applied to steel structures with very great accuracy. The explanation of this is that the crystals are very small; usually there are millions of them in one cubic inch of steel. While the elastic properties of a single crystal may be very different in different directions, the crystals arc ordinarily distributed at random and the elastic properties of larger pieces of metal represent averages of properties of the crystals. So long as the geometrical dimensions defining the form of a body are large in comparison with the dimensions of a single crystal the assumption of homogeneity can be used with great accuracy, and if the crystals are orientated at random the material can be treated as isotropic.

When, due to certain technological processes such as rolling, a certain orientation of the crystals in a metal prevails, the elastic properties of the metal become different in different directions and the condition of *anisotropy* must be considered. We have such a condition, for instance, in the case of cold-rolled copper.

1.2. Objectives

After studying this unit we are able to understand

- The State of Stress at a Point
- The State of Strain at a Point

- Basic Equations of Elasticity
- Methods of Solution of Elasticity Problems
- Plane Stress
- Plane Strain
- Spherical Co-ordinates

1.3. The State of Stress at a Point

Knowing the stress components σ_x, σ_y, τ_{xy} at any point of a plate in a condition of plane stress or plane strain, thestress acting on any plane through this point perpendicular to the plateand inclined to the x- and y-axes can be calculated from the equations of statics. Let O be a point of the stressed plate and suppose the stress components σ_x, σ_y, τ_{xy} are known (Fig. 1).

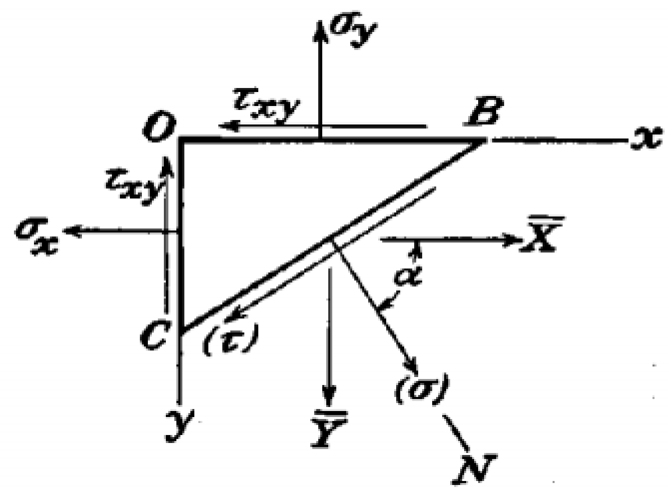


Fig. 1

To find the stress for anyplane through the z-axis and inclined to the x- and y-axes, we take a plane BC parallel to it, at a small distancefrom O, so that this latter planetogether with the coordinate planescuts out from the plate a very small triangular prism OBC. Since thestresses vary continuously over the volume of the body the stress actingon the plane BC will approach the stress on the parallel plane through$O$ as the element is made smaller.

In discussing the conditions of equilibrium of the small triangular prism, the body force can be neglected as a small quantity of a higher order. Likewise, if the element is very small, we can neglect the variation of the stresses over the sides and assume that the stresses are uniformly distributed. The forces acting on the triangular prism can therefore be determined by multiplying the stress components by the areas of the sides. Let *N* be the direction of the normal to the plane *BC*, and denote the cosines of the angles between the normal *N* and the axes *x* and *y* by

$$\cos Nx = l, \qquad \cos Ny = m$$

Then, if *A* denotes the area of the side *BC* of the element, the areas of the other two sides are *Al* and *Am*.

If we denote by *X* and □ the components of stress acting on the side *BC*, the equations of equilibrium of the prismatical element give

$$\begin{aligned} \bar{X} &= l\sigma_x + m\tau_{xy} \\ \bar{Y} &= m\sigma_y + l\tau_{xy} \end{aligned} \tag{1}$$

Thus the components of stress on any plane defined by direction cosines*l* and *m* can easily be calculated from Eqs. (1), provided thethree components of stress σ_x, σ_y, τ_{xy}at the point *O* are known.

Letting α be the angle between the normal *N* and the *x*-axis, so that $l = \cos \alpha$ and $m = \sin \alpha$, the normal and shearing components of stress on the plane *BC* are (from Eqs. 1)

$$\begin{aligned} \sigma &= \bar{X} \cos \alpha + \bar{Y} \sin \alpha = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha \\ &\qquad + 2\tau_{xy} \sin \alpha \cos \alpha \\ \tau &= \bar{Y} \cos \alpha - \bar{X} \sin \alpha = \tau_{xy}(\cos^2 \alpha - \sin^2 \alpha) \\ &\qquad + (\sigma_y - \sigma_x) \sin \alpha \cos \alpha \end{aligned} \tag{2}$$

It may be seen that the angle α can be chosen in such a manner that the shearing stress τbccomes equal to zero. For this case we have

$$\tau_{xy}(\cos^2 \alpha - \sin^2 \alpha) + (\sigma_y - \sigma_x) \sin \alpha \cos \alpha = 0$$

or

$$\frac{\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\sin \alpha \cos \alpha}{\cos^2 \alpha - \sin^2 \alpha} = \frac{1}{2} \tan 2\alpha \tag{3}$$

From this equation two perpendicular directions can be found for which the shearing stress is zero. These directions are called *principal directions* and the corresponding normal stresses *principal stresses*.

If the principal directions are taken as the x- and y-axes, τ_{xy} is zero and Eqs. (2) are simplified to

$$\begin{aligned} \sigma &= \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha \\ \tau &= \tfrac{1}{2} \sin 2\alpha(\sigma_y - \sigma_x) \end{aligned} \quad (4)$$

The variation of the stress components σ and τ, as we vary the angle α, can be easily represented graphically by making a diagram in which σ and τ are taken as coordinates. For each plane there will correspond a point on this diagram, the coordinates of which represent the values of σ and τ for this plane. Fig. 2 represents such a diagram. For the planes perpendicular to the principal directions we obtain points *A*and *B* with abscissas σ_x and σ_y respectively. Now it can be proved that the stress components for any plane *BC* with an angle α (Fig. 2)will be represented by coordinates of a point on the circle having*AB*as a diameter. To find this point it is only necessary to measure from the point A in the same direction as α is measured in Fig. 2 an arc subtending an angle equal to 2α. If *D*is the point obtained in this manner, then, from the figure,

$$\begin{aligned} OF &= OC + CF = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha \\ DF &= CD \sin 2\alpha = \tfrac{1}{2}(\sigma_x - \sigma_y) \sin 2\alpha \end{aligned}$$

Comparing with Eqs. (4) it is seen that the coordinates of point *D* give the numerical values of stress components on the plane *BC* at The angle α. To bring into coincidence the sign of the shearing component we take τ positive in the upward direction (Fig. 2) and consider shearing stresses as positive when they give a couple in the clockwise direction, as on the sides *bc* and *ad* of the element *abcd* (Fig. 2b). Shearingstresses of opposite direction, as on the sides*ab* and *dc* of the element,are considered as negative.

As the plane *BC* rotates about an axis perpendicular to the *xy*-plane (Fig. 1) in the clockwise direction, and a varies from 0 to $\pi/2$, the

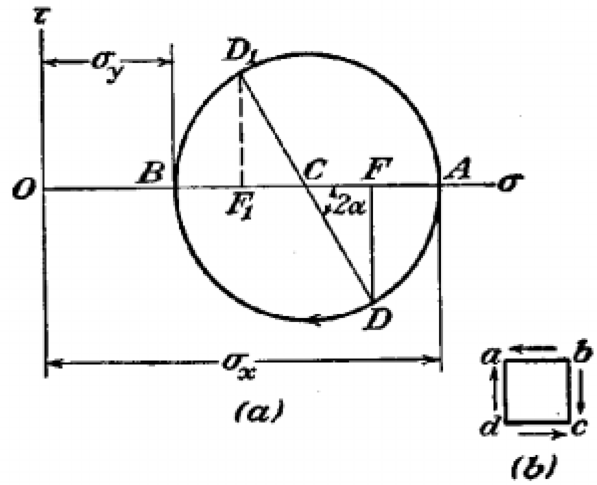


Fig. 2

point$D$ in Fig. 2 moves from A to B, so that the lower half circle determines the stress variation for all values of α within these limits. The upper half of the circle gives stresses for $\pi/2 \leq \alpha \leq \pi$.

Prolonging the radius CD to the point D_1 (Fig. 2), i.e., taking the angle $\pi + 2\alpha$, instead of 2α, the stresses on the plane perpendicular to BC (Fig. 1) are obtained. This shows that the shearing stresses on two perpendicular planes are numerically equal as previously proved. As for normal stresses, we see from the figure that $OF_1 + OF = 20C$, i.e., the sum of the normal stresses over two perpendicular cross sections remains constant when the angle α changes.

The maximum shearing stress is given in the diagram (Fig. 2) by the maximum ordinate of the circle, i.e., is equal to the radius of the circle. Hence

$$\tau_{max.} = \frac{\sigma_x - \sigma_y}{2}$$

It acts on the plane for which $\alpha = \pi/4$, i.e., on the plane bisecting the angle between the two principal stresses.

1.4. The State of Strain at a Point

When the strain components ϵ_x, ϵ_y, γ_{xy} at a point are known, the unit elongation for any direction, and the decrease of a right angle the shearing strain of any orientation at the pointcan be found. A line element PQ (Fig. 3a) between the points (x,y),$\{x + dx, y + dy)$ is translated, stretched (or contracted) and rotated into the line element $P'Q'$ when the deformation occurs. The displacement components of $P$are u, v, and those of Q are

$$u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

If *P'Q'* in Fig. 3a is now translated so that*P'* is brought back to P, it is in the position *PQ''* of Fig. 3b, and *QR*, *RQ''* represent the components of the displacement of *Q* relative to *P*. Thus

$$QR = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad RQ'' = \frac{\partial v}{\partial x} dx + \frac{\partial v}{dy} \partial y \qquad (a)$$

The components of this relative displacement *QS*, *SQ''*, normal to *PQ''* and along *PQ''*, can be found from these as

$$QS = -QR \sin\theta + RQ'' \cos\theta, \qquad SQ'' = QR \cos\theta + RQ'' \sin\theta \quad (b)$$

ignoring the small angle QPS in comparison with *θ*. Since the short line *QS* may be identified with an arc of a circle with center *P*, *SQ''*

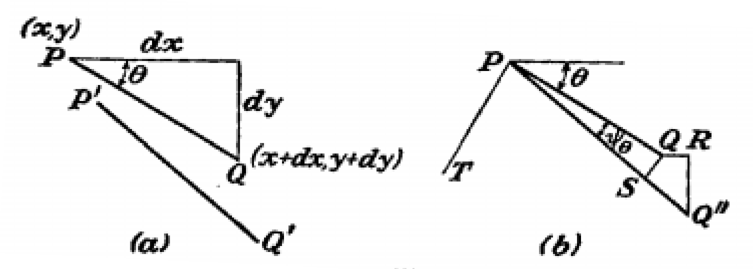


Fig. 3

gives the stretch of *PQ*. The unit elongation of *P'Q'*, denoted by ϵ_θ is *SQ''/PQ*. Using (b) and (a) we have

$$\begin{aligned}\epsilon_\theta &= \cos\theta \left(\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}\right) + \sin\theta \left(\frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds}\right) \\ &= \frac{\partial u}{\partial x}\cos^2\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta + \frac{\partial v}{\partial y}\sin^2\theta\end{aligned}$$

or

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \gamma_{xy} \sin\theta \cos\theta + \epsilon_y \sin^2\theta \qquad (c)$$

which gives the unit elongation for any direction θ.

The angle ψ_θ through which PQ is rotated is QS/PQ. Thus from (b) and (a),

$$\psi_\theta = -\sin\theta\left(\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}\right) + \cos\theta\left(\frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds}\right)$$

or

$$\psi_\theta = \frac{\partial v}{\partial x}\cos^2\theta + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta - \frac{\partial u}{\partial y}\sin^2\theta \qquad (d)$$

The line element PT at right angles to PQ makes an angle $\theta + (\pi/2)$ with the x-direction, and its rotation $\psi_\theta + (\pi/2)$is therefore given by (d) when
$\theta + (\pi/2)$is substituted for θ. Since $\cos[\theta + (\pi/2)] = -\sin\theta$, $\sin[\theta + (\pi/2)] = \cos\theta$, we find

$$\psi_{\theta+\frac{\pi}{2}} = \frac{\partial v}{\partial x}\sin^2\theta - \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta - \frac{\partial u}{\partial y}\cos^2\theta \qquad (e)$$

The shear strain γ_θ for the directions PQ, PT is ψ_θ- $\psi_\theta + (\pi/2)$ so

$$\gamma_\theta = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)(\cos^2\theta - \sin^2\theta) + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)2\sin\theta\cos\theta$$

or

$$\tfrac{1}{2}\gamma_\theta = \tfrac{1}{2}\gamma_{xy}(\cos^2\theta - \sin^2\theta) + (\epsilon_y - \epsilon_x)\sin\theta\cos\theta \qquad (f)$$

Comparing (c) and (f) with (2), we observe that they may be obtained from (2) by replacing σ by $\square_\theta$, $\tau$by $\gamma_\theta/2$, σ_x by $\square_x$, σ_y by $\square_y$, $\tau_{xy}$by $\gamma_{xy}/2$,and α by θ. Consequently for each deduction made from (2) as to σ and τ, there is a corresponding deduction from (c) and (f) as to $\square_\theta$ and $\gamma_\theta/2$. Thus there are two values of θ, differing by 90 deg., for which γ_θ is zero. They are given by

$$\frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = \tan 2\theta$$

The corresponding strains $\square_\theta$ are *principal strains*. A Mohr circle diagram analogous to Fig. 2 may be drawn, the ordinates representing $\gamma_\theta/2$ and the abscissas $\square_\theta$. The principal strains $\square_1$, $\square_2$ will be the algebraically greatest and least values of $\square_\theta$ as a function of θ. The greatest value of $\gamma_\theta/2$ will be represented by the radius of the circle. Thus the greatest shearing strain

$$\gamma_{\theta \text{ max.}} = \epsilon_1 - \epsilon_2$$

1.5. Basic Equations of Elasticity

The general form of a constitutive equation tor a linearly elastic material is

stress = (a constant) x strain

Since strain is dimensionless, the constant of proportionality has the dimensions of stress. Thus, under uniaxial tensile load.

stress = *E* x strain

or

$$\sigma_{xx} = Ee_{xx}$$

Where *E* is Young's modulus or the modulus of elasticity of the material. It was soon found that, as a result of stress, strains are produced in directions normal to the direction of the stress and that these strains are proportional to the strain in the direction of the stress. Thus the stress σ_{xx} produces a strain $e_{xx} = \sigma_{xx}/E$ in the x direction and strainsin orthogonal directions, the negative sign indicating that these strains are of the opposite sense to e_{xx}. The proportionality factor, υ, is called Poisson's ratio and is dimensionless. The elastic constants E and υ apply to both tensile and compressive loading.

From tests on the torsion of circular bars, the proportionality between shear stress and shear strain was established as

$$\sigma_{xy} = Ge_{xy}$$

where*G* is the modulus of rigidity or shear modulus.

Again, from consideration of the dilatation resulting from a hydrostatic state of stress, a fourth constant was introduced

$$\bar{\sigma} = K\Delta$$

where

$$\bar{\sigma} = \tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\Delta = e_{xx} + e_{yy} + e_{zz} = e_1 + e_2 + e_3$$

And K is the bulk or compressibility modulus.

Relation between K, G and E

$$G = \frac{E}{2(1+\upsilon)}$$

$$K = \frac{E}{3(1-2\upsilon)}$$

1.6. Methods of Solution of Elasticity Problems

Unfortunately, solving directly the equations of elasticity derived may be a formidable task, and it is often advisable to attempt a solution by the *inverse* or *semi-inverse* method.The inverse method requires examination of the assumed solutionswith a view toward finding one that will satisfy the governing equations and boundary conditions The semi-inverse method requires the assumption of a partial solution formed by expressing stress, strain, displacement, or stress function in terms ofknown or undetermined coefficients. The governing equations are thus renderedmore manageable.

It is important to note that the preceding assumptions, based on the mechanicsof a particular problem, are subject to later verification. This is in contrast with themechanics of materials approach, in which analytical verification does not occur.

A number of problems may be solved by using a linear combination of polynomials in x and y and undetermined coefficients of the stress function ϕ . Clearly, an assumed polynomial form must satisfy the biharmonic equation and must be of second degree or higher in order to yield a nonzero stress solution of Eq.

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2\Phi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\, \partial y}$$

In general, finding the desirable polynomial form is laborious and requires a systematic approach. The*Fourier series*, indispensible in the analytical treatment of many problemsin the field of applied mechanics.

1.7. Plane Stress

If a thin plate is loaded by forces applied at the boundary, parallel to the plane of the plate and distributed uniformly over the thickness (Fig. 4), the stress components σ_z,τ_{xz}, τ_{yz} are zero on both faces of the plate, and it may be assumed, tentatively, that they are zero also within the plate. The state of stress is then specified by σ_x, σ_y, τ_{xy}. only, and is called plane dress. It may also be assumed thatthese three components are independent of z, i.e., they do not vary through the thickness. They are then functions of x and y only.

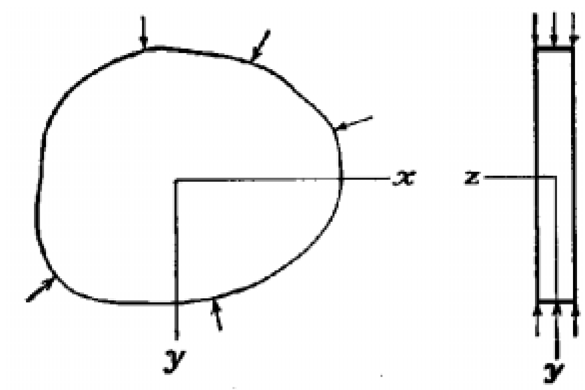


Fig. 4

1.8. Plane Strain

A similar simplification is possible at the other extreme when the dimension of the body in the z-direction is very large. If a long cylindrical or prismatical body is loaded by forces which are perpendicular to the longitudinal elements and do not vary along the length, it may be assumed that all cross sections are in the same condition. It is simplest to suppose at first that the end sections are confined between fixed smooth rigid planes, so that displacement in theaxial direction is prevented. The effect of removing these will beexamined later. Since there is no axial displacement at the ends, and,by symmetry, at the mid-section, it may be assumed that the same holds at every cross section.

There are many important problems of this kind—a retaining wall with lateral pressure (Fig. 5), a culvert or tunnel (Fig. 6), a cylindrical tube with internal pressure, a cylindrical roller compressed by forces in a diametral plane as in a roller bearing (Fig. 7). In each case of course the loading must not vary along the length. Since conditions are the same at all cross sections, it is sufficient to consider only a slice between two sections unit distance apart.

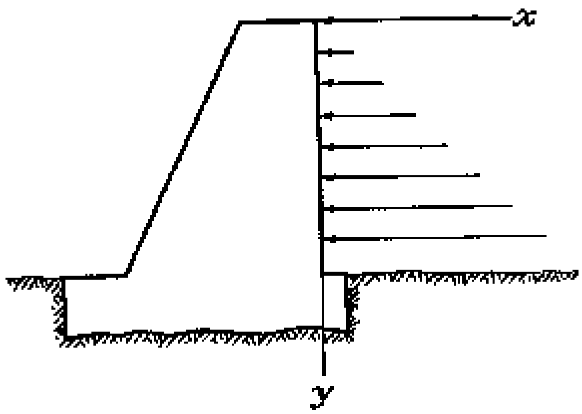


Fig. 5

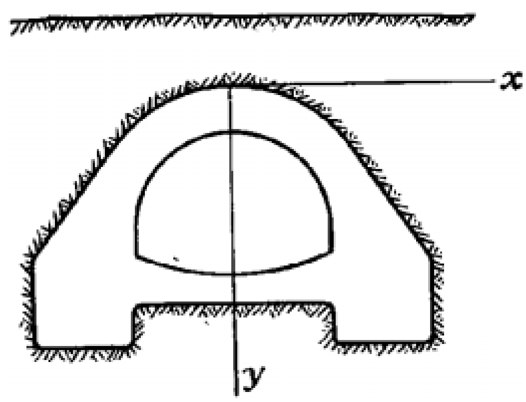


Fig. 6

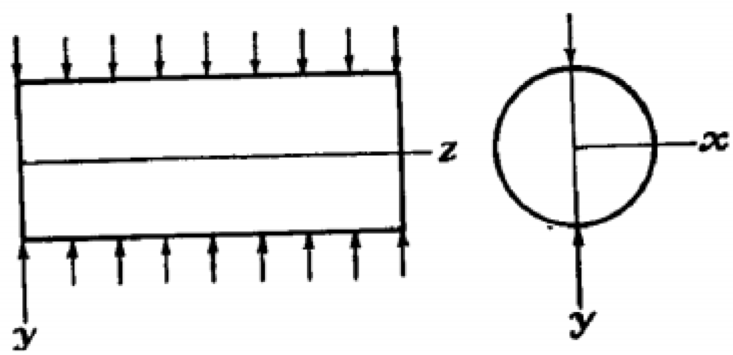


Fig. 7

The components *u* and *v* ofthe displacement are functions of *x* and *y* but are independent of thelongitudinal coordinate *z*. Since the longitudinal displacement *w* is zero, Eqs.

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

Give

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \qquad (a)$$
$$\epsilon_z = \frac{\partial w}{\partial z} = 0$$

The longitudinal normal stress σ_z can be found in terms of σ_x and σ_y by means of Hooke's law,

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Since $\epsilon_z = 0$ we find

$$\sigma_z - \nu(\sigma_x + \sigma_y) = 0$$
$$\sigma_z = \nu(\sigma_x + \sigma_y) \qquad (b)$$

These normal stresses act over the cross sections, including the ends, where they represent forces required to maintain the plane strain, and . provided by the fixed smooth rigid planes.By Eq. (a), the stress components τ_{xz} and τ_{yz} are zero, because

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$

and,by Eq. (b), σ_z can be found from σ_x and σ_y. Thus the plane strain problem, like the plane stress problem, reduces to the determination of σ_x, σ_y, and τ_{xy} as functions of x and y only.

1.9. Spherical Coordinates

The coordinate system is defined by (r, ϕ , θ). where$r$ is the length of the radiusvector ϕ is the angle made by the radius vector with a fixed axis, and θ is the anglemeasured round this axis. If the velocity components in the coordinate directions are denoted by (u,v,w), then the components of the true strain rate are

$$\dot{\varepsilon}_r = \frac{\partial u}{\partial r}, \qquad \dot{\gamma}_{r\phi} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \phi}\right),$$

$$\dot{\varepsilon}_\phi = \frac{1}{r}\left(u + \frac{\partial v}{\partial \phi}\right), \qquad \dot{\gamma}_{r\phi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \phi} - \frac{w}{r}\cot\phi + \frac{1}{r\sin\phi}\frac{\partial v}{\partial \theta}\right),$$

$$\dot{\varepsilon}_\theta = \frac{1}{r}\left(u + v\cot\phi + \operatorname{cosec}\phi\frac{\partial w}{\partial \theta}\right), \qquad \dot{\gamma}_{r\theta} = \frac{1}{2}\left(\frac{\partial w}{\partial r} - \frac{w}{r} + \frac{1}{r\sin\phi}\frac{\partial u}{\partial \theta}\right).$$

Denoting the normal stresses by σ_r,σ_ϕ and σ_θ and the shear stresses by $\tau_{r\phi}$, $\tau_{\phi\,\theta}$ and $\tau_{r\theta}$, the equations of equilibrium in the absence of body forces can be written as

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\phi}}{\partial \phi}\frac{1}{r\sin\phi}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}\left(2\sigma_r - \sigma_\phi - \sigma_\theta + \tau_{r\phi}\cot\phi\right) = 0,$$

$$\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\phi}{\partial \phi}\frac{1}{r\sin\phi}\frac{\partial \tau_{\phi\theta}}{\partial \theta} + \frac{1}{r}\left\{\left(\sigma_\phi - \sigma_\theta\right)\cot\phi + 3\tau_{r\phi}\right\} = 0,$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r}\left(3\tau_{r\theta} + 2\tau_{\phi\theta}\cot\phi\right) = 0.$$

When the deformation is infinitesimal, the preceding expressions for the components of the strain rate may be regarded as those for the strain itself, provided thecomponents of the velocity are interpreted as those of the displacement.

1.10. Summary

In this unit we have studied

- The State of Stress at a Point
- The State of Strain at a Point
- Basic Equations of Elasticity
- Methods of Solution of Elasticity Problems
- Plane Stress
- Plane Strain
- Spherical Co-ordinates

1.11. Keywords

Plane

Stress

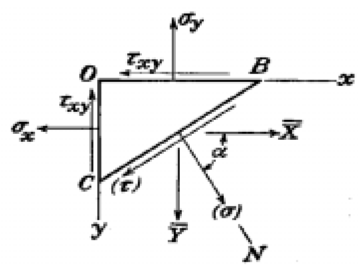
Strain

Elasticity

Anisotropy

1.12. Exercise

1. Write short notes on elasticity and basic equations of elasticity.
2. Find the maximum shearing stress under a condition of plane stress for an element with state of stress at a point.
3. Show that equations $\bar{X} = l\sigma_x + m\tau_{xy}$ and $\bar{Y} = m\sigma_y + l\tau_{xy}$ remains valid when the element as shown in the fig. below has acceleration.



4. Derive an expression for maximum shearing strain for a state of strain at a point on an element.
5. Write short notes on
   a) Plane stress
   b) Plane strain
   c) Spherical coordinates

Unit 2
Two-Dimensional Problems in Cartesian Co-Ordinates

Structure

2.1. Introduction

The Two-Dimensional Cartesian coordinate System

In a two-dimensional plane, we can pick any point and single it out as a reference point called the origin. Through the origin we construct two perpendicular number lines called axes. These are traditionally labeled the x axis and the y axis. An orientation or sense of the place is determined by the positions of the positive sides of the x and y axes. If a counterclockwise rotation of 90° about the origin aligns the positive x axis with the positive y axis, the coordinate system is said to have a right-handed orientation; otherwise the coordinate system is called left handed.

2.2. Objectives

After studying this unit we are able to understand

- Airy's Stress Function
- Direct method for determining Airy polynomial
- Cantilever having Udl and concentrated load of the free end
- Bending of a Cantilever Loaded at the End
- Bending of a Beam by Uniform Load
- Fourier Series
- Complex Potentials

- Cauchy Integral Method
- The Fourier Transform

2.3. Airy's Stress Function

It has been shown that a solution of two-dimensional problems reduces to the integration of the differentialequations of equilibrium together with the compatibility equation andthe boundary conditions. If we begin with the case when the weightof the body is the only body force, the equations to be satisfied are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g = 0 \qquad (a)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \qquad (b)$$

To these equations boundary equations

$$\bar{X} = l\sigma_x + m\tau_{xy}$$
$$\bar{Y} = m\sigma_y + l\tau_{xy}$$

are added

The usual method of solving these equations is by introducing a new function, called the *stress function*. As is easily checked, Eqs. (a) are satisfied by taking any function ϕ of x and y and putting the following expressions for the stress components:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \rho g y, \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \rho g y, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \, \partial y} \qquad 1)$$

In this manner we can get a variety of solutions of the equations of equilibrium (a). The true solution of the problem is that which satisfies also the compatibility equation (b). Substituting expressions (1)for the stress components into Eq. (b) we find that the stress functionϕ must satisfy the equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (2)$$

Thus the solution of a two-dimensional problem, when the weight of the body is the only body force, reduces to finding a solution of Eq. (2) which satisfies the boundary conditions of the problem.

$$X = -\frac{\partial V}{\partial x}$$
$$Y = -\frac{\partial V}{\partial y} \qquad (c)$$

in which V is the potential function. Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0$$

become

$$\frac{\partial}{\partial x}(\sigma_x - V) + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial}{\partial y}(\sigma_y - V) + \frac{\partial \tau_{xy}}{\partial x} = 0$$

These equations are of the same form as Eqs. (a) and can be satisfied by taking

$$\sigma_x - V = \frac{\partial^2 \phi}{\partial y^2}, \qquad \sigma_y - V = \frac{\partial^2 \phi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y} \qquad (3)$$

in which ϕ is the stress function. Substituting expressions (3) in the compatibility equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1 + \nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)$$

for plane stress distribution, we find

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1 - \nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \qquad (4)$$

An analogous equation can be obtained for the case of plane strain.

When the body force is simply the weight, the, potential V is $-\varrho gy$. In this case the right-hand side of Eq. (4) reduces to zero. By taking the solution $\phi = 0$ of (4), or of (2), we find the stress distribution from (3), or (1),

$$\sigma_x = -\rho gy, \qquad \sigma_y = -\rho gy, \qquad \tau_{xy} = 0 \qquad (d)$$

as a possible state of stress due to gravity. This is a state of hydrostatic pressure ϱgy in two dimensions, with zero stress at $y = 0$. It can exist in a plate or cylinder of any shape provided the corresponding boundary forces are applied.

2.4. Direct method for determining Airy polynomial

2.4.3. Cantilever having Udl and concentrated load of the free end

It has been shown that the solutionof two-dimensional problems, when body forces are absent or are constant, is reduced to the integration of the differential equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (a)$$

having regard to boundary conditions

$$\bar{X} = l\sigma_x + m\tau_{xy}$$
$$\bar{Y} = m\sigma_y + l\tau_{xy}$$

In the case of longrectangular strips, solutions of Eq. (a) in the form of polynomials areof interest. By taking polynomialsof various degrees, and suitably adjusting their coefficients, a number ofpractically important problems can besolved.

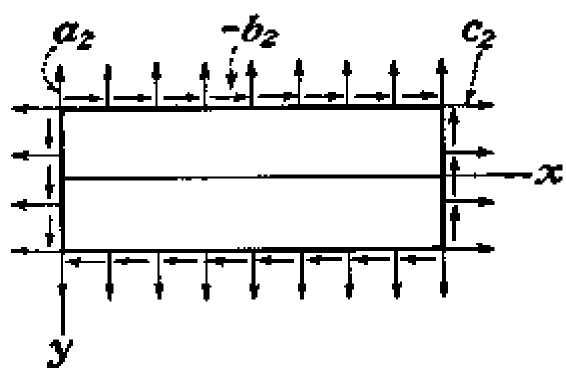


Fig. 1

Beginning with a polynomial of thesecond degree

$$\phi_2 = \frac{a_2}{2} x^2 + b_2 xy + \frac{c_2}{2} y^2 \qquad (b)$$

which evidently satisfies Eq. (a), we find from Eqs. (1), putting $\varrho g = 0$

$$\sigma_x = \frac{\partial^2 \phi_2}{\partial y^2} = c_2, \qquad \sigma_y = \frac{\partial^2 \phi_2}{\partial x^2} = a_2, \qquad \tau_{xy} = -\frac{\partial^2 \phi_2}{\partial x\, \partial y} = -b_2$$

All three stress components are constant, throughout the body, i.e., the stress function (b) represents a combination of uniform tensions or compressions in two perpendicular directions and a uniform shear. The forces on the boundaries must equal the stresses at these points; in the case of a rectangular plate with sidesparallel to the coordinate axes these forces are shown in Fig. 1.

Let us consider now a stress function in the form of a polynomial of the third degree:

$$\phi_3 = \frac{a_3}{3 \cdot 2} x^3 + \frac{b_3}{2} x^2 y + \frac{c_3}{2} x y^2 + \frac{d_3}{3 \cdot 2} y^3 \qquad (c)$$

This also satisfies Eq. (a). Using Eqs. (1) and putting $\varrho g = 0$, we find

$$\sigma_x = \frac{\partial^2 \phi_3}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 \phi_3}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x\, \partial y} = -b_3 x - c_3 y$$

For a rectangular plate, taken as in Fig. 2, assuming all coefficients except d_3 equal to zero, we obtain pure bending. If only coefficient a_3 is different from zero, we obtain pure bending by normal stresses applied to the sides $y = \pm c$ of the plate. If coefficient b_3 or c_3 is taken

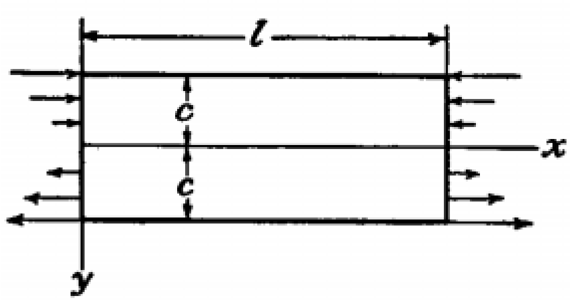


Fig. 2

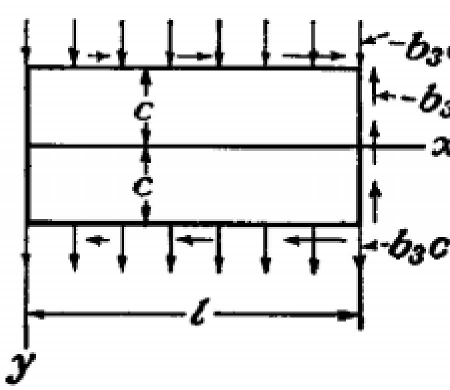


Fig. 3

different from zero, we obtain not only normal but also shearing stresses acting on the sides of the plate. Fig.3 represents, for instance, the case in which all coefficients, except b_3 in function (c), are equal to zero. The directions of stresses indicated are for b_3 positive. Along the sides $y = \pm c$ we have uniformly distributed tensile and compressive stresses, respectively, and shearing stresses proportional to x. On the side $x=l$ we have only the constant shearing stress — $b_3 l$, and there are no stresses acting on the side $x= 0$. An analogous stress distribution is obtained if coefficient c_3 is taken different from zero.

In taking the stress function in the form of polynomials of the second and third degrees we are completely free in choosing the magnitudes of the coefficients, since Eq. (a) is satisfied whatever values they may have. In the case of polynomials of higher degrees Eq. (a) is satisfied only if certain relations between the coefficients are satisfied. Taking, for instance, the stress function in the form of a polynomial of the fourth degree,

$$\phi_4 = \frac{a_4}{4 \cdot 3} x^4 + \frac{b_4}{3 \cdot 2} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{3 \cdot 2} x y^3 + \frac{e_4}{4 \cdot 3} y^4 \qquad (d)$$

and substituting it into Eq. (a), we find that the equation is satisfied only if

$$e_4 = -(2c_4 + a_4)$$

The stress components in this case are

$$\sigma_x = \frac{\partial^2 \phi_4}{\partial y^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4) y^2$$
$$\sigma_y = \frac{\partial^2 \phi_4}{\partial x^2} = a_4 x^2 + b_4 xy + c_4 y^2$$
$$\tau_{xy} = \frac{\partial^2 \phi_4}{\partial x \, \partial y} = -\frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2$$

Coefficients a_4,... ,d_4 in these expressions are arbitrary, and by suitably adjusting them we obtain various conditions of loading of a rectangular plate. For instance, taking all coefficients except d_4 equal to zero, we find

$$\sigma_x = d_4 xy, \qquad \sigma_y = 0, \qquad \tau_{xy} = -\frac{d_4}{2} y^2 \qquad (e)$$

Assuming d_4 positive, the forces acting on the rectangular plate shown in Fig. 4 and producing the stresses (e) are as given. On the longitudinal sides $y = \pm c$ are uniformly distributed shearing forces; on theends shearing forces are distributed according to a parabolic law. The shearing forces acting on the boundary of the plate reduce to the couple

$$M = \frac{d_4c^2l}{2} \cdot 2c - \frac{1}{3}\frac{d_4c^2}{2} \cdot 2c \cdot l = \frac{2}{3} d_4c^3l$$

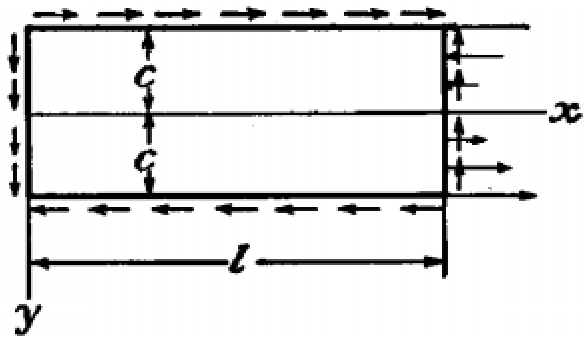


Fig. 4

This couple balances the couple produced by the normal forces along the side $x = l$ of the plate.

Let us consider a stress function in the form of a polynomial of the fifth degree.

$$\phi_5 = \frac{a_5}{5 \cdot 4} x^5 + \frac{b_5}{4 \cdot 3} x^4y + \frac{c_5}{3 \cdot 2} x^3y^2 + \frac{d_5}{3 \cdot 2} x^2y^3 + \frac{e_5}{4 \cdot 3} xy^4 + \frac{f_5}{5 \cdot 4} y^5 \tag{f}$$

Substituting in Eq. (a) we find that this equation is satisfied if

$$e_5 = -(2c_5 + 3a_5)$$
$$f_5 = -\tfrac{1}{3}(b_5 + 2d_5)$$

The corresponding stress components are:

Again coefficients a_5,... ,$d_5$are arbitrary, and in adjusting them we obtain solutions for various loading conditions of a plate. Taking,

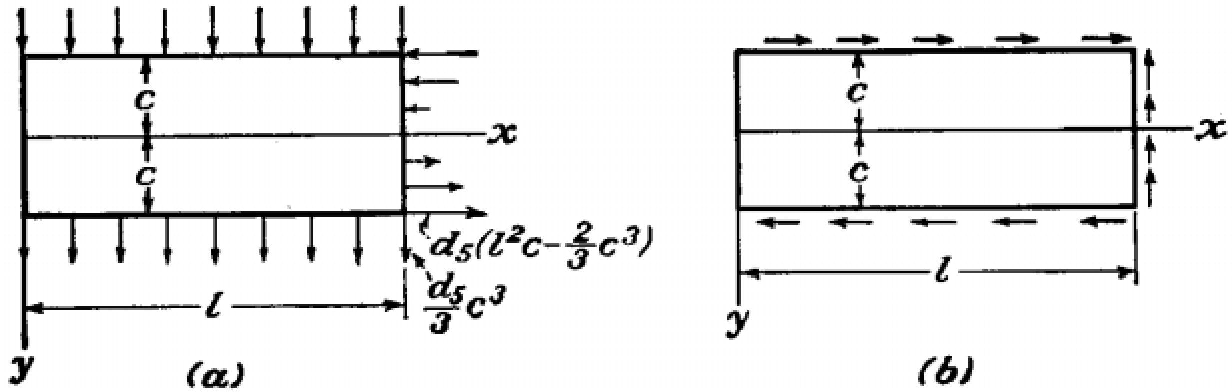


Fig. 5

for instance, all coefficients, except d_5, equal to zero we find

$$\begin{aligned} \sigma_x &= d_5(x^2y - \tfrac{2}{3}y^3) \\ \sigma_y &= \tfrac{1}{3}d_5y^3 \\ \tau_{xy} &= -d_5xy^2 \end{aligned} \qquad (g)$$

The normal forces are uniformly distributed along the longitudinal sides of the plate (Fig. 5a). Along the side $x=l$, the normal forces consist of two parts, one following a linear law and the other following the law of a cubic parabola. The shearing forces are proportional to x on the longitudinal sides of the plate and follow a parabolic law along the side $x=l$. The distribution of these stresses is shown in Fig. 5b.Since Eq. (a) is a linear differential equation, it may be concluded that a sum of several solutions of this equation is also a solution.

2.4.4. Bending of a Cantilever Loaded at the End

Consider a cantilever having a narrow rectangular cross section of unit width bent by force P applied at the end (Fig. 6).

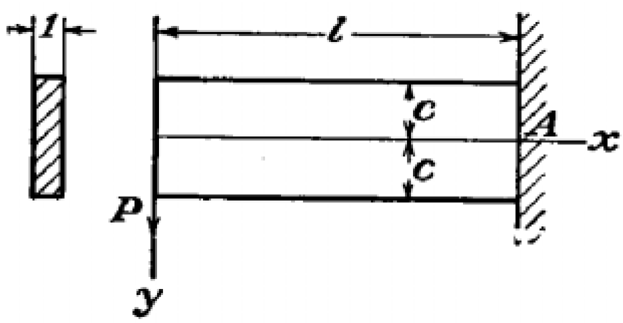


Fig. 6

The upper and lower edges arefree from load, and shearing forces, having a resultant P, are distributed along the end x= 0. These conditions can be satisfied by aproper combination of pure shear,with the stresses (e) of previous article represented in Fig. 8. Superposing the pure shear $\tau_{xy}= b_2$ on the stresses (e), we find

$$\sigma_x = d_4 xy, \qquad \sigma_y = 0$$
$$\tau_{xy} = -b_2 - \frac{d_4}{2} y^2 \qquad (a)$$

To have the longitudinal sides $y = \pm c$ free from forces we must have

$$(\tau_{xy})_{y=\pm c} = -b_2 - \frac{d_4}{2} c^2 = 0$$

from which

$$d_4 = -\frac{2b_2}{c^2}$$

To satisfy the condition on the loaded end the sum of the shearing forces distributed over this end must be equal to P. Hence

$$-\int_{-c}^{c} \tau_{xy} \cdot dy = \int_{-c}^{c} \left(b_2 - \frac{b_2}{c^2} y^2\right) dy = P$$

from which

$$b_2 = \frac{3}{4}\frac{P}{c}$$

Substituting these values of d_4 and b_2 in Eqs. (a) we find

$$\sigma_x = -\frac{3}{2}\frac{P}{c^3} xy, \qquad \sigma_y = 0$$
$$\tau_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

Noting that $2/3c^3$ is the moment of inertia I of the cross section of the cantilever, we have

$$\sigma_x = -\frac{Pxy}{I}, \qquad \sigma_y = 0$$
$$\tau_{xy} = -\frac{P}{I}\frac{1}{2}(c^2 - y^2) \qquad (b)$$

This coincides completely with the elementary solution as given in books on the strength of materials. It should be noted that this solution represents an exact solution only if the shearing forces on the ends are distributed according to the same parabolic law as the shearing stress τ_{xy} and the intensity of the normal forces at the built-in end is proportional to *y*. If the forces at the ends are distributed in any other manner, the stress distribution (b) is not a correct solution for the ends of the cantilever, but, by virtue of Saint-Venant's principle, it can be considered satisfactory for cross sections at a considerable distance from the ends.

Let us consider now the displacement corresponding to the stresses(b). Applying Hooke's law we find

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{Pxy}{EI}, \qquad \epsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu\sigma_x}{E} = \frac{\nu Pxy}{EI} \tag{c}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = -\frac{P}{2IG}(c^2 - y^2) \tag{d}$$

The procedure for obtaining the components *u* and *v* of the displacement consists in integrating Eqs. (c) and (d). By integration of Eqs.(c) we find

$$u = -\frac{Px^2y}{2EI} + f(y), \qquad v = \frac{\nu Pxy^2}{2EI} + f_1(x)$$

in which $f(y)$ and $f_1(x)$ are as yet unknown functions of *y* only and *x* only. Substituting these values of *u* and *v* in Eq. (d) we find

$$-\frac{Px^2}{2EI} + \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} + \frac{df_1(x)}{dx} = -\frac{P}{2IG}(c^2 - y^2)$$

In this equation some terms are functions of *x* only, some are functions of *y* only, and one is independent of both *x* and*y*. Denoting these groups by $F(x)$, $G(y)$, K, we have

$$F(x) = -\frac{Px^2}{2EI} + \frac{df_1(x)}{dx}, \qquad G(y) = \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG}$$

$$K = -\frac{Pc^2}{2IG}$$

and the equation may be written

$$F(x) + G(y) = K$$

Such an equation means that $F(x)$ must be some *constant d* and $G(y)$ some constant *e*. Otherwise $F(x)$ and $G(y)$ would vary with *x* and *y*, respectively, and by varying *x* alone, or *y* alone, the equality would be violated. Thus

$$e + d = -\frac{Pc^2}{2IG} \tag{e}$$

and

$$\frac{df_1(x)}{dx} = \frac{Px^2}{2EI} + d, \qquad \frac{df(y)}{dy} = -\frac{Py^2}{2EI} + \frac{Py^2}{2IG} + e$$

Functions $f(y)$ and $f_1(x)$ are then

$$f(y) = -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$
$$f_1(x) = \frac{Px^3}{6EI} + dx + h$$

Substituting in the expressions for u and v we find

$$\begin{aligned} u &= -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g \\ v &= \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} + dx + h \end{aligned} \tag{g}$$

The constants d, e, g, h may now be determined from Eq. (e) and from the three conditions of constraint which are necessary to prevent the beam from moving as a rigid body in the xy-plane. Assume that the point A, the centroid of the end cross section, is fixed. Then u and v are zero for $x = l$, $y = 0$, and we find from Eqs. (g),

$$g = 0, \qquad h = -\frac{Pl^3}{6EI} - dl$$

The deflection curve is obtained by substituting $y = 0$ into thesecond of
Eqs. (g). Then

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^3}{6EI} - d(l - x) \tag{h}$$

For determining the constant d in this equation we must use the third condition of constraint, eliminating the possibility of rotation of the beam in the xy-plane about the fixed point A. This constraint can be realized in various ways. Let us consider two cases:

(1) When an element of the axis of the beam is fixed at the end A. Then the conditionof constraint is

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=l\\y=0}} = 0 \qquad (k)$$

(2) When a vertical element of the cross section at the point *A* is fixed.Then the condition of constraint is

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} = 0 \qquad (l)$$

In the first case we obtain from Eq. (h)

$$d = -\frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$e = \frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting all the constants in Eqs. (g), we find

$$\begin{aligned} u &= -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y \\ v &= \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \end{aligned} \qquad (m)$$

The equation of the deflection curve is

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \qquad (n)$$

which gives for the deflection at the loaded end ($x = 0$) the value $Pl^3/3EI$. This coincides with the value usually derived in elementary books on the strength of materials.

To illustrate the distortion of cross sections produced by shearing stresses let us consider the displacement u at the fixed end ($x = l$). For this end we have from Eqs. (m),

$$\begin{aligned} (u)_{x=l} &= -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} - \frac{Pc^2y}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{x=l} &= -\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} - \frac{Pc^2}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} &= -\frac{Pc^2}{2IG} = -\frac{3}{4}\frac{P}{cG} \end{aligned} \qquad (o)$$

The shape of the cross section after distortion is as shown in Fig. 7a. Due to the shearing stress $\tau_{xy} = -3P/4c$ at the point A, an element of the cross section at A rotates in the xy-plane about the point A through an angle $3P/4cG$ in the clockwise direction.

If a vertical element of the cross section is fixed at A (Fig. 7b) instead of a horizontal element of the axis, we find from condition (l)and the first of Eqs. (g)

$$e = \frac{Pl^2}{2EI}$$

and from Eq. (e) we find

Substituting in the second of Eqs. (g) we find

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} + \frac{Pc^2}{2IG}(l - x) \qquad (r)$$

Comparing this with Eq. (n) it can be concluded that, due to rotationof the end of the axis at A (Fig. 7b),

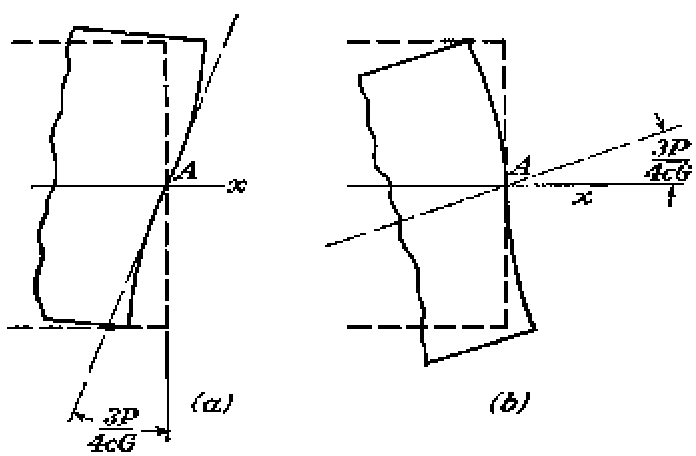


Fig. 7

the deflections of the axis of thecantilever are increased by the quantity

$$\frac{Pc^2}{2IG}(l - x) = \frac{3P}{4cG}(l - x)$$

This is the so-called effect of shearing force on the deflection of the beam. In practice, at the built-in end we have conditions different from those shown in Fig. 7. The fixed section is usually not free to distort and the distribution of forces at this end is different from that given by Eqs. (b). Solution (b) is, however, satisfactory for comparatively long cantilevers at considerable distances from the terminals.

2.5. Bending of a Beam by Uniform Load

Let a beam of narrowrectangular cross section of unit width, supported at the ends, be bent by a uniformly distributed load of intensity *q*, as shown in Fig. 8 The conditions at the upper and lower edges of the beam are:

$$(\tau_{xy})_{y=\pm c} = 0, \qquad (\sigma_y)_{y=+c} = 0, \qquad (\sigma_y)_{y=-c} = -q \qquad (a)$$

The conditions at the ends $x = \pm l$ are

$$\int_{-c}^{c} \tau_{xy}\, dy = \mp ql, \qquad \int_{-c}^{c} \sigma_x\, dy = 0, \qquad \int_{-c}^{c} \sigma_x y\, dy = 0 \qquad (b)$$

The last two of Eqs. (b) state that there is no longitudinal force and no bending couple applied at the ends of the beam. All the conditions (a) and (b) can be satisfied by combining certain solutions in the form

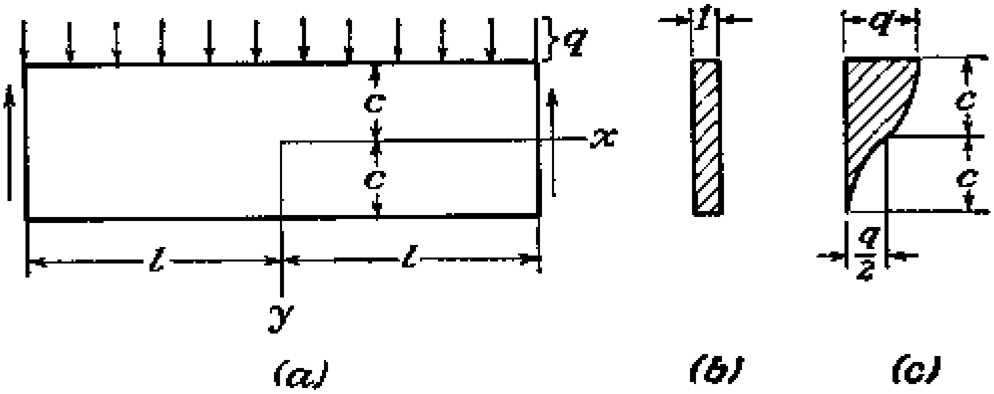


Fig. 8

of polynomials. We begin with solution (g)illustrated by Fig. 5. To remove the tensile stresses along the side$y = c$ and the shearing stresses along the sides $y = \pm c$ we superpose asimple compression $\sigma_y = a_2$ from solution (b), solution by polynomials, and the stresses$\sigma_y = b_3 y$ and $\tau_{xy} = -b_3 x$ in Fig. 3. In this manner wefind

$$\begin{aligned} \sigma_x &= d_5\left(x^2 y - \tfrac{2}{3} y^3\right) \\ \sigma_y &= \tfrac{1}{3} d_5 y^3 + b_3 y + a_2 \\ \tau_{xy} &= -d_5 x y^2 - b_3 x \end{aligned} \qquad (c)$$

From the conditions (a) we find

$$-d_5c^2 - b_3 = 0$$
$$\tfrac{1}{3}d_5c^3 + b_3c + a_2 = 0$$
$$-\tfrac{1}{3}d_5c^3 - b_3c + a_2 = -q$$

from which

$$a_2 = -\frac{q}{2}, \qquad b_3 = \frac{3}{4}\frac{q}{c}, \qquad d_5 = -\frac{3}{4}\frac{q}{c^3}$$

Substituting in Eqs. (c) and noting that $2c^3/3$ is equal to the moment of inertia I of the rectangular cross-sectional area of unit width, we find

$$\begin{aligned} \sigma_x &= -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) = -\frac{q}{2I}\left(x^2y - \frac{2}{3}y^3\right) \\ \sigma_y &= -\frac{3q}{4c^3}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) = -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) \qquad (d) \\ \tau_{xy} &= -\frac{3q}{4c^3}(c^2 - y^2)x = -\frac{q}{2I}(c^2 - y^2)x \end{aligned}$$

It can easily be checked that these stress components satisfy not only conditions (a) on the longitudinal sides but also the first two conditions (b) at the ends. To make the couples at the ends of the beam vanish we superpose on solution (d) a pure bending, $\sigma_x = d_3y$, $\sigma_y = \tau_{xy} = 0$, shown in Fig. 2, and determine the constant d_3 from the condition at $x = \pm l$

$$\int_{-c}^{c} \sigma_x y\, dy = \int_{-c}^{c}\left[-\frac{3}{4}\frac{q}{c^3}\left(l^2y - \frac{2}{3}y^3\right) + d_3y\right] y\, dy = 0$$

From which

$$d_3 = \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)$$

Hence, finally,

$$\begin{aligned} \sigma_x &= -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) + \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)y \\ &= \frac{q}{2I}(l^2 - x^2)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right) \end{aligned} \qquad (5)$$

The first term in this expression represents the stresses given by the usual elementary theory of bending, and the second term gives the necessary correction. This correction does not depend on

x and is small in comparison with the maximum bending stress, provided the span of the beam is large in comparison with its depth. For such beams the elementary theory of bending gives a sufficiently accuratevalue for the stresses σ_x. It should be noted that expression (5) is an exact solution only if at the ends $x = \pm l$ the normal forces are distributed according to the law

$$\bar{X} = \frac{3}{4}\frac{q}{c^3}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right)$$

i.e., if the normal forces at the ends are the same as σ_x for $x = \pm l$ from Eq. (5). These forces have a resultant force and a resultant couple equal to zero. Hence, from Saint-Venant's principle we can concludethat their effects on the stresses at considerable distances from the ends, say at distances larger than the depth of the beam, can be neglected. Solution (5) at such points is therefore accurate enough for the case when there are no forces X.

The discrepancy between the exact solution (5) and the approximate solution, given by the first term of (5), is due to the fact that in deriving the approximate solution it is assumed that the longitudinal fibers of the beam are in a condition of simple tension. From solution(d) it can be seen that there are compressive stresses σ_y, between the fibers. These stresses are responsible for the correction represented by the second term of solution (5). The distribution of the compressive stresses σ_y over the depth of the beam is shown in Fig. 8c.The distribution of shearing stress τ_{xy}, given by the third of Eqs. (d),over a cross section of the beam coincides with that given by the usualelementary theory.

When the beam is loaded by its own weight instead of the distributed load q, the solution must be modified by putting q=2ϱ gc in (5) and the last two of Eqs. (d), and adding the stresses

$$\sigma_x = 0, \qquad \sigma_y = \rho g(c - y), \qquad \tau_{xy} = 0 \tag{e}$$

For the stress distribution (e) can be obtained from Eqs. (1) ofAiry's stress function by taking

$$\phi = \tfrac{1}{2}\rho g(cx^2 + y^3/3)$$

and therefore represents a possible state of stress due to weight and boundary forces. On the upper edge y=-c we have σ_y =2ϱ gc, and on the lower edge y =c, σ_y = 0. Thus when the stresses (e) are added to the previous solution, with q=2ϱ gc, the stress on both horizontal edges is zero, and the load on the beam consists only of its own weight.

The displacements u and v can be calculated by the method indicated in the previous article. Assuming that at the centroid of the middle cross section (x = 0, y= 0) the horizontal displacement is zero and the vertical displacement is equal to the deflection δ, we find, using solutions (d) and (5),

$$u = \frac{q}{2EI}\left[\left(l^2x - \frac{x^3}{3}\right)y + x\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right) + \nu x\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right)\right]$$

$$v = -\frac{q}{2EI}\left\{\frac{y^4}{12} - \frac{c^2y^2}{2} + \frac{2}{3}c^3y + \nu\left[(l^2 - x^2)\frac{y^2}{2} + \frac{y^4}{6} - \frac{1}{5}c^2y^2\right]\right\}$$
$$- \frac{q}{2EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} - \frac{1}{5}c^2x^2 + \left(1 + \frac{1}{2}\nu\right)c^2x^2\right] + \delta$$

It can be seen from the expression for u that the neutral surface of thebeam is not at the center line. Due to the compressive stress

$$(\sigma_y)_{y=0} = -\frac{q}{2}$$

the center line has a tensile strain $\nu q/2E$, and we find

$$(u)_{y=0} = \frac{\nu qx}{2E}$$

From the expression for v we find the equation of the deflection curve,

$$(v)_{y=0} = \delta - \frac{q}{2EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} - \frac{1}{5}c^2x^2 + \left(1 + \frac{1}{2}\nu\right)c^2x^2\right] \qquad (f)$$

Assuming that the deflection is zero at the ends ($x = \pm l$) of the center line, we find

$$\delta = \frac{5}{24}\frac{ql^4}{EI}\left[1 + \frac{12}{5}\frac{c^2}{l^2}\left(\frac{4}{5} + \frac{\nu}{2}\right)\right] \qquad (6)$$

The factor before the brackets is the deflection which is derived by the elementary analysis, assuming that cross sections of the beam remain plane during bending. The second term in the brackets represents the correction usually called the *effect of shearing force*.

By differentiating Eq. (f) for the deflection curve twice with respect to x, we find the following expression for the curvature:

$$\left(\frac{d^2v}{dx^2}\right)_{y=0} = \frac{q}{EI}\left[\frac{l^2 - x^2}{2} + c^2\left(\frac{4}{5} + \frac{\nu}{2}\right)\right] \qquad (7)$$

It will be seen that the curvature is not exactly proportional to the bending moment $q(l^2-x^2)/2$. The additional term in the brackets represents the necessary correction to the usual elementary formula. A more general investigation of the curvature of beams shows that the correction term given in expression (7) can also be used for any case of continuously varying intensity of load.

2.6. Fourier Series

Certain problems in the analysis of structural deformation mechanical vibration, heat transfer, and the like, are amenable to solution by means of trigonometric series. This approach offers as an important advantage the fact that a single expression may apply to the entire length of the member. The method is now illustratedusing the case of a simply supported beam subjected to a moment at point*A* (Fig. 9a). The solution by trigonometric series can also be employed in the analysisof beams having any other type of end condition and beams under combined loading.

The deflection curve can be represented by a Fourier sine series:

$$v = a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{2\pi x}{L} + \cdots = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \tag{8}$$

The end conditions of the beam ($v = 0$, $v'' = 0$ at $x = 0$, $x = L$) are observed to be satisfied by each term of this infinite series. The first and second terms of the series are represented by the curves in Fig. 9b and c, respectively. As a physical interpretation of Eq. (8), consider the true deflection curve of the beam to be the superposition of sinusoidal curves of n different configurations. The coefficients a_n of the series are the maximum coordinates of the sine curves, and the *n's* indicate the number of half-waves in the sine curves. It is demonstrable that, when the coefficients $a_n$are determined properly, the series given by Eq. (8) can be used to represent any deflection curve. By increasing the number of terms in the series, the accuracy can be improved.

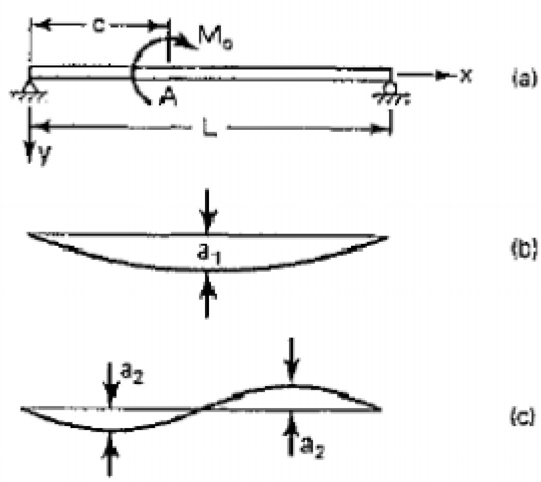


Fig. 9

To evaluate the coefficients, the principle of virtual work will be applied. The strain energy of the system, is

$$U = \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left[\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}\right]^2 dx \qquad \textbf{(a)}$$

Expanding the term in brackets,

$$\left[\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}\right]^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m a_n \left(\frac{m\pi}{L}\right)^2 \left(\frac{n\pi}{L}\right)^2 \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}$$

Since for the orthogonal functions $sin(m\pi x/L)$ and $sin(n\pi x/L)$ it can be shown by direct integration that

$$\int_0^l \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n \end{cases}$$

Eq. (a) gives then

$$\delta U = \frac{\pi^4 EI}{2L^3}\sum_{n=1}^{\infty} n^4 a_n \delta a_n \qquad (9)$$

The virtual work done by a moment M_o acting through a virtual rotation at A increases the strain energy of the beam by δU:

$$M_o\left(\delta\frac{\partial w}{\partial x}\right)_A = \delta U \qquad \textbf{(b)}$$

Therefore, from Eqs. (9) and (b), we have

$$M_o \sum_{n=1}^{\infty} \frac{n\pi}{L}\cos\frac{n\pi c}{L}\,\delta a_n = \frac{\pi^4 EI}{2L^3}\sum_{n=1}^{\infty} n^4 a_n \delta a_n$$

which leads to

$$a_n = \frac{2M_o L^2}{\pi^3 EI}\frac{1}{n^3}\cos\frac{n\pi c}{L}$$

Upon substitution of this for a_n in the series given by Eq. (8), the equation for the deflection curve is obtained in the form

$$v = \frac{2M_0L^2}{\pi^3EI}\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\frac{n\pi c}{L}\sin\frac{n\pi x}{L}$$

Through the use of this infinite series, the deflection for any given value of x can becalculated.

2.7. Complex Potentials

So far the stress and displacement components have been expressed in terms of the stress function ϕ . But since Eq.

$$\phi = \operatorname{Re}\left[\bar{z}\psi(z) + \chi(z)\right] \tag{10}$$

expressesϕ in termsof two functions ψ(z),χ(z), it is possible to express the stress and displacement in terms of these two "complex potentials."

Any complex function f(z) can be put into the form $\alpha+i\ \beta$ where α and β are real. To this there corresponds the *conjugate*, α-$i\beta$, the value taken by f(z) when i is replaced, wherever it occurs in f(z), by - i. This change is indicated by the notation

$$\bar{f}(\bar{z}) = \alpha - i\beta \tag{a}$$

Thus if $f(z) = e^{inz}$we have

$$\bar{f}(\bar{z}) = e^{-in\bar{z}} = e^{-in(x-iy)} = e^{-inx}\cdot e^{-ny} \tag{b}$$

This may be contrasted with

$$f(\bar{z}) = e^{in\bar{z}}$$

to illustrate the significance of the bar over the f in Eq. (a).Evidently

$$f(z) + \bar{f}(\bar{z}) = 2\alpha = 2\operatorname{Re} f(z)$$

In the same way if we add to the function in brackets in Eq. (10) its conjugate, the sum will be twice the real part of this function. Thus Eq. (10) may be replaced by

$$2\phi = \bar{z}\psi(z) + \chi(z) + z\bar{\psi}(\bar{z}) + \bar{\chi}(\bar{z}) \tag{11}$$

and by differentiation

$$2\frac{\partial \phi}{\partial x} = \bar{z}\psi'(z) + \psi(z) + \chi'(z) + z\bar{\psi}'(\bar{z}) + \bar{\psi}(\bar{z}) + \bar{\chi}'(\bar{z})$$

$$2\frac{\partial \phi}{\partial y} = i[\bar{z}\psi'(z) - \psi(z) + \chi'(z) - z\bar{\psi}'(\bar{z}) + \bar{\psi}(\bar{z}) - \bar{\chi}'(\bar{z})]$$

These two equations may be combined into one by multiplying the second by *i* and adding. Then

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = \psi(z) + z\bar{\psi}'(\bar{z}) + \bar{\chi}'(\bar{z}) \qquad (c)$$

2.8. Cauchy Integral Method

Cauchy- Riemann Equations in Cartesian and polar co-ordinates are as follows:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (12)$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \qquad (13)$$

It can be observed that relations (12) allow the differential of *u* to be expressed in terms of variable *v*, that is,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \qquad (14)$$

and so if we know *v*, we could calculate *u*by integrating relation (14). In this discussion the roles of *u* and *v* could be interchanged and therefore if we know one of these functions, theother can be determined. This behavior establishes *u* and *v* as *conjugate functions*.

Next consider some concepts and results related to integration in the complex plane shown in Fig. l0. The line integral over a curve*C* from z_1 to z_2 is given by

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C ((udx - vdy) + i(udy + vdx)) \qquad (15)$$

Using the Cauchy-Riemann relations, we can show that if the function *f*is analytic in a region*D* that encloses the curve*C*, then the line integral is independent of the path taken between theend points z_1 and z_2. This fact leads to two useful theorems in complex variable theory.

Cauchy Integral Theorem: If a function *f(z)* is analytic at all points interior to and on a closed curve *C*, then

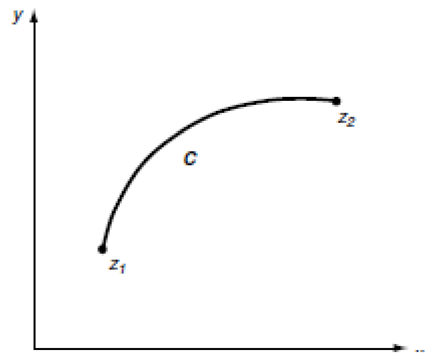


Fig. 10

$$\oint_C f(z)dz = 0$$

Cauchy Integral Formula: If*f(z)* is analytic everywhere within and on a closed curve *C*,and if $z_o$is any point interior to *C*, then

$$f(z_o) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_o}dz$$

2.9. The Fourier Transform

The Fourier transform, in essence, decomposes or separates a waveform or functioninto sinusoids of different frequency which sum to the original waveform. It identifies ordistinguishes the different frequency sinusoids and their respective amplitudes. The Fourier transform of *f*(*x*) is defined as

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi xs}dx.$$

Applying the same transform to *F(s)* gives

$$f(w) = \int_{-\infty}^{\infty} F(s) e^{-i2\pi x s} dx.$$

If *f(x)* is an even function of *x*, that is *f(x)*= *f(-x)*, then *f(w)* = *f(x)*. If *f(x)* is an odd function of *x*, that is *f(x)*= *-f(-x)*, then *f(w)* = *f(-x)*. When *f(x)* is neither even nor odd, it can often be split into even or odd parts.

To avoid confusion, it is customary to write the Fourier transform and its inverse so thatthey exhibit reversibility:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi x s} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi x s} dx$$

So that

$$f(x) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x) e^{-i2\pi x s} dx \right] e^{i2\pi x s} ds$$

as long as the integral exists and any discontinuities, usually represented by multiple integrals of the form 1/2[$f(x_+) + f(x_-)$], are finite. The transform quantity *F(s)* is often representedas *f(s)* and the Fourier transform is often represented by the operator *F*.

There are functions for which the Fourier transform does not exist; however, most physical functions have a Fourier transform, especially if the transform represents a physicalquantity. Other functions can be treated with Fourier theory as limiting cases. Many of thecommon theoretical functions are actually limiting cases in Fourier theory.

Usually functions or waveforms can be split into even and odd parts as follows

$$f(x) = E(x) + O(x)$$

Where

$$E(x) = \frac{1}{2}[f(x) + f(-x)]$$

$$O(x) = \frac{1}{2}[f(x) - f(-x)]$$

and*E(x)*, *O(x)* are, in general, complex. In this representation, the Fourier transform of*f(x)* reduces to

$$2\int_0^\infty E(x)\cos(2\pi xs)\,dx - 2i\int_0^\infty O(x)\sin(2\pi xs)\,dx$$

It follows then that an even function has an even transform and that an odd function hasan odd transform.

The *cosine transform* of a function *f(x)* is defined as

$$F_c(s) = 2\int_0^\infty f(x)\cos\ 2\pi sx\ dx.$$

This is equivalent to the Fourier transform if *f(x)* is an even function. In general, the evenpart of the Fourier transform of *f(x)*is the cosine transform of the even part of *f(x)*. Thecosine transform has a reverse transform given by

$$f(x) = 2\int_0^\infty F_c(s)\cos\ 2\pi sx\ ds$$

Likewise, the *sine transform* of *f(x)* is defined by

$$F_s(s) = 2\int_0^\infty f(x)\sin\ 2\pi sx\ dx.$$

As a result, *i*times the odd part of the Fourier transform of *f(x)* is the sine transform ofthe odd part of *f(x)*.

Combining the sine and cosine transforms of the even and odd parts of *f(x)* leads to theFourier transform of the whole of *f(x)*:

$$\mathscr{F} f(x) = \mathscr{F}_c E(x) - i\mathscr{F}_s O(x)$$

Where F, F_c and F_s stand for -*i* times the Fourier transform, the cosine transform, andthe sine transform respectively, or

$$F(s) = \frac{1}{2}F_c(s) - \frac{1}{2}iF_s(s)$$

2.10. Summary

In this unit we have studied

- Airy's Stress Function
- Direct method for determining Airy polynomial
- Cantilever having Udl and concentrated load of the free end
- Bending of a Cantilever Loaded at the End
- Bending of a Beam by Uniform Load
- Fourier Series
- Complex Potentials
- Cauchy Integral Method
- The Fourier Transform

2.11. Keywords

Cantilever

Fourier series

Complex potentials

Cauchy Integral method

Airy's Stress

2.12. Exercise

1. Derive an equation for the deflection curve with the use of Fourier Series.
2. Write short notes on Complex Potentials.
3. In what means is Fourier Transform helpful and show that $F(s) = \frac{1}{2}F_c(s) - \frac{1}{2}iF_s(s)$.
4. Write short notes on Airy's Stress Function
5. Using solution by polynomials in case of a cantilever loaded at the end, show that

   $$\frac{Pc^2}{2IG}(l - x) = \frac{3P}{4cG}(l - x)$$

6. Determine Airy polynomial for a cantilever having uniformly distributed load.
7. Find out the equation for deflection δ due to bending of beam by uniform load.
8. Write a short note on Cauchy Integral Method.

Unit 3

Two-Dimensional Problems in Polar Co-Ordinates

Structure

3.1. Introduction

The problem addressed in this work is two-dimensional elastic wave propagation in the vicinity of cylindrical objects. The motivation for such a study is to simulate phenomena associated with boreholes. A two dimensional study, in which the cylindrical geometry is tackled, is a first step towards constructing a full 3-D simulator for borehole measurement techniques such as vertical seismic profiling.

The algorithm described here is based on a direct solution in polar coordinates of the equations of momentum conservation and the stress strain relations for an isotropic solid. Solving in polar coordinates appears necessary because of the cylindrical geometry, since representing the cylindrical cavity using Cartesian coordinates would require a prohibitively fine spatial grid.

3.2. Objectives

After studying this unit we are able to understand

- Basic equations

- Biharmonic equation
- Solution of Biharmonic Equation for Axial Symmetry
- General Solution of Biharmonic Equation
- Saint Venant's Principle
- Thick Cylinder

3.3. Basic Equation

In discussing stresses in circular rings and disks, curved bars of narrow rectangular cross section with a circular axis, etc., it is advantageous to use polar coordinates. The position of a point in the middle plane of a plate is then defined by the distance from the origin *O* (Fig. 1) and by the angle θ between *r* and a certain axis *Ox* fixed in the plane.

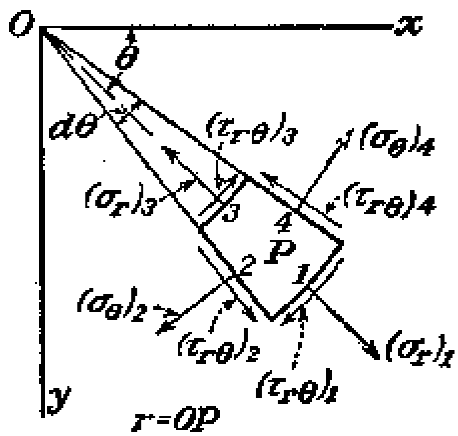


Fig. 1

Let us now consider the equilibrium of a small element 1234 cut out from the plate by the radial sections 04,02, normal to the plate, and by two cylindrical surfaces 3,1, normal to theplate.The normal stress component in the radial direction is denoted by σ_r, thenormal component in the circumferential direction by σ_θ, and the shearing-stresscomponent by$\tau_{r\theta}$, each symbol representing stress at the point *r*, θ, which is themid-point P of the element.On account of the variation of stress the values at themid-points of the sides 1, 2, 3, 4 are not quite the same as the values σ_r, σ_θ,$\tau_{r\theta}$, andare denoted by $(\sigma_r)_1$, etc., in Fig..The radii of the sides 3, 1 aredenoted by r_3, r_1. The radial force on the side 1 is $\sigma_{r1} r_1 d\theta$which maybe written $(\sigma_r r)_1 d\theta$, and similarly the radial force on side 3 is $(\sigma_r r)_3 d\theta$. The normal force on side 2 has a component along the radius through P of$(\sigma_\theta)_2(r_1-r_3)\ \sin\ (d\ \theta\ /2)$, which may be replaced by$(\sigma_\theta)_2\ dr\ (d\ \theta\ /2)$. The corresponding component from side 4 is $(\sigma_\theta)_4 dr\ (d\theta/2)$. The shearing forces on sides 2 and 4 give $[(\tau_{r\theta})_2 - (\tau_{r\theta})_4]dr$.

Summing up forces in the radial direction, including body force R per unit volume in the radial direction, we obtain the equation of equilibrium

$$(\sigma_r r)_1\, d\theta - (\sigma_r r)_3\, d\theta - (\sigma_\theta)_2\, dr\,\frac{d\theta}{2} - (\sigma_\theta)_4\, dr\,\frac{d\theta}{2} + [(\tau_{r\theta})_2 - (\tau_{r\theta})_4]\, dr + Rr\, d\theta\, dr = 0$$

Dividing by $drd\theta$ this becomes

$$\frac{(\sigma_r r)_1 - (\sigma_r r)_3}{dr} - \frac{1}{2}[(\sigma_\theta)_2 + (\sigma_\theta)_4] + \frac{(\tau_{r\theta})_2 - (\tau_{r\theta})_4}{d\theta} + Rr = 0$$

If the dimensions of the element are now taken smaller and smaller, to the limit zero, the first term of this equation is in the limit $\partial(\sigma_r r)/\partial r$. The second becomes σ_θ, and the third $\partial\tau_{r\theta}/\partial r$. The equation of equilibrium in the tangential direction may be derived in the same manner. The two equations take the final form

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0$$

These equations take the place of equations,

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + X = 0$$

$$\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + Y = 0$$

when we solve the two-dimensional problems by means of polar coordinates. When the body force R is zero they are satisfied by putting

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\,\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

where ϕ is the stress function as a function of r and θ.

3.4. Biharmonic Equation

We have seen that in the case of two-dimensional problems of elasticity, in the absence of volume forces and with given forces at the boundary, the stresses are defined by a stress function ϕ , which satisfies the biharmonic equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

and the boundary conditions

$$l\frac{\partial^2 \phi}{\partial y^2} - m\frac{\partial^2 \phi}{\partial x \partial y} = \bar{X}$$
$$m\frac{\partial^2 \phi}{\partial x^2} - l\frac{\partial^2 \phi}{\partial x \partial y} = \bar{Y}$$

Knowing the forces distributed along the boundary we may calculate atthe boundary by integration of boundary conditions. Then the problem is reduced to that of finding a function ϕ which satisfies biharmonic eq. at every pointwithin the boundary and at the boundary has, together with its first derivatives, the prescribed values.

Using the finite difference method, let us take a square net

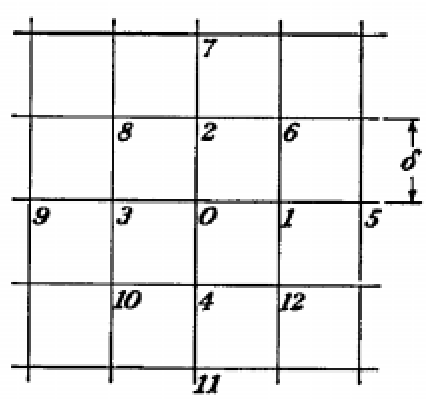


Fig. 2

and transform biharmonic eq. to a finite-difference equation. Knowing thesecond derivatives,

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 \approx \frac{1}{\delta^2}(\phi_1 - 2\phi_0 + \phi_3)$$
$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_1 \approx \frac{1}{\delta^2}(\phi_5 - 2\phi_1 + \phi_0)$$
$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_3 \approx \frac{1}{\delta^2}(\phi_0 - 2\phi_3 + \phi_9)$$

We conclude that,

$$\left(\frac{\partial^4\phi}{\partial x^4}\right)_0 = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\phi}{\partial x^2}\right) \approx \frac{1}{\delta^2}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_1 - 2\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \left(\frac{\partial^2\phi}{\partial x^2}\right)_3\right]$$
$$\approx \frac{1}{\delta^4}(6\phi_0 - 4\phi_1 - 4\phi_3 + \phi_5 + \phi_9)$$

Similarly we find

$$\frac{\partial^4\phi}{\partial y^4} \approx \frac{1}{\delta^4}(6\phi_0 - 4\phi_2 - 4\phi_4 + \phi_7 + \phi_{11})$$
$$\frac{\partial^4\phi}{\partial x^2\,\partial y^2} \approx \frac{1}{\delta^4}[4\phi_0 - 2(\phi_1 + \phi_2 + \phi_3 + \phi_4) + \phi_6 + \phi_8 + \phi_{10} + \phi_{12}]$$

Substituting into biharmonic eq. we obtain the required finite-difference equation

$$20\phi_0 - 8(\phi_1 + \phi_2 + \phi_3 + \phi_4) + 2(\phi_6 + \phi_8 + \phi_{10} + \phi_{12}) + \phi_5 + \phi_7 + \phi_9 + \phi_{11} = 0$$

This equation must be satisfied at every nodal point of the net within the boundary of the plate. To find the boundary values of the stress function ϕ we integrate boundary conditions. Assuming that

$$l = \cos\alpha = \frac{dy}{ds} \qquad \text{and} \qquad m = \sin\alpha = -dx/ds$$

We write the boundary equations in the following form:

$$\frac{dy}{ds}\frac{\partial^2\phi}{\partial y^2} + \frac{dx}{ds}\frac{\partial^2\phi}{\partial x\,\partial y} = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right) = \bar{X}$$
$$-\frac{dx}{ds}\frac{\partial^2\phi}{\partial x^2} - \frac{dy}{ds}\frac{\partial^2\phi}{\partial x\,\partial y} = -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right) = \bar{Y}$$

And by integration we obtain

$$-\frac{\partial\phi}{\partial x}=\int\bar{Y}\,ds$$
$$\frac{\partial\phi}{\partial y}=\int\bar{X}\,ds$$

To findϕ we use the equation

$$\frac{\partial\phi}{\partial s}=\frac{\partial\phi}{\partial x}\frac{dx}{ds}+\frac{\partial\phi}{\partial y}\frac{dy}{ds}$$

which, after integration by parts, gives

$$\phi=x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}-\int\left(x\frac{\partial^2\phi}{\partial s\,\partial x}+y\frac{\partial^2\phi}{\partial s\,\partial y}\right)ds$$

3.5. Solution of Biharmonic Equation for Axial Symmetry

If the stress distribution is symmetrical with respect to the axis through*O* perpendicular to the xy-plane (Fig. 3),

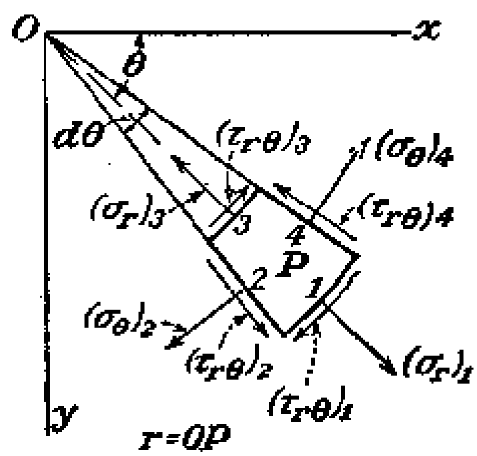


Fig. 3

the stress components do notdepend on θ and are functions of *r* only. From symmetry it followsalso that the shearing stress $\tau_{r\theta}$ must vanish. Then only the first ofthe two equations of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0$$

remains, and we have

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \qquad (1)$$

If the body force R is zero, we may use the stress function ϕ . When this function depends only on r, the equation of compatibility

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0$$

Becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right) = \frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0 \qquad (2)$$

This is an ordinary differential equation, which can be reduced to a linear differential equation with constant coefficients by introducing a new variable t such that $r = e^t$. In this manner the general solution of Eq. (2) can easily be obtained. This solution has four constants of integration, which must be determined from the boundary conditions. By substitution it can be checked that

$$\phi = A \log r + Br^2 \log r + Cr^2 + D \qquad (3)$$

is the general solution. The solutions of all problems of symmetrical stress distribution and no body forces can be obtained from this. The corresponding stress components from Eqs.

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\, \partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

are

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} = \frac{A}{r^2} + B(1 + 2\log r) + 2C$$
$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2\log r) + 2C$$
$$\tau_{r\theta} = 0 \qquad (4)$$

If there is no hole at the origin of coordinates, constants A and B vanish, since otherwise the stress components (4) become infinite when$r = 0$. Hence, for a plate without a hole at the origin and with nobody forces, only one case of stress distribution symmetrical withrespect to the axis may exist, namely that when

$\sigma_r = \sigma_\theta =$ constantand the plate is in a condition of uniform tension or uniform compression in all directions in its plane.

If there is a hole at the origin, other solutions than uniform tensionor compression can be derived from expressions (4).Taking B as zero, for instance, Eqs. 4 become

$$\sigma_r = \frac{A}{r^2} + 2C$$
$$\sigma_\theta = -\frac{A}{r^2} + 2C \qquad (5)$$

3.6. General Solution of the Two-dimensional Problem in Polar Coordinates

Having discussed various particular cases of the two dimensional problem in polarcoordinates we are now in a position to write down the general solution of theproblem. The general expression for the stress function ϕ , satisfying the compatibility equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0$$

$$\phi = a_0 \log r + b_0 r^2 + c_0 r^2 \log r + d_0 r^2\theta + a_0'\theta + \frac{a_1}{2} r\theta \sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r \log r) \cos\theta - \frac{c_1}{2} r\theta \cos\theta + (d_1 r^3 + c_1' r^{-1} + d_1' r \log r) \sin\theta + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}) \cos n\theta + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}) \sin n\theta \quad (6)$$

The first three terms in the first line of this expression represent the solution for the stress distribution symmetrical with, respect to the origin of coordinates. The fourth term gives the stress distribution on the straight edge of the plate. The fifth term gives the solution for pure shear. The firstterm in the second line is the simple radial distribution for a load in the direction$\theta = 0$. The remaining terms of the second line represent the solution for aportion of a circular ring bent by a radial force. By a combination ofall the terms of the second line the solution for force acting on an infinite plate was obtained. Analogous solutions are obtained also from the third lineof expression (6), the only difference being that the direction of the force is changedby $\pi/2$. The further terms of (6) represent solutions for shearing and normalforces, proportional to $\sin n\theta$ and $\cos n\theta$, acting on the inner and outer boundariesof a circular ring.

In the case of a portion of a circular ring the constants of integration in expression (6) can be calculated without any difficulty from the boundary conditions.If we have a complete ring, certain additional investigations of the displacementsare sometimes necessary in determining these constants. We shall consider thegeneral case of a complete ring and assume that the intensities of the normal andshearing forces at the boundaries $r = a$ and $r = b$ are given by the following trigonometrical series:

$$
\begin{aligned}
(\sigma_r)_{r=a} &= A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \\
(\sigma_r)_{r=b} &= A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta + \sum_{n=1}^{\infty} B_n' \sin n\theta \\
(\tau_{r\theta})_{r=a} &= C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta + \sum_{n=1}^{\infty} D_n \sin n\theta \\
(\tau_{r\theta})_{r=b} &= C_0' + \sum_{n=1}^{\infty} C_n' \cos n\theta + \sum_{n=1}^{\infty} D_n' \sin n\theta
\end{aligned}
\qquad (a)
$$

in which the constants A_o, A_n, B_n, . . . , are to be calculated in the usual manner from the given distribution of forces at the boundaries. Calculating the stress components from expression (6) by using Eqs.

$$
\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\
\sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} \\
\tau_{r\theta} &= \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\, \partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)
\end{aligned}
$$

and comparingthe values of these components for $r = a$ and $r = b$with those given by Eqs. (a),we obtain a sufficient number of equations to determine the constants of integration in all cases with $n \geq 2$. For $n = 0$, i.e., for the terms in the first line of expression (6), and for $n = 1$, i.e., for the terms in the second and third lines, further investigations are necessary.

Taking the first line of expression (6) as a stress function, the constant a_0' is determined by the magnitude of the shearing forces uniformly distributed along the boundaries. The stress distribution given by the term with the factor d_o is many valued and, in a complete ring, we must assume $d_o = 0$. For the determination of the remaining three constants a_o, b_o and c_o we have only two equations,

$$
(\sigma_r)_{r=a} = A_0 \qquad \text{and} \qquad (\sigma_r)_{r=b} = A_0'
$$

The additional equation for determining these constants is obtained from the consideration of displacements. The displacements in a complete ring should be*single-valued*functions of θ. Our previous investigation shows that this condition is fulfilled if we put $c_o = 0$. Then the remaining two constants a_o and b_o are determined from the two boundary conditions stated above.

Let us consider now, the terms for which $n= 1$. For determining the eight constants$a_1, b_1, \ldots, d_1'$ entering into the second and the third lines of expression (6), we calculate the stress components σ_r and $\tau_{r\theta}$ using this portion of ϕ. Then using conditions (a) and equating corresponding coefficients

Of *sin nθ* and *cosnθ* at the inner and outer boundaries, we obtain the following eight equations:

$$
\begin{aligned}
(a_1 + b_1')a^{-1} + 2b_1a - 2a_1'a^{-3} &= A_1 \\
(a_1 + b_1')b^{-1} + 2b_1b - 2a_1'b^{-3} &= A_1' \\
(c_1 + d_1')a^{-1} + 2d_1a - 2c_1'a^{-3} &= B_1 \\
(c_1 + d_1')b^{-1} + 2d_1b - 2c_1'b^{-3} &= B_1'
\end{aligned}
\qquad (b)
$$

$$
\begin{aligned}
2d_1a - 2c_1'a^{-3} + d_1'a^{-1} &= -C_1 \\
2d_1b - 2c_1'b^{-3} + d_1'b^{-1} &= -C_1' \\
2b_1a - 2a_1'a^{-3} + b_1'a^{-1} &= D_1 \\
2b_1b - 2a_1'b^{-3} + b_1'b^{-1} &= D_1'
\end{aligned}
\qquad (c)
$$

Comparing Eqs. (b) with (c) it can be seen that they are compatible only if

$$
\begin{aligned}
a_1a^{-1} &= A_1 - D_1 \\
a_1b^{-1} &= A_1' - D_1' \\
c_1a^{-1} &= B_1 + C_1 \\
c_1b^{-1} &= B_1' + C_1'
\end{aligned}
\qquad (d)
$$

From which it follows that

$$
a(A_1 - D_1) = b(A_1' - D_1'), \qquad a(B_1 + C_1) = b(B_1' + C_1') \qquad (e)
$$

It can be shown that Eq. (e) are always fulfilled if the forces acting on the ring are in equilibrium. Taking, for instance, the sum of the components of all the forces in the direction of the *x*-axis as zero, we find

$$\int_0^{2\pi} \{[b(\sigma_r)_{r=b} - a(\sigma_r)_{r=a}] \cos\theta - [b(\tau_{r\theta})_{r=b} - a(\tau_{r\theta})_{r=a}] \sin\theta\} \, d\theta = 0$$

Substituting for σ_r and $\tau_{r\theta}$ from (a), we arrive at the first of Eqs. (e). In the same manner, by resolving all the forces along the y-axis, we obtain the second of Eqs(e).

When a_1 and c_1; are determined from Eqs. (d) the two systems of Eqs. (b) and(c) become identical, and we have only four equations for determining the remaining six constants. The necessary two additional equations are obtained by considering the displacements. The terms in the second line in expression (6) represent the stress function for a combination of a simple radial distribution and the bending stresses in a curved bar. By superposing the general expressions for the displacements in these two cases, namely Eqs.

$$\begin{aligned} u &= -\frac{2P}{\pi E}\cos\theta \log r - \frac{(1-\nu)P}{\pi E}\theta\sin\theta + A\sin\theta + B\cos\theta \\ v &= \frac{2\nu P}{\pi E}\sin\theta + \frac{2P}{\pi E}\log r \sin\theta - \frac{(1-\nu)P}{\pi E}\theta\cos\theta \\ &\qquad + \frac{(1-\nu)P}{\pi E}\sin\theta + A\cos\theta - B\sin\theta + Cr \end{aligned} \tag{7}$$

andEqs.

$$\begin{aligned} u &= -\frac{2D}{E}\theta\cos\theta + \frac{\sin\theta}{E}\left[D(1-\nu)\log r + A(1-3\nu)r^2 \right. \\ &\qquad \left. + \frac{B(1+\nu)}{r^2}\right] + K\sin\theta + L\cos\theta \\ v &= \frac{2D}{E}\theta\sin\theta - \frac{\cos\theta}{E}\left[A(5+\nu)r^2 + \frac{B(1+\nu)}{r^2} \right. \\ &\qquad \left. - D(1-\nu)\log r\right] + \frac{D(1+\nu)}{E}\cos\theta + K\cos\theta - L\sin\theta + Hr \end{aligned} \tag{8}$$

and, substituting $a_1/2$ for $-P/\pi$in Eqs. (7) and b_1' for D inEqs. (8), we find the following many-valued terms in the expressions for the displacements u and v, respectively:

$$\frac{a_1}{2}\frac{1-\nu}{E}\theta\sin\theta + \frac{2b_1'}{E}\theta\sin\theta$$

$$\frac{a_1}{2}\frac{1-\nu}{E}\theta\cos\theta + \frac{2b_1'}{E}\theta\cos\theta$$

These terms must vanish in the case of a complete ring, hence

$$\frac{a_1}{2}\frac{1-\nu}{E} + \frac{2b_1'}{E} = 0$$

Or

$$b_1' = -\frac{a_1(1-\nu)}{4} \qquad (f)$$

Considering the third line of expression (6) in the same manner, we find

$$d_1' = -\frac{c_1(1-\nu)}{4} \qquad (g)$$

Equations (f) and (g), together with Eqs. (b) and (c), are now sufficient for determining all the constants in the stress function represented by the second and the third lines of expression (6).

We conclude that in the case of a complete ring the boundary conditions (a) arenot sufficient for the determination of the stress distribution, and it is necessary toconsider the displacements. The displacements in a complete ring must be single valued and to satisfy this condition we must have

$$c_0 = 0, \qquad b_1' = -\frac{a_1(1-\nu)}{4}, \qquad d_1' = -\frac{c_1(1-\nu)}{4} \qquad (9)$$

We see that the constants b_1' and d_1' depend on Poisson's ratio. Accordingly the stress distribution in a complete ring will usually depend on the elastic properties of the material. It becomes independent of the elastic constants only when a_1 and c_1 vanish so that, from Eq. (9), $b_1' = d_1' = 0$. This particular case occurs if [see Eqs. (d)]

$$A_1 = D_1 \qquad \text{and} \qquad B_1 = -C_1$$

We have such a condition when the resultant of the forces applied to each boundaryof the ring vanishes. Take, for instance, the resultant component in the

x-direction of forces applied to the boundary $r = a$. This component, from (a), is

$$\int_0^{2\pi} (\sigma_r \cos\theta - \tau_{r\theta} \sin\theta) a\, d\theta = a\pi(A_1 - D_1)$$

If it vanishes we find $A_1 = D_1$. In the same manner, by resolving the forces in the *y*-direction, we obtain $B_1 = -C_1$ when the *y*-component is zero. From this we may conclude that the stress distribution in a complete ring is independent of the elastic constants of the material if the resultant of the forces applied to each boundary is zero. The moment of these forces need not be zero.

3.7. Saint-Venant's Principle

In the previous article several caseswere discussed in which exact solutions for rectangular plates wereobtained by taking very simple forms for the stress function ϕ . Ineach case all the equations of elasticity are satisfied, but the solutionsare exact only if the surface forces are distributed in the manner given.

In the case of pure bending, for instance in the figure

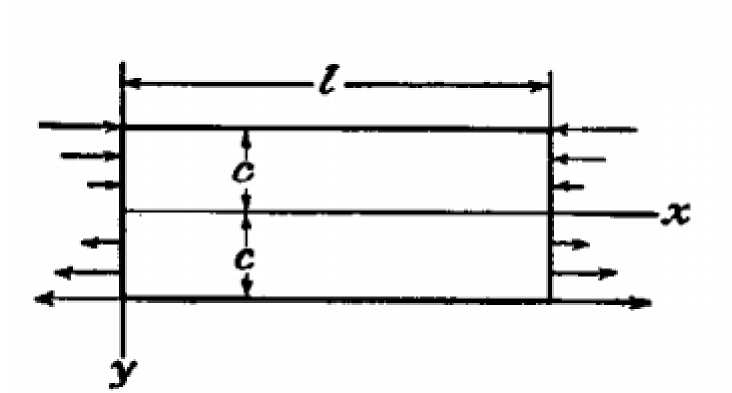


Fig. 4

the bending momentmust be produced by tensions and compressions on the ends, thesetensions and compressions being proportional to the distance from the neutral axis. The fastening of the end, if any, must be such as not to interfere with distortion of the plane of the end. If the above conditions are not fulfilled, i.e., the bending moment is applied in some different manner or the constraint is such that it imposes other forces on the end section. The practical utility of the solution however is not limited to such a specialized case. It can be applied with sufficient accuracy to cases of bending in which the conditions at the ends are not rigorously satisfied. Such an extension in the application of the solution is usually based on the so-called principle of

Saint-Venant.

This principle states that if the forces acting on a small portion of the surface of an elastic body are replaced by another statically equivalent system of forces acting on the same portion of the surface, this redistribution of loading produces substantial changes in the stresses locally but has a negligible effect on the stresses at distances which are large in comparison with the linear dimensions of the surface on which the forces are changed. For instance, in the case of pure bending of a rectangular strip (Fig. 4) the cross-sectional dimensions of which aresmall in comparison with its length, the manner of application of the external bending moment affects the stress distribution only in the vicinity of the ends and is of no consequence for distant cross sections.

The same is true in the case of axial tension. Only near the loaded end does the stress distribution depend on the manner of applying the tensile force, and in cross sections at a distance from the end the stresses are practically uniformly distributed.

3.8. Thick Cylinder

The circular cylinder, of special importance in engineering, is usually divided intothin-walled and thick-walled classifications. A thin-walled cylinder is defined as onein which the tangential stress may, within certain prescribed limits, be regarded asconstant with thickness. The following familiar expression applies to the case of athin-walled cylinder subject to internal pressure:

$$\sigma_\theta = \frac{pr}{t}$$

Here p is the internal pressure, r the mean radius and t the thickness. If the wall thickness exceeds the inner radius by more than approximately 10%, the cylinder is generally classified as thick walled and the variation of stress with radius can no longer be disregarded.

In the case of a thick-walled cylinder subject to uniform internal or external pressure, the deformation is symmetrical about the axial (z) axis. Therefore, the equilibrium and strain-displacement equations, Eqs.

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (10)$$

and

$$\varepsilon_r = \frac{du}{dr}, \qquad \varepsilon_\theta = \frac{u}{r}, \qquad \gamma_{r\theta} = 0$$

apply to any

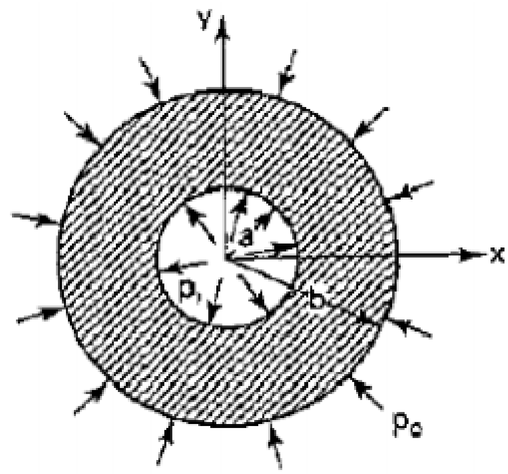


Fig. 5

point on a line of unit length cut from the cylinder (Fig. 5). Assuming that the ends of the cylinder are open and unconstrained, $\sigma_z = 0$, as shall be subsequently demonstrated. Thus, the cylinder is in a condition of plane stress and according to Hooke's law, the strains are given by

$$\frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

From these, σ_r and σ_θ are as follows:

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) \qquad (11)$$

Substituting this into Eq. (10) results in the following *equidimensional equation* in radial displacement:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

having a solution

$$u = c_1 r + \frac{c_2}{r} \qquad \text{(a)}$$

The radial and tangential stresses may now be written in terms of the constants of integration c_1 and c_2 by combining Eqs. (a) and (11):

$$\sigma_r = \frac{E}{1-\nu^2}\left[c_1(1+\nu) - c_2\left(\frac{1-\nu}{r^2}\right)\right] \qquad \text{(b)}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[c_1(1+\nu) + c_2\left(\frac{1-\nu}{r^2}\right)\right] \qquad \text{(c)}$$

The constants are determined from consideration of the conditions pertaining to the inner and outer surfaces.

Observe that the sum of the radial and tangential stresses is constant, regardless of radial position: $\sigma_r + \sigma_\theta = 2Ec_1/(1 - \nu)$. Hence, the longitudinal strain is constant:

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = \text{constant}$$

We conclude therefore that *plane sectionsremain plane* subsequent to loading. Then $\sigma_z = E\varepsilon_z$ = constant = c. But if the ends of the cylinder are open and free,

$$\int_a^b \sigma_z \cdot 2\pi r\, dr = \pi c(b^2 - a^2) = 0$$

or $c = \sigma_z = 0$, as already assumed previously.

For a cylinder subjected to internal and external pressures p_i and p_o, respectively, the boundary conditions are

$$(\sigma_r)_{r=a} = -p_i, \qquad (\sigma_r)_{r=b} = -p_o \qquad \text{(d)}$$

where the negative sign connotes compressive stress. The constants are evaluated by substitution of Eqs. (d) into (b):

$$c_1 = \frac{1-\nu}{E}\,\frac{a^2 p_i - b^2 p_o}{b^2 - a^2}, \qquad c_2 = \frac{1+\nu}{E}\,\frac{a^2 b^2 (p_i - p_o)}{b^2 - a^2} \qquad \text{(e)}$$

Leading finally to

$$\sigma_r = \frac{a^2p_i - b^2p_o}{b^2 - a^2} - \frac{(p_i - p_o)a^2b^2}{(b^2 - a^2)r^2}$$

$$\sigma_\theta = \frac{a^2p_i - b^2p_o}{b^2 - a^2} + \frac{(p_i - p_o)a^2b^2}{(b^2 - a^2)r^2}$$

$$u = \frac{1 - \nu}{E}\frac{(a^2p_i - b^2p_o)r}{b^2 - a^2} + \frac{1 + \nu}{E}\frac{(p_i - p_o)a^2b^2}{(b^2 - a^2)r}$$

The maximum numerical value of σ_r is found at $r = a$ to be p_i, provided that p_i exceeds p_o. If $p_o > p_i$ the maximum σ_r, occurs at $r = b$ and equals p_o. On the other hand, the maximum σ_θ occurs at either the inner or outer edge according to the pressure ratio.

Recall that the maximum shearing stress at any point equals one-half the algebraic difference between the maximum and minimum principal stresses. At anypoint in the cylinder, we may therefore state that

$$\tau_{max} = \frac{1}{2}(\sigma_\theta - \sigma_r) = \frac{(p_i - p_o)a^2b^2}{(b^2 - a^2)r^2}$$

The largest value of τ_{max} is found at $r = a$, the inner surface. The effect of reducing p_o is clearly to increase τ_{max}. Consequently, the greatest τ_{max} corresponds to $r = a$ and $p_0 = 0$.

$$\tau_{max} = \frac{p_i b^2}{b^2 - a^2} \qquad (12)$$

Because σ_r and σ_θ are principal stresses, τ_{max} occurs on planes making an angle of 45^0 with the plane on which σ_r and σ_θ act. This is quickly confirmed by a Mohr's circle construction. The pressure p_{yp} that initiates yielding at the inner surface is obtained by setting $\tau_{max} = \sigma_{yp}/2$ in Eq. (12):

$$p_{yp} = \frac{(b^2 - a^2)\sigma_{yp}}{2b^2}$$

Here σ_{yp} is the yield stress in uniaxial tension.

3.9. Summary

In this unit we have studied

- Basic equations

- Biharmonic equation
- Solution of Biharmonic Equation for Axial Symmetry
- General Solution of Biharmonic Equation
- Saint Venant’s Principle
- Thick Cylinder

3.10. Keywords

Biharmonic equation

Saint Venant’s Principle

Thick Cylinder

3.11. Exercise

1. State and explain Saint Venant’s principle.
2. Derive an expression for maximum shearing stress in case of thick cylinders.
3. Show that maximum tensile stress is three times the uniform stress applied at ends of the plate.
4. In case of Rotating Disks show that maximum tangential stress doubles when a small circular hole is made at the center of it.
5. Determine stress induced due bending of curved bar due to load at the end.

Unit 4

Two-Dimensional Problems in Polar Co-Ordinates – Part II

Structure

4.1. Introduction

The famous solution of Stress Concentration Factor (SCF) for a circular hole in a plate subjected to uniform tensile loading by Kirsch is valid only for an infinite plate with a finite hole. Most of the structures of practical importance have finite geometry, hence the SCF will not be 3 and usually one finds it difficult to solve it by theory of elasticity. Photo elasticity comes in handy to evaluate SCF for finite body problems. Here the evaluation of SCF reduces to finding the ratio of maximum fringe order to the far field fringe order.

4.2. Objectives

After studying this unit we are able to understand

- Stress-concentration due to a Circular Hole in a Stressed Plate (Kirsch Problem)
- Rotating Disk
- Bending of a Curved Bar by a Force at the End

4.3. Stress-concentration due to a Circular Hole in a Stressed Plate (Kirsch Problem)

Figure 1 represents a plate submitted to a uniform tension of magnitude S in the *x*-direction. If a small circular hole is made in the middle of the plate, the stress distribution in the neighborhood of the hole will be changed, but we can conclude from Saint-Venant's principle that the change is negligible at distances which are large compared with a, the radius of the hole.

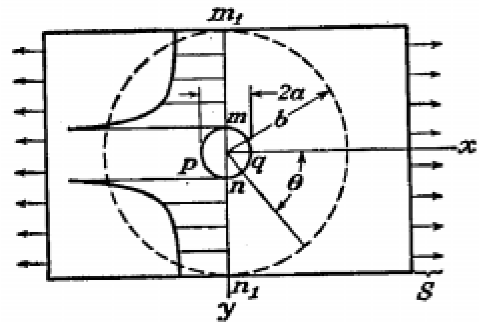


Fig. 1

Consider the portion of the plate within a concentric circle of radius *b*, large in comparison with *a*. The stresses at the radius *b* are effectively the same as in the plate without the hole and are therefore given by

$$\begin{aligned}(\sigma_r)_{r=b} &= S\cos^2\theta = \tfrac{1}{2}S(1+\cos 2\theta) \\ (\tau_{r\theta})_{r=b} &= -\tfrac{1}{2}S\sin 2\theta\end{aligned} \qquad (a)$$

These forces, acting around the outside of the ring having the inner and outer radii $r = a$ and $r = b$, give a stress distribution within the ring which we may regard as consisting of two parts. The first is due to the constant component 1/2*S* of the normal forces. The stresses it produces can be calculated by means of Eqs.

$$\begin{aligned}\sigma_r &= \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\cdot\frac{1}{r^2} + \frac{p_ia^2 - p_ob^2}{b^2 - a^2} \\ \sigma_\theta &= -\frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\cdot\frac{1}{r^2} + \frac{p_ia^2 - p_ob^2}{b^2 - a^2}\end{aligned} \qquad (13)$$

The remaining part, consisting of the normal forces 1/2*Scos* 2*θ*, together with the shearing forces -1/2*S sin* 2*θ*, produces stresses which may be derived from a stress function of the form

$$\phi = f(r)\cos 2\theta \qquad (b)$$

Substituting this into the compatibility equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0$$

we find the following ordinary differential equation to determine *f(r)*:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right) = 0$$

The general solution is

$$f(r) = Ar^2 + Br^4 + C\frac{1}{r^2} + D$$

The stress function is therefore

$$\phi = \left(Ar^2 + Br^4 + C\frac{1}{r^2} + D\right)\cos 2\theta \qquad (c)$$

and the corresponding stress components, from Eqs.

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$
$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$
$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\,\partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

are

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta$$
$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta \qquad (d)$$
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta$$

The constants of integration are now to be determined from conditions (a) for the outer boundary and from the condition that the edge of the hole is free from external forces. These conditions give

$$2A + \frac{6C}{b^4} + \frac{4D}{b^2} = -\frac{1}{2}S$$
$$2A + \frac{6C}{a^4} + \frac{4D}{a^2} = 0$$
$$2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2} = -\frac{1}{2}S$$
$$2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2} = 0$$

Solving these equations and putting a/b = 0, i.e., assuming an infinitely large plate, we obtain

$$A = -\frac{S}{4}, \qquad B = 0, \qquad C = -\frac{a^4}{4}S, \qquad D = \frac{a^2}{2}S$$

Substituting these values of constants into Eqs. (d) and adding the stresses produced by the uniform tension 1/2*S* on the outer boundary calculated from Eqs. (13) we find

$$\sigma_r = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$
$$\sigma_\theta = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$
$$\tau_{r\theta} = -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta \qquad (14)$$

If *r* is very large, σ_r and $\tau_{r\theta}$ approach the values given in Eqs. (a). At the edge of the hole, $r = a$ and we find

$$\sigma_r = \tau_{r\theta} = 0, \qquad \sigma_\theta = S - 2S\cos 2\theta$$

It can be seen that σ_θ is greatest when $\theta = \pi/2$ or $\theta = 3\pi/2$, i.e., at the ends *m* and *n* of the diameter perpendicular to the direction of the tension (Fig. 6). At these points $(\sigma_\theta)_{max} = 3S$. This is the maximum tensile stress and is three times the uniform stress *S*, applied atthe ends of the plate.

At the points *p* and *q*, θ is equal to π and *0* and we find

$$\sigma_\theta = -S$$

So that there is a compression stress in the tangential direction at these points

4.4. Rotating Disks

The stress distribution in rotating circular disks is of great practical importance. If the thickness of the disk is small in comparison with its radius, the variation of radial and tangential stresses over the thickness can be neglected and the problem can be easily solved. If the thickness of the disk is constant Eq.

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \tag{15}$$

can be applied, and it is only necessary to put the body force equal to the inertia force. Then

$$R = \rho\omega^2 r \tag{a}$$

Where ϱ is the mass per unit volume of the material of the disk and ω the angular velocity of the disk.

Equation (15) can then be written in the form

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0 \tag{b}$$

This equation is satisfied if we derive the stress components from a stress function F in the following manner:

$$r\sigma_r = F, \qquad \sigma_\theta = \frac{dF}{dr} + \rho\omega^2 r^2 \tag{c}$$

The strain components in the case of symmetry are,

$$\epsilon_r = \frac{du}{dr}, \qquad \epsilon_\theta = \frac{u}{r}$$

Eliminating *u* between these equations, we find

$$\epsilon_\theta - \epsilon_r + r\frac{d\epsilon_\theta}{dr} = 0 \tag{d}$$

Substituting for the strain components their expressions in terms of the stress components,

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

and using Eqs. (c), we find that the stress function F should satisfy the following equation:

$$r^2 \frac{d^2F}{dr^2} + r\frac{dF}{dr} - F + (3 + \nu)\rho\omega^2 r^3 = 0 \qquad (e)$$

It can be verified by substitution that the general solution of this equation is

$$F = Cr + C_1 \frac{1}{r} - \frac{3 + \nu}{8} \rho\omega^2 r^3 \qquad (f)$$

And from Eqs, (c) we find

$$\begin{aligned} \sigma_r &= C + C_1 \frac{1}{r^2} - \frac{3 + \nu}{8} \rho\omega^2 r^2 \\ \sigma_\theta &= C - C_1 \frac{1}{r^2} - \frac{1 + 3\nu}{8} \rho\omega^2 r^2 \end{aligned} \qquad (g)$$

The integration constants C and C_1 are determined from the boundary conditions.

For a *solid disk* we must take $C_1 = 0$ since otherwise the stresses (g) become infinite at the center. The constant C is determined from the condition at the periphery ($r = b$) of the disk. If there are no forces applied there, we have

$$(\sigma_r)_{r=b} = C - \frac{3 + \nu}{8} \rho\omega^2 b^2 = 0$$

from which

$$C = \frac{3 + \nu}{8} \rho\omega^2 b^2$$

and the stress components, from Eqs. (g), are

$$\begin{aligned} \sigma_r &= \frac{3 + \nu}{8} \rho\omega^2 (b^2 - r^2) \\ \sigma_\theta &= \frac{3 + \nu}{8} \rho\omega^2 b^2 - \frac{1 + 3\nu}{8} \rho\omega^2 r^2 \end{aligned} \qquad (16)$$

These stresses are greatest at the center of the disk, where

$$\sigma_r = \sigma_\theta = \frac{3 + \nu}{8} \rho\omega^2 b^2 \qquad (17)$$

In the case of a disk with a *circular hole* of radius *a* at the center, the constants of integration in Eqs. (g) are obtained from the conditions at the inner and outer boundaries. If there are no forces acting on these boundaries, we have

$$(\sigma_r)_{r=a} = 0, \qquad (\sigma_r)_{r=b} = 0 \tag{h}$$

from which we find that

$$C = \frac{3+\nu}{8}\rho\omega^2(b^2+a^2); \qquad C_1 = -\frac{3+\nu}{8}\rho\omega^2 a^2 b^2$$

Substituting in Eqs. (g),

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right) \tag{18}$$

We find the maximum radial stress at $r = \sqrt{ab}$, where

$$(\sigma_r)_{max.} = \frac{3+\nu}{8}\cdot\rho\omega^2(b-a)^2 \tag{19}$$

The maximum tangential stress is at the inner boundary, where

$$(\sigma_\theta)_{max.} = \frac{3+\nu}{4}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right) \tag{20}$$

It will be seen that this stress is larger than $(\sigma_r)_{max}$.

When the radius *a* of the hole approaches zero, the maximum tangential stress approaches a value twice as great as that for a solid disk (17); i.e., by making a small circular hole at the center of a solid rotating disk we double the maximum stress.

4.5. Bending of a Curved Bar by a Force at the End

We begin with the simple case shown in Fig. 2

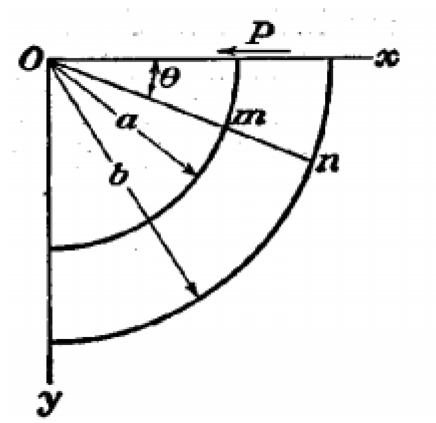


Fig.2

A bar of a narrow rectangular cross section and with a circular axis is constrained at the lower end and bent by a force *P* applied at the upper end in the radial direction. The bending moment at any cross section *mn* is proportional to $\sin\theta$, and the normal stress σ_θ, according to elementary theory of the bending of curved bars, is proportional to the bending moment. Assuming that this holds also for, the exact solution, an assumption which the results will justify, we find from the equation

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2}$$

that the stress function ϕ satisfying the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0 \qquad (a)$$

should be proportional to $\sin\theta$. Taking

$$\phi = f(r)\sin\theta \qquad (b)$$

and substituting in Eq. (a), we find that *f(r)* must satisfy the following ordinary differential equation:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2}\right) = 0 \qquad (c)$$

This equation can be transformed into a linear differential equation with constant coefficients, and its general solution is

$$f(r) = Ar^3 + B\frac{1}{r} + Cr + Dr\log r \tag{d}$$

in which A, B, C, and D are constants of integration, which are determined from the boundary conditions. Substituting solution (d) in expression (b) for the stress function, and using the general formulas, we find the following expressions for the stress components:

$$\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta \\
\sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta \\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = -\left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta
\end{aligned} \tag{21}$$

From the conditions that the outer and inner boundaries of the curved bar (Fig. 2) are free from external forces, we require that

$$\sigma_r = \tau_{r\theta} = 0 \text{ for } r = a \text{ and } r = b$$

or, from eqs. (21)

$$\begin{aligned}
2Aa - \frac{2B}{a^3} + \frac{D}{a} = 0 \\
2Ab - \frac{2B}{b^3} + \frac{D}{b} = 0
\end{aligned} \tag{e}$$

The last condition is that the sum of the shearing forces distributed over the upper end of the bar should equal the force *P*. Taking the width of the cross section as unity or *P* as the load per unit thickness

of the plate we obtain for $\theta = 0$,

$$\begin{aligned}
\int_a^b \tau_{r\theta}\,dr &= -\int_a^b \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)dr = \left|\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_b^a \\
&= \left|Ar^2 + \frac{B}{r^2} + C + D\log r\right|_b^a = P
\end{aligned}$$

or,

$$-A(b^2 - a^2) + B\frac{(b^2 - a^2)}{a^2b^2} - D\log\frac{b}{a} = P \tag{f}$$

From Eqs. (e) and (f) we find

$$A = \frac{P}{2N}, \qquad B = -\frac{Pa^2b^2}{2N}, \qquad D = -\frac{P}{N}(a^2 + b^2) \qquad (g)$$

in which

$$N = a^2 - b^2 + (a^2 + b^2)\log\frac{b}{a}$$

Substituting the values (g) of the constants of integration in Eqs. (21),we obtain the expressions for the stress components. For the upper end of the bar, $\theta = 0$, we find

$$\begin{aligned} \sigma_\theta &= 0 \\ \tau_{r\theta} &= -\frac{P}{N}\left[r + \frac{a^2b^2}{r^3} - \frac{1}{r}(a^2 + b^2)\right] \end{aligned} \qquad (h)$$

For the lower end $\theta = \pi/2$,

$$\begin{aligned} \tau_{r\theta} &= 0 \\ \sigma_\theta &= \frac{P}{N}\left[3r - \frac{a^2b^2}{r^3} - (a^2 + b^2)\frac{1}{r}\right] \end{aligned} \qquad (k)$$

The expressions (21) constitute an exact solution of the problem only when the forces at the ends of the curved bar are distributed in the manner given by Eqs. (h) and (k). For any other distribution of forces the stress distribution near the ends will be different from that given by solution (21), but at larger distances this solution will be valid by Saint-Venant's principle. Calculations show that the simple theory, based on the assumption that cross sections remain plane during bending, again gives very satisfactory results.

4.6. Summary

In this unit we have studied

- Stress-concentration due to a Circular Hole in a Stressed Plate (Kirsch Problem)
- Rotating Disk
- Bending of a Curved Bar by a Force at the End

4.7. Keywords

Rotating disk

Curved bar

4.8. Exercise

1. What is Biharmonic equation? Solve for Biharmonic equation for the case of symmetrical stress distribution.
2. What is the general solution for a two dimensional problem in polar coordinates?

Unit 1
Torsion of Prismatic Bars

Structure

1.1. Introduction

In this chapter, consideration is given to stresses and deformations in prismaticmembers subject to equal and opposite end torques. In general, these bars are assumed free of end constraint. Usually, members that transmit torque, such as propeller shafts and torque tubes of power equipment, are circular or tubular in cross section. For circular cylindrical bars, the torsion formulas are readily derived employing the method of mechanics of materials, as illustrated in the next section.

Slender members with other than circular cross sections are also often used. Intreating noncircular prismatic bars, cross sections initially plane (Fig. 1a) experience out-of-plane deformation or warping (Fig. 1b), and the basic kinematic assumptions of the elementary theory are no longer appropriate. Consequently, the theory of elasticity, a general analytic approach is employed.The governing differential equations derived using this method are applicable to both the linear elastic and the fully plastic torsion problems.

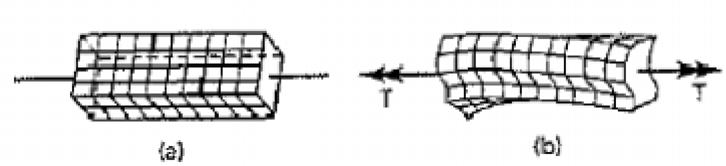


Fig. 1

1.2. Objectives

After studying this unit we are able to understand

- St. Venant's Theory
- Torsion of Hollow Shafts
- Torsion of thin-walled tubes
- Analogous Methods
- Torsion of Bars of Variable Diameter

1.3. St. Venant's Theory

Consider a torsion bar or shaft of circular cross section (Fig. 2). Assume that theright end twists relative to the left end so that longitudinal line *AB* deforms to *AB'*. This results in a shearing stressτ and an angle of twist or angular deformation θ. The basic assumptions underlying the formulations for the torsional loading of circular bars:

1. All plane sections perpendicular to the longitudinal axis of the bar remain plane following the application of torque; that is, points in a given cross-sectional planeremain in that plane after twisting.

2. Subsequent to twisting, cross sections are undistorted in their individual planes; that is, the shearing strain γ varies linearly from zero at the center to a maximumon the outer surface.

The preceding assumptions hold for both elastic and inelastic material behavior. In the elastic case, the following also applies:

3. The material is homogeneous and obeys Hooke's law; hence, the magnitude of the maximum shear angle γ_{max} must be less than the yield angle.

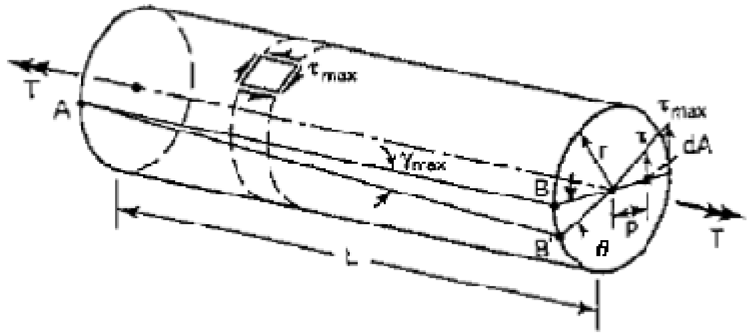


Fig. 2

1.4. Torsion of Hollow Shafts

Let us consider now hollow shafts whose cross sections have two or more boundaries. The simplest problem of this kind is a hollow shaft with an inner boundary coinciding with one of the stress lines of the solid shaft, having the same boundary as the outer boundary of the hollow shaft.

Take, for instance, an elliptic cross section (Fig. 3).

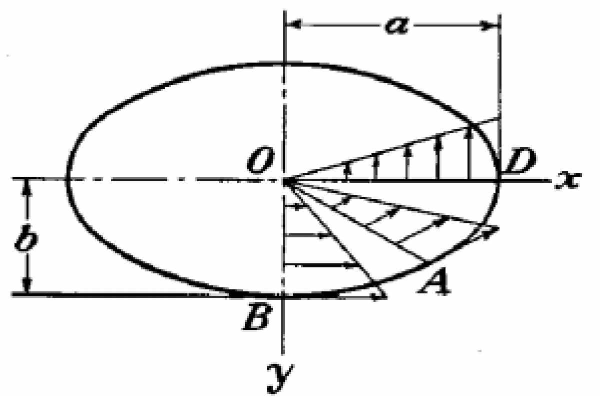


Fig. 3

The stressfunction for the solid shaft is

$$\phi = \frac{a^2 b^2 F}{2(a^2 + b^2)} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \qquad (a)$$

The curve

$$\frac{x^2}{(ak)^2} + \frac{y^2}{(bk)^2} = 1 \qquad (b)$$

is an ellipse which is geometrically similar to the outer boundary of the cross section. Along this ellipse the stress function (a) remains constant, and hence, for k less than unity, this ellipse is a stress line for the solid elliptic shaft. The shearing stress at any point of this line is in the direction of the tangent to the line. Imagine now a cylindrical surface generated by this stress line with its axis parallel to the axis ofthe shaft. Then, from the above conclusion regarding the direction of the shearing stresses, it follows that there will be no stresses acting across this cylindrical surface. We can imagine the material bounded by this surface removed without changing the stress distribution in the outer portion of the shaft. Hence the stress function (a) applies to the hollow shaft also.

For a given angle θ of twist the stresses in the hollow shaft are the same as in the corresponding solid shaft. But the torque will be smaller by the amount which in the case of the solid shaft is carried by the portion of the cross section corresponding to the hole. From Eq. for the angle of twist

$$\theta = M_t \cdot \frac{a^2 + b^2}{\pi a^3 b^3 G} \qquad (1)$$

we see that the latter portion is in the ratio k^4: 1 to the totaltorque. Hence, for the hollow shaft., instead of Eq. (1), we will have

$$\theta = \frac{M_t}{1 - k^4} \frac{a^2 + b^2}{\pi a^3 b^3 G}$$

and the stress function (a) becomes

$$\phi = -\frac{M_t}{\pi ab(1 - k^4)} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$$

The formula for the maximum stress will be

$$\tau_{\max.} = \frac{2M_t}{\pi ab^2} \frac{1}{1 - k^4}$$

1.5. Torsion of Thin Walled Tubes

An approximate solution of the torsional problem for thin tubes can easily be obtained by using the membrane analogy. Let AB and CD (Fig. 4)

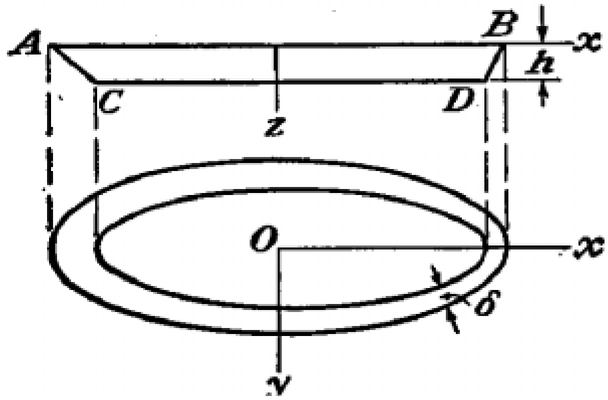


Fig. 4

represent the levels of theouter and the inner boundaries, and *AC* and *DB* be the cross section ofthe membrane stretched between these boundaries. In the case of a thin wall, we can neglect the variation in the slope of the membrane across the thickness and assume that *AC* and *BD* are straight lines. This is equivalent to the assumption that the shearing stresses are uniformly distributed over the thickness of the wall. Then denoting by h the difference in level of the two boundaries and by δ the variable thickness of the wall, the stress at any point, given by the slope of the membrane, is

$$\tau = \frac{h}{\delta} \qquad (a)$$

It is inversely proportional to the thickness of the wall and thus greatest where the thickness of the tube is least.

To establish the relation between the stress and the torque M_t we apply again the membrane analogy and calculate the torque from the volume *ACDB*. Then

$$M_t = 2Ah = 2A\delta\tau \qquad (b)$$

in which A is the mean of the areas enclosed by the outer and the inner boundaries of the cross section of the tube. From (b) we obtain a simple formula for calculating shearing stresses,

$$\tau = \frac{M_t}{2A\delta} \qquad (2)$$

For determining the angle of twist θ, we apply Eq.

$$\int \tau \, ds = 2G\theta A$$

Then

$$\tau\, ds = \frac{M_t}{2A} \int \frac{ds}{\delta} = 2G\theta A \qquad (c)$$

from which

$$\theta = \frac{M_t}{4A^2G} \int \frac{ds}{\delta} \qquad (3)$$

In the case of a tube of uniform thickness, δ is constant and (3) gives

$$\theta = \frac{M_t s}{4A^2 G \delta} \qquad (4)$$

in which s is the length of the center line of the ring section of the tube.

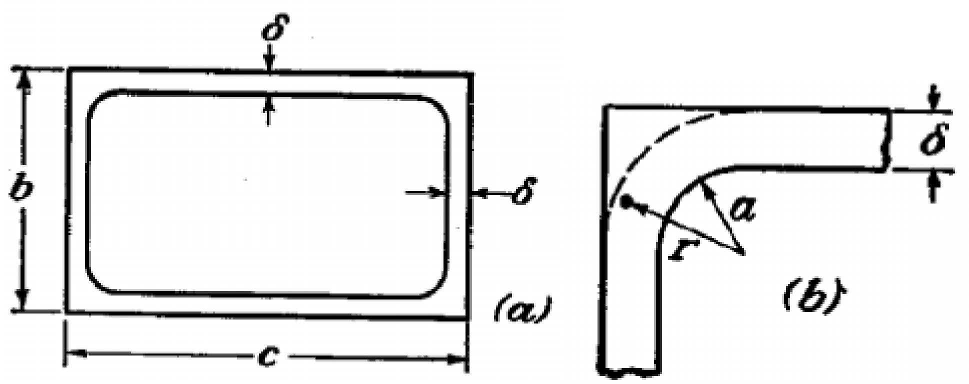


Fig. 5

If the tube has reentrant corners, as in the case represented in Fig. 5, a considerable stress concentration may take place at these corners. The maximum stress is larger than the stress given by Eq. (2)and depends on the radius *a* of the fillet of the reentrant corner (Fig. 5b). In calculating this maximum stress we shall use the membraneanalogy.The equation of the membrane at the reentrant corner may be takenin the form

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} = -\frac{q}{S}$$

Replacing q/S by $2G\theta$ and noting that $\tau = -dz/dr$ (see Fig. 4), we find

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = 2G\theta \qquad (d)$$

Assuming that we have a tube of a constant thickness δ and denoting by τ_0 the stress at a considerable distance from the corner calculated from Eq. (2), we find, from (c),

$$2G\theta = \frac{\tau_0 s}{A}$$

Substituting in (d),

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = \frac{\tau_0 s}{A} \quad (e)$$

The general solution of this equation is

$$\tau = \frac{C}{r} + \frac{\tau_0 s r}{2A} \quad (f)$$

Assuming that the projecting angles of the cross section have fillets with the radius *a*, as indicated in the figure, the constant of integration *C* can be determined from the equation

$$\int_a^{a+\delta} \tau \, dr = \tau_0 \delta \quad (g)$$

which follows from the hydro dynamical analogy, viz.: if anideal fluid circulates in a channel having the shape of the ring crosssection of the tubular member, the quantity of fluid passing each crosssection of the channel must remain constant. Substituting expression(f) for τ into Eq. (g), and integrating, we find that

$$C = \tau_0 \delta \frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)}$$

and, from Eq. (f), that

$$\tau = \frac{\tau_0 \delta}{r} \frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)} + \frac{\tau_0 s r}{2A} \quad (h)$$

For a thin-walled tube the ratios $s(2a + \delta)/A$, sr/A, will be small, and (h) reduces to

$$\tau = \tau_0 \cdot \frac{\delta}{r} \Big/ \log_e \left(1 + \frac{\delta}{a}\right) \quad (i)$$

1.6. Analogous Method

In the solution of torsional problems themembrane analogy, introduced by L. Prandtl, has proved very valuable. Imagine a homogeneous membrane (Fig. 6) supported at theedges, with the same outline as that of the cross section of the twistedbar, subjected to a uniform tension at the edges and a uniform lateralpressure. If *q* is the pressure per unit area of the membrane and *S* isthe uniform tension per unit length of its boundary, the tensile forcesacting on the sides *ad* and *bc* of an infinitesimal element *abcd* (Fig. 6)give, in the case of small deflections of the membrane, a resultant in the upward direction $S(\partial^2 z/\partial x^2)\, dx\, dy$.

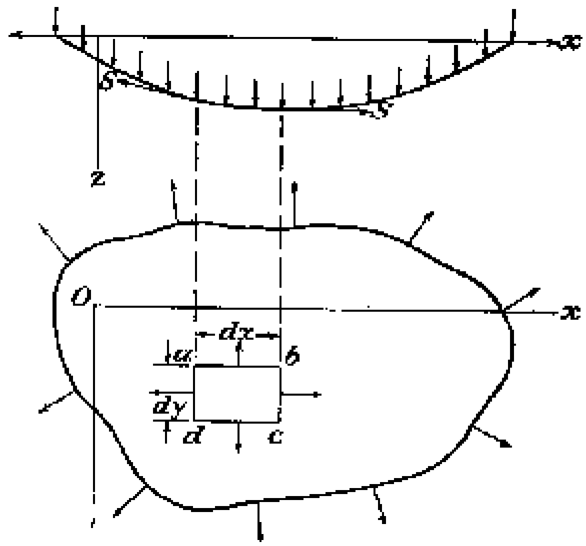


Fig. 6

In the same manner the tensileforces acting on the other two sides of the element give the resultant $S(\partial^2 z/\partial y^2)\, dx\, dy$ and the equation of equilibrium of the element is

$$q\, dx\, dy + S\frac{\partial^2 z}{\partial x^2}\, dx\, dy + S\frac{\partial^2 z}{\partial y^2}\, dx\, dy = 0$$

from which

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{q}{S} \tag{5}$$

At the boundary the deflection of the membrane is zero. Comparing Eq. (5) and the boundary condition for the deflections z of the membrane with Eq.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = F \tag{6}$$

and the boundary condition

$$\frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = \frac{d\phi}{ds} = 0$$

for the stress function ϕ , we conclude that these two problems are identical. Hence from the deflections of the membrane we can obtain values of ϕ by replacing the quantity (q/S) of Eq. (5) with the quantity $F = -2G\theta$ of Eq. (6).

Having the deflection surface of the membrane represented by con tour lines (Fig. 7), several important conclusions regarding stress distribution in torsion can be obtained. Consider any point *B* on the

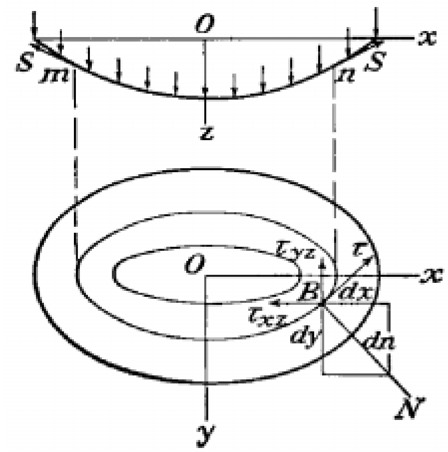


Fig. 7

membrane. The deflection of the membrane along the contour line through this point is constant, and we have

$$\frac{\partial z}{\partial s} = 0$$

The corresponding equation for the stress function ϕ is

$$\frac{\partial \phi}{\partial s} = \left(\frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds}\right) = \tau_{xz}\frac{dy}{ds} - \tau_{yz}\frac{dx}{ds} = 0$$

This expresses that the projection of the resultant shearing stress at a point *B* on the normal *N* to the contour line is zero and therefore we may conclude that the shearing stressat a point *B* in the twisted bar is in the direction of the tangent to the contour line through this point. Thecurves drawn in the cross section of a twisted bar, in such a manner thatthe resultant shearing stress at any point of the curve is in the directionof the tangent to the curve, are called *lines of shearing stress*. Thus thecontour lines of the membrane are the lines of shearing stress for the crosssection of the twisted bar. The magnitude of the resultant stress τ at *B* (Fig. 7) is obtained by projecting on the tangent, the stress components τ_{xz} and τ_{yz}. Then

$$\tau = \tau_{yz} \cos(Nx) - \tau_{xz} \cos(Ny)$$

Substituting

$$\tau_{xz} = \frac{\partial \phi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}, \qquad \cos(Nx) = \frac{dx}{dn}, \qquad \cos(Ny) = \frac{dy}{dn}$$

we obtain

$$\tau = -\left(\frac{\partial \phi}{\partial x}\frac{dx}{dn} + \frac{\partial \phi}{\partial y}\frac{dy}{dn}\right) = -\frac{d\phi}{dn}$$

Thus the magnitude of the shearing stress at B is given by the maximum slope of the membrane at this point, It is only necessary in the expression for the slope to replace q/S by $2G\theta$. From this it can be concluded that the maximum shear acts at the points where the contour lines are closest to each other.

1.7. Torsion of Bars of Variable Diameter

Let usconsider a shaft in the form of a body of revolution twisted by couples applied at the ends (Fig. 8). We may take the axis of the shaft asthe z-axis and use polar coordinates r and θ for defining the position of an element in the plane of a cross section. The notations for stress components in such a case are σ_r, σ_θ, σ_z, τ_{rz}, $\tau_{r\theta}$, $\tau_{\theta z}$. The components of displacements in the radial and tangential directions we may denote by u and v and the component in the z-direction by w. Then, using the

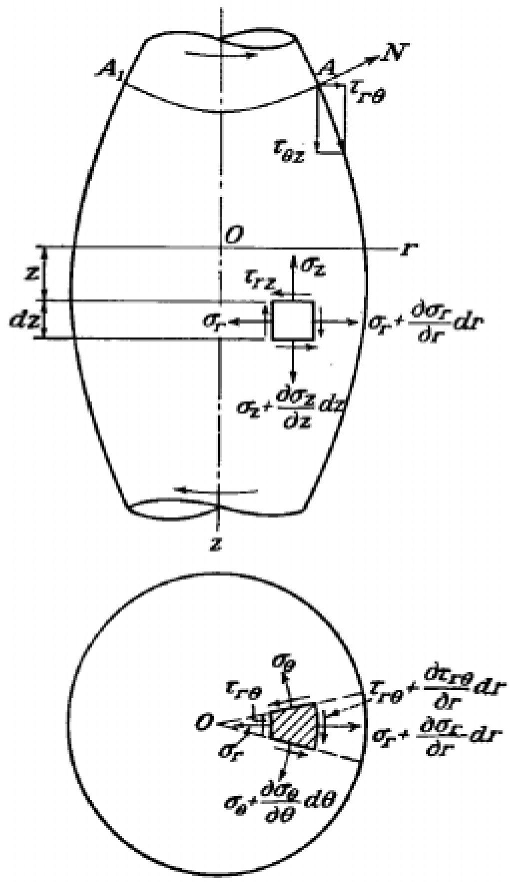


Fig. 8

formulas obtained previously for two-dimensional problems, we find the following expressions for the strain components:

$$\epsilon_r = \frac{\partial u}{\partial r}, \qquad \epsilon_\theta = \frac{u}{r} + \frac{\partial v}{r\,\partial\theta}, \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial u}{r\,\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \qquad \gamma_{z\theta} = \frac{\partial v}{\partial z} + \frac{\partial w}{r\,\partial\theta} \tag{7}$$

Writing down the equations of equilibrium of an element (Fig. 8), as was done before for the case of two-dimensional problems (Art. 25),and assuming that there are no body forces, we arrive at the following differential equations of equilibrium:

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \tag{8}$$

In the application of these equations to the torsional problem we use the *semi-inverse method* and assume that u and w are zero, i.e., that during twist the particles move only in tangential directions. This assumption differs from that for a circular shaft of constant diameter in that these tangential displacements are no longerproportional to the distance from the axis, i.e., the radii of a cross section become curved during twist.

Substituting in (7) $u = w = 0$, and taking into account the fact that, from symmetry the displacement v does not depend on the angle θ, we find that

$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{rz} = 0, \qquad \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}, \qquad \gamma_{\theta z} = \frac{\partial v}{\partial z} \tag{a}$$

Hence, of all the stress components, only $\tau_{r\theta}$ and $\tau_{\theta z}$, are different from zero. The first two of Eqs. (8) are identically satisfied, and thethird of these equations gives

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \tag{b}$$

This equation can be written in the form

$$\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{\partial}{\partial z}(r^2\tau_{\theta z}) = 0 \tag{c}$$

It is seen that this equation is satisfied by using a stress function ϕ of r and z, such that

$$r^2\tau_{r\theta} = -\frac{\partial \phi}{\partial z}, \qquad r^2\tau_{\theta z} = \frac{\partial \phi}{\partial r} \tag{d}$$

To satisfy the compatibility conditions it is necessary to consider the fact that $\tau_{r\theta}$ and $\tau_{\theta z}$, are functions of the displacement v. From Eqs. (a) and (d) we find

$$\begin{aligned} \tau_{r\theta} &= G\gamma_{r\theta} = G\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = Gr\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = -\frac{1}{r^2}\frac{\partial \phi}{\partial z} \\ \tau_{\theta z} &= G\gamma_{\theta z} = G\frac{\partial v}{\partial z} = Gr\frac{\partial}{\partial z}\left(\frac{v}{r}\right) = \frac{1}{r^2}\frac{\partial \phi}{\partial r} \end{aligned} \tag{e}$$

From these equations it follows that

$$\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial \phi}{\partial z}\right) = 0 \tag{f}$$

Or

$$\frac{\partial^2\phi}{\partial r^2} - \frac{3}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (g)$$

Let us consider now the boundary conditions for the function ϕ. From the condition that the lateral surface of the shaft is free from external forces we conclude that at any point A at the boundary of an axial section (Fig. 8) the total shearing stress must be in the direction of the tangent to the boundary and its projection on the normal N to the boundary must be zero. Hence

$$\tau_{r\theta}\frac{dz}{ds} - \tau_{\theta z}\frac{dr}{ds} = 0$$

where ds is an element of the boundary. Substituting from (d), we find that

$$\frac{\partial\phi}{\partial z}\frac{dz}{ds} + \frac{\partial\phi}{\partial r}\frac{dr}{ds} = 0 \qquad (h)$$

from which we conclude that ϕ is constant along the boundary of the axial section of the shaft. Equation (g) together with the boundary condition (h) completely determines the stress function ϕ, from which we may obtain the stresses satisfying the equations of equilibrium, the compatibility equations, and the condition at the lateral surface of the shaft.

The magnitude of the torque is obtained by taking a cross section and calculating the moment given by the shearing stresses $\tau_{\theta z}$. Then

$$M_t = \int_0^a 2\pi r^2 \tau_{\theta z}\, dr = 2\pi \int_0^a \frac{\partial\phi}{\partial r}\, dr = 2\pi \left|\phi\right|_0^a \qquad (k)$$

where a is the outer radius of the cross section. The torque is thus easily obtained if we know the difference between the values of the stress function at the outer boundary and at the center of the cross section.

1.8. Summary

In this unit we have studied

- St. Venant's Theory
- Torsion of Hollow Shafts
- Torsion of thin-walled tubes
- Analogous Methods

– Torsion of Bars of Variable Diameter

1.9. Keywords

Torsion

St.Venant's Theory

Hollow shafts

Thin walled tubes

1.10. Exercise

1. Write a short note on Saint Venant's theory.
2. Derive expression for moment and max. stress due to torsion of a hollow shaft.
3. Determine the equation for angle of twist and stress induced in thin walled tube due to torsion.
4. With the help of membrane analogy determine the equation to find the stress induced due to torsion.
5. For torsion of a bar of variable diameter find out the equation to determine magnitude of moment.

Unit 2
Bending of Prismatic Bars

Structure

2.1. Introduction

Consider a, prismatical bar bent in one of its principal planes by two equal and opposite couples *M*(Fig.1). Taking the origin of the coordinates at the centroid of the cross section and the *xz*-plane in the

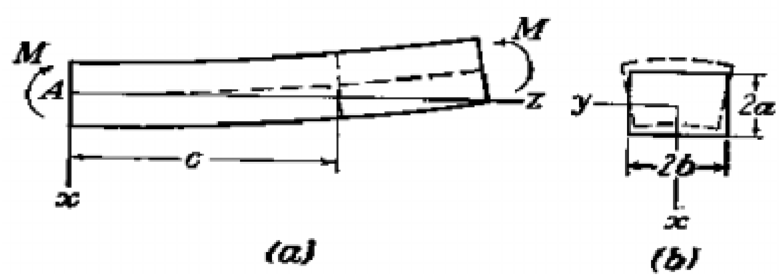


Fig. 1

principal plane of bending, thestress components given by the usual elementary theory of bending are

$$\sigma_z = \frac{Ex}{R}, \qquad \sigma_y = \sigma_x = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \qquad (a)$$

in which *R* is the radius of curvature of the bar after bending. Substituting expressions (a) for the stress components in the equations of equilibrium,

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z &= 0 \end{aligned}$$

it is found that these equations are satisfied if thereare no body forces. The boundary conditions

$$\begin{aligned} \bar{X} &= \sigma_x l + \tau_{xy} m + \tau_{xz} n \\ \bar{Y} &= \sigma_y m + \tau_{yz} n + \tau_{xy} l \\ \bar{Z} &= \sigma_z n + \tau_{xz} l + \tau_{yz} m \end{aligned}$$

for the lateralsurface of the bar, which is free from external forces, are also satisfied.The boundary conditions at the ends require that the surface forces must be distributed over the ends in the same manner as the stresses σ_z. Only under this condition do the stresses (a) represent the exact solution of the problem. The bending moment *M* is given by the equation

$$M = \int \sigma_z x \, dA = \int \frac{Ex^2 \, dA}{R} = \frac{EI_y}{R}$$

in which I_y is the moment of inertia of the cross section of the beam with respect to the neutral axis parallel to the *y*-axis. From this equation we find

$$\frac{1}{R} = \frac{M}{EI_y}$$

which is a well-known formula of the elementary theory of bending.

2.2. Objectives

After studying this unit we are able to understand

- Unsymmetrical Bending
- Shear Centre
- Solution of Bending of Bars by Harmonic Functions
- Solution of Bending Problems by Soap-Film Method

2.3. Unsymmetrical Bending

Let us consider thecase of an isosceles triangle (Fig. 2). The boundary of the cross section is givenby the equation

$$(y - a)[x + (2a + y)\tan\alpha][x - (2a + y)\tan\alpha] = 0$$

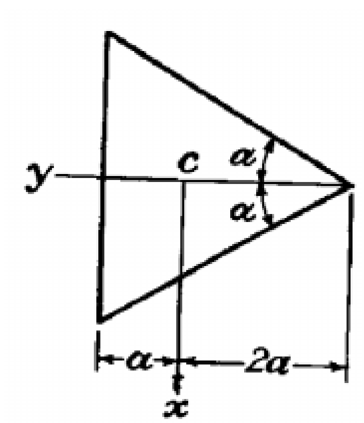


Fig. 2

The right side of Eq.

$$\frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = \frac{\partial \phi}{\partial s} = \left[\frac{Px^2}{2I} - f(y)\right]\frac{dy}{ds}$$

is zero if we take

$$f(y) = \frac{P}{2I}(2a + y)^2 \tan^2 \alpha$$

Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Py}{I} - \frac{df}{dy}$$

for determining the stress function ϕ thenbecomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Py}{I} - \frac{P}{I}(2a + y)\tan^2 \alpha \qquad (a)$$

An approximate solution may be obtained by using the energy method. In the particular case when

$$\tan^2 \alpha = \frac{\nu}{1+\nu} = \frac{1}{3} \qquad (b)$$

an exact solution of Eq. (a) is obtained by taking for the stress function theexpression

$$\phi = \frac{P}{6I}\left[x^2 - \frac{1}{3}(2a + y)^2\right](y - a)$$

The stress components are then obtained from Eqs.

$$\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + f(y), \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

which are

$$\begin{aligned} \tau_{xz} &= \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + \frac{P}{6I}(2a + y)^2 = \frac{2\sqrt{3}P}{27a^4}[-x^2 + a(2a + y)] \\ \tau_{yz} &= -\frac{\partial \phi}{\partial x} = \frac{2\sqrt{3}P}{27a^4}x(a - y) \end{aligned} \qquad (c)$$

Along the *y*-axis, *x* = *0*, and the resultant shearing stress is vertical and is represented by the linear function

$$(\tau_{xz})_{x=0} = \frac{2\sqrt{3}P}{27a^3}(2a + y)$$

The maximum value of this stress, at the middle of the vertical side of the cross section, is

$$\tau_{max.} = \frac{2\sqrt{3}P}{9a^2} \qquad (d)$$

By calculating the moment with respect to the *z*-axis of the shearing forces given by the stresses (c), it can be shown that in this case the resultant shearing forcepasses through the centroid *C* of the cross section.

2.4. Shear Center

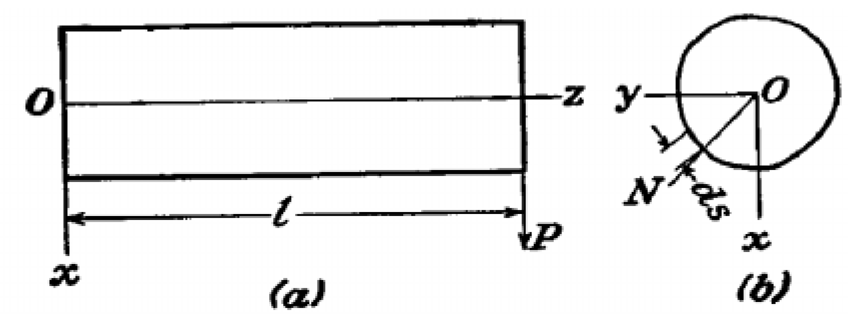


Fig. 3

In the cantilever problem (Fig. 3) we choose for *z*-axis the centroidal axis of the bar and for *x* and *y* axes the principal centroidal axes of the cross section. We assume that the force *P*is parallel to the *x*-axis and at such a distance from the centroid that twisting of the bar does not occur. This distance, which is of importance in practical calculations, can readily be found once the stresses represented by Eqs.

$$\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + f(y), \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

are known. For this purpose we evaluatethe moment about the centroid produced by the shear stresses τ_{xz} and τ_{yz}. This moment evidently is

$$M_z = \iint (\tau_{xz}y - \tau_{yz}x)\, dx\, dy \qquad (a)$$

Observing that the stresses distributed over the end cross section of the beam are statically equivalent to the acting force *P* we conclude thatthe distance *d* of the force *P* from the centroid of the cross section is

$$d = \frac{|M_z|}{P} \qquad (b)$$

For positive M_z the distance d must be taken in the direction of positive y. In the preceding discussion the assumption was made that the force is acting parallel to the x-axis.

When the force P is parallel to the y-axis instead of the x-axis we can, by a similar calculation, establish the position of the line of action of P for which no rotation of centroidal elements of cross sections occurs. The intersection point of the two lines of action of the bending forces has an important significance. If a force, perpendicular to the axis of the beam, is applied at that point we can resolve it into two components parallel to the x and $y$axes and on the basis of the above discussion we conclude that it does not produce rotation of centroidal elements of cross sections of the beam. This point is called the *shear center*sometimes also the center of flexure, or flexural center.

2.5. Solution of Bending of Bars by Harmonic Functions

Consider a general case of bending of a cantilever of a constant cross section of any shape by a force P applied at the end and parallel to one of the principal axes of the cross section (Fig. 3). Take the origin of the coordinates at the centroid of the fixed end. The z-axis coincides with the center line of the bar, and the x- and y-axes coincide with the principal axes of the cross section. In the solution of the problem we apply Saint-Venant's semi-inverse method and at the very beginning make certain assumptions regarding stresses. We assume that normal stresses over a cross section at a distance z from the fixed end are distributed in the same manner as in the case of pure bending:

$$\sigma_z = -\frac{P(l-z)x}{I} \qquad (a)$$

We assume also that there are shearing stresses, acting on the same cross sections, which we resolve at each point into components τ_{xz} and τ_{yz}. We assume that the remaining three stress components σ_x, σ_y, τ_{xy} are zero. It will now be shown that by using these assumptions we arrive at a solution which satisfies all of the equations of the theory of elasticity and which is hence the exact solution of the problem.

With these assumptions, neglecting body forces, the differential equations of equilibrium become

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \qquad \frac{\partial \tau_{yz}}{\partial z} = 0 \tag{b}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{Px}{I} \tag{c}$$

From (b) we conclude that shearing stresses do not depend on *z* and are the same in all cross sections of the bar.

Considering now the boundary conditions and applying them to the lateral surface of the bar, which is free from external forces, we find that the first two of these equations are identically satisfied and the third one gives

$$\tau_{xz} l + \tau_{yz} m = 0$$

From Fig. 3b we see that

$$l = \cos(Nx) = \frac{dy}{ds}, \qquad m = \cos(Ny) = -\frac{dx}{ds}$$

in which *ds* is an element of the bounding curve of the cross section. Then the condition at the boundary is

$$\tau_{xz}\frac{dy}{ds} - \tau_{yz}\frac{dx}{ds} = 0 \tag{d}$$

Turning to the compatibility equations

$$\begin{aligned}
(1+\nu)\nabla^2\sigma_x + \frac{\partial^2\Theta}{\partial x^2} &= 0, & (1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2\Theta}{\partial y\,\partial z} &= 0\\
(1+\nu)\nabla^2\sigma_y + \frac{\partial^2\Theta}{\partial y^2} &= 0, & (1+\nu)\nabla^2\tau_{xz} + \frac{\partial^2\Theta}{\partial x\,\partial z} &= 0\\
(1+\nu)\nabla^2\sigma_z + \frac{\partial^2\Theta}{\partial z^2} &= 0, & (1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2\Theta}{\partial x\,\partial y} &= 0
\end{aligned} \tag{1}$$

we see that the firstthree of these equations, containing normal stress components, and the last equation, containing τ_{xy}, are identically satisfied. The system (1) then reduces to the two equations

$$\nabla^2\tau_{yz} = 0, \qquad \nabla^2\tau_{xz} = -\frac{P}{I(1+\nu)} \tag{e}$$

Thus the solution of the problem of bending of a prismatical cantilever of any cross section reduces to finding, for τ_{xz} and τ_{yz}, functions of x and y which satisfy the equation of equilibrium (c), the boundary condition (d), and the compatibility equations (e).

$$\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + f(y), \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x} \tag{2}$$

in which ϕ is the stress function of x and y, and $f(y)$ is a function of y only, which will be determined later from the boundary condition.

Substituting (2) in the compatibility equations (e), we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = \frac{\nu}{1+\nu}\frac{P}{I} - \frac{d^2 f}{dy^2}$$

From these equations we conclude that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Py}{I} - \frac{df}{dy} + c \tag{f}$$

where c is a constant of integration. This constant has a very simple physical meaning. Consider the rotation of an element of area in the plane of a cross section of the cantilever. This rotation is expressed by the equation

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The rate of change of this rotation in the direction of the z-axis can be written in the following manner:

$$\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}$$

and, by using Hooke's law and expressions (2) for the stress components, we find

$$\frac{\partial}{\partial z}(2\omega_z) = \frac{1}{G}\left(\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y}\right) = -\frac{1}{G}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{df}{dy}\right)$$

Substituting in Eq. (f),

$$-G\frac{\partial}{\partial z}(2\omega_z) = \frac{\nu}{1+\nu}\frac{Py}{I} + c$$

2.6. Solution of Bending Problems by the Soap-film Method

The exact solutions of bending problems are known for only a few special cases in which the cross sections have certain simple forms. For practical purposes it is important to have means of solving the problem for any assigned shape of the cross section. This can be accomplished by numerical calculations based on equations of finite differences, or experimentally by the soap-film method. For deriving the theory of the soap-film method we use Eqs.

$$\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + f(y), \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Py}{I} - \frac{df}{dy}$$

$$\frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = \frac{\partial \phi}{\partial s} = \left[\frac{Px^2}{2I} - f(y)\right]\frac{dy}{ds}$$

Taking

$$f(y) = \frac{\nu}{2(1+\nu)}\frac{Py^2}{I}$$

Eq. for the stress function is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (a)$$

The boundary condition becomes

$$\frac{\partial \phi}{\partial s} = \left[\frac{Px^2}{2I} - \frac{\nu}{2(1+\nu)}\frac{Py^2}{I}\right]\frac{dy}{ds} \qquad (b)$$

Integrating along the boundary *s* we find the expression

$$\phi = \frac{P}{I}\int \frac{x^2\,dy}{2} - \frac{\nu}{2(1+\nu)}\frac{Py^3}{3I} + \text{constant} \qquad (c)$$

from which the value of ϕ for every point of the boundary can be calculated.

2.7. Summary

In this unit we have studied

- Unsymmetrical Bending

- Shear Centre
- Solution of Bending of Bars by Harmonic Functions
- Solution of Bending Problems by Soap-Film Method

2.8. Keywords

Unsymmetrical Bending
Shear Centre
Harmonic functions
Soap-Film Method

2.9. Exercise

1. What do you mean by Shear Center? Explain.
2. With the help of Harmonic Functions find a solution for bending of bars.
3. Write a short note on Solution of bending problems by Soap Film Method.
4. Derive an expression for relation between radius of curvature "R" of the bar after bending and bending moment "M".
5. Find out the stress components due to bending of bar of
   a) Circular Cross Section
   b) Elliptical Cross Section
   c) Rectangular Cross Section or
   d) Unsymmetrical Cross Section.

Unit 3
Bending of Plates

Structure

3.1. Introduction

If stresses $\sigma_x = Ez/R$ are distributed over the edges of the plate parallel to the *y*-axis (Fig. 1),

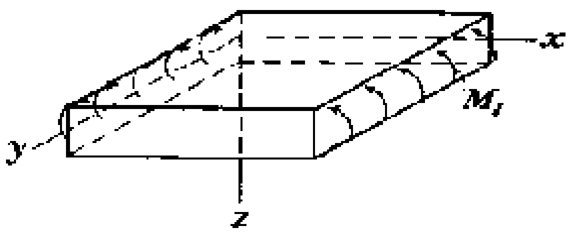


Fig. 1

the surface of the plate will become an antielastic surface, the curvature of which in planes parallel to the*xz*-plane is $1/R$ and in the perpendicular direction is-ν/R.If h denotes the thickness of the plate, M_1 the bending moment per unit length on the edges parallel tothe *y*-axis and

$$I_y = \frac{1 \cdot h^3}{12}$$

the moment of inertia per unit length, the relation between M_1 and R, is

$$\frac{1}{R} = \frac{M_1}{EI_y} = \frac{12M_1}{Eh^3} \qquad (a)$$

When we have bending moments in two perpendicular directions (Fig. 2), the curvatures of the deflection surface may be obtained by superposition.

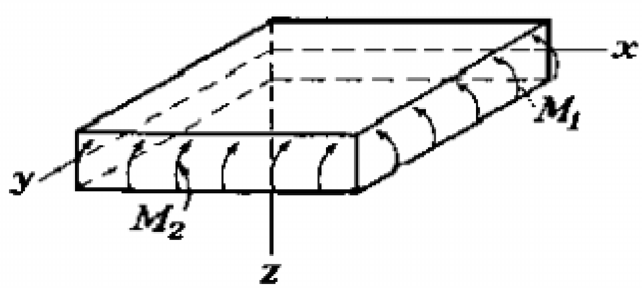


Fig. 2

Let $1/R_1$ and $1/R_2$ be the curvatures of the deflection surface in planes parallel to the coordinate planes *zx* and *zy*, respectively; and let M_1 and M_2 be the bendingmoments per unit length on theedges parallel to the *y*- and *x*-axes, respectively. Then, using Eq. (a) and applying the principle of superposition, we find

$$\frac{1}{R_1} = \frac{12}{Eh^3}(M_1 - \nu M_2)$$
$$\frac{1}{R_2} = \frac{12}{Eh^3}(M_2 - \nu M_1) \qquad (b)$$

The moments are considered positive if they produce a deflection of the plate which is convex down. Solving Eqs. (b) forM_1 and M_2, we find

$$M_1 = \frac{Eh^3}{12(1-\nu^2)}\left(\frac{1}{R_1} + \nu\frac{1}{R_2}\right)$$
$$M_2 = \frac{Eh^3}{12(1-\nu^2)}\left(\frac{1}{R_2} + \nu\frac{1}{R_1}\right) \qquad (c)$$

For small deflections we can use the approximations

$$\frac{1}{R_1} = -\frac{\partial^2 w}{\partial x^2}, \qquad \frac{1}{R_2} = -\frac{\partial^2 w}{\partial y^2}$$

Then, writing

$$\frac{Eh^3}{12(1-\nu^2)} = D$$

We find

$$M_1 = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$
$$M_2 = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

The constant D is called the *flexural rigidity* of a plate.

3.2. Objectives

After studying this unit we are able to understand

- Cylindrical Bending of Rectangular Plates
- Slope and Curvatures
- Determination of Bending and Twisting Moments on any plane

- Membrane Analogy for Bending of a Plate
- Symmetrical Bending of a Circular Plate
- Navier's Solution for simply supported Rectangular Plates
- Combined Bending and Stretching of Rectangular Plates

3.3. Cylindrical Bending of Rectangular Plates

We shall begin the theory of bending of plates with the simple problem of the bending of a long rectangular plate that is subjected to a transverse load that does not vary along the length of the plate. The deflected surface of a portion of such a plate at a considerable distance from the ends can be assumed cylindrical, with the axis of the cylinder parallel to the length of the plate. We can therefore restrict ourselves to the investigation of the bending of an elemental strip cut from the plate by two planes perpendicular to the length of the plate and a unit distance (say 1 in.) apart. The deflection of this strip is given by a differential equation which is similar to the deflectionequation of a bent beam.

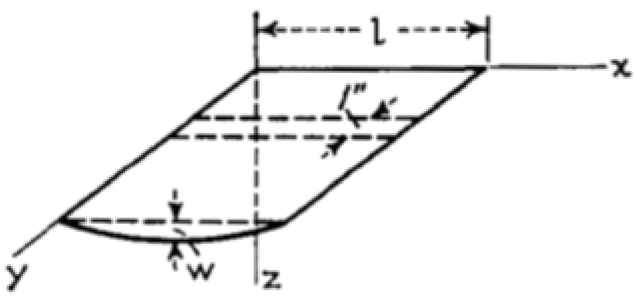


Fig. 3

To obtain the equation for the deflection, we consider a plate of uniform thickness, equal to h, and take the xy plane as the middle plane ofthe plate before loading, i.e., as theplane midway between the faces of the plate. Let the y axis coincide with one of the longitudinal edges of the plate and let the positive direction of the z axis be downward, as shown in Fig. 3. Then if the width of the plate is denoted by l, the elemental strip may be considered as a bar of rectangular cross section which has a length of l and a depth of h. In calculating the bending stresses in such a bar we assume, as in the ordinary theory of beams, that cross sections of the bar remain plane during bending, so that they undergo only a rotation with respect to their neutral axes. If no normal forces are applied to the end sections of the bar, the neutral surface of the bar coincides with the middle surface of the plate, and the unit elongation of a fiber parallel to the x axis is proportional to its distance $z$from the middle surface. The curvature of the deflection curve can be taken equal to d^2w/dx^2, where w, the deflection of the bar in the z direction, is assumed to be small compared with the length of the bar l. The unit elongation ϵ_x of a fiber at a distance z from the middle surface (Fig. 4) is then $z\, d^2w/dx^2$.

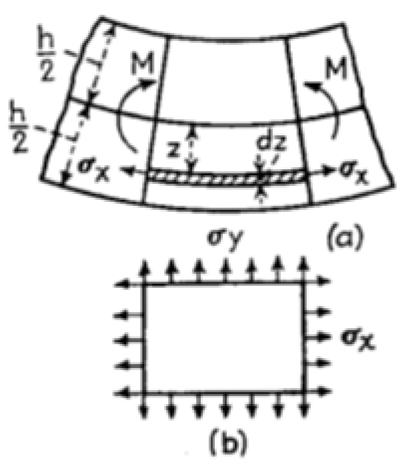


Fig. 4

Making use of Hooke's law, the unit elongations ϵ_x and ϵ_y in terms of the normal stresses σ_x and σ_y acting on the element shown shaded in Fig. 4a are

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$
$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} = 0 \qquad (1)$$

where$E$ is the modulus of elasticity of thematerial and v is Poisson's ratio. The lateralstrain in the y direction must be zero in order to maintain continuityin the plate during bending, from which it follows by the second of theequations (1) that$\sigma_y = v\sigma_x$. Substituting this value in the first of theequations (1), we obtain

$$\epsilon_x = \frac{(1 - \nu^2)\sigma_x}{E}$$

and

$$\sigma_x = \frac{E\epsilon_x}{1 - \nu^2} = -\frac{Ez}{1 - \nu^2}\frac{d^2w}{dx^2} \qquad (2)$$

If the plate is submitted to the action of tensile or compressive forces acting in the x direction and uniformly distributed along the longitudinal sides of the plate, the corresponding direct stress must be added to the stress (2) due to bending.

Having the expression for bending stress σ_x, we obtain by integration the bending moment in the elemental strip:

$$M = \int_{-h/2}^{h/2} \sigma_x z \, dz = -\int_{-h/2}^{h/2} \frac{Ez^2}{1-\nu^2} \frac{d^2w}{dx^2} dz = -\frac{Eh^3}{12(1-\nu^2)} \frac{d^2w}{dx^2}$$

Introducing the notation

$$\frac{Eh^3}{12(1-\nu^2)} = D \tag{3}$$

we represent the equation for the deflection curve of the elemental strip in the following form:

$$D \frac{d^2w}{dx^2} = -M \tag{4}$$

in which the quantity *D*, taking the place of the quantity *EI* in the caseof beams, is called the flexural rigidity of the plate. It is seen that thecalculation of deflections of the plate reduces to the integration of Eq. (4), which has the same form as the differential equation for deflection of beams. If there is only a lateral load acting on the plate and the edgesare free to approach each other as deflection occurs, the expression forthe bending moment *M* can be readily derived, and the deflection curveis then obtained by integrating Eq. (4). In practice the problem is more complicated, since the plate is usually attached to the boundary and its edges are not free to move. Such a method of support sets up tensile reactions along the edges as soon as deflection takes place. These reactions depend on the magnitude of the deflection and affect the magnitudeof the bending moment *M* entering in Eq. (4). The problem reduces tothe investigation of bending of an elemental strip submitted to the actionof a lateral load and also an axial force which depends on the deflectionof the strip.In the following we consider this problem for the particularcase of uniform load acting on a plate and for various conditions alongthe edges.

3.4. Slope and Curvatures

In discussing small deflections of a plate we take the middle plane of the plate, before bending occurs, as the *xy* plane. During bending, the particles that were in the *xy* plane undergo small displacements *w* perpendicular to the *xy* plane and form the middle surface of the plate. These displacements of themiddle surface are called deflections of a plate in our further discussion. Taking a normal section of the plate parallel to the *xz* plane (Fig. 5a),

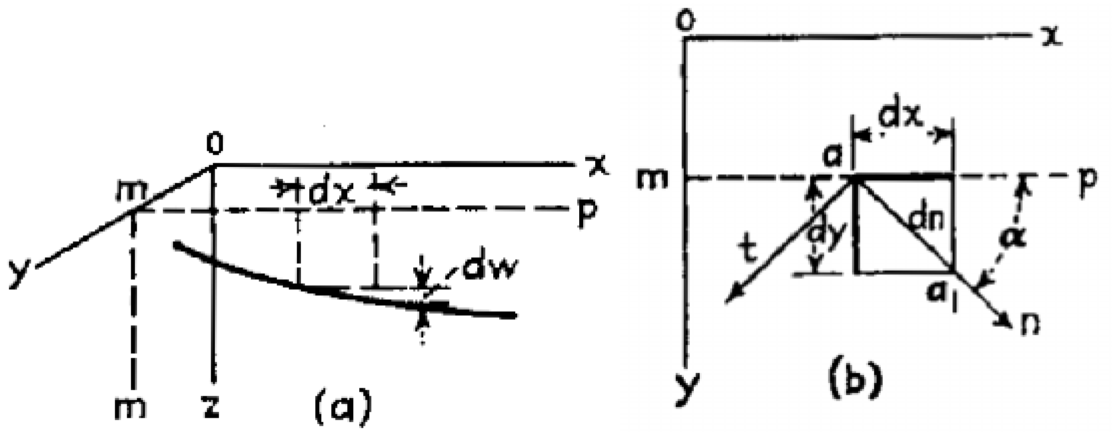


Fig. 5

we find that theslope of the middle surface in the x directionis $i_x = \partial w/\partial x$. In the same manner the slope in the y direction is $i_y = \partial w/\partial y$. Takingnow any direction an in the xy plane (Fig. 5b) making an angle a with the x axis, we findthat the difference in the deflections of the twoadjacent points a and a_1 in the an direction is

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy$$

and that the corresponding slope is

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{dx}{dn} + \frac{\partial w}{\partial y}\frac{dy}{dn} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha \qquad (a)$$

To find the direction α_1 for which the slope is a maximum we equate to zero the derivative with respect to α of expression (a). In this way weobtain

$$\tan\alpha_1 = \frac{\partial w}{\partial y}\bigg/\frac{\partial w}{\partial x} \qquad (b)$$

Substituting the corresponding values of sin α_1 and cosα_1 in (a), we obtain for the maximum slope the expression

$$\left(\frac{\partial w}{\partial n}\right)_{max} = \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} \qquad (c)$$

By setting expression (a) equal to zero we obtain the direction for whichthe slope of the surface is zero. The corresponding angle α_2 is determined from the equation

$$\tan\alpha_2 = -\frac{\partial w}{\partial x}\bigg/\frac{\partial w}{\partial y} \qquad (d)$$

which shows that the directions of zero slope and of maximum slope are perpendicular to each other.

In determining the curvature of the middle surface of the plate we observe that the deflections of the plate are very small. In such a case the slope of the surface in any direction can be taken equal to the angle that the tangent to the surface in that direction makes with the *xy* plane, and the square of the slope may be neglected compared to unity. The curvature of the surface in a plane parallel to the *xz* plane (Fig. 5) isthen numerically equal to

$$\frac{1}{r_x} = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = -\frac{\partial^2 w}{\partial x^2} \qquad (e)$$

We consider a curvature positive if it is convex downward. The minus sign is taken in Eq. (e), since for the deflection convex downward, as shown in the figure, the second derivative $\partial^2 w/\partial x^2$ is negative.

In the same manner we obtain for the curvature in a plane parallel to the *yz* plane

$$\frac{1}{r_y} = -\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -\frac{\partial^2 w}{\partial y^2} \qquad (f)$$

These expressions are similar to those used in discussing the curvature of a bent beam. In considering the curvature of the middle surface in any direction *an* (Fig. 5) we obtain

$$\frac{1}{r_n} = -\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial n}\right)$$

Substituting expression (a) for $\partial w/\partial n$ and observing that

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha$$

We find

$$\begin{aligned}\frac{1}{r_n} &= -\left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha\right)\\ &= -\left(\frac{\partial^2 w}{\partial x^2}\cos^2\alpha + 2\frac{\partial^2 w}{\partial x\,\partial y}\sin\alpha\cos\alpha + \frac{\partial^2 w}{\partial y^2}\sin^2\alpha\right)\\ &= \frac{1}{r_x}\cos^2\alpha - \frac{1}{r_{xy}}\sin 2\alpha + \frac{1}{r_y}\sin^2\alpha \qquad (g)\end{aligned}$$

It is seen that the curvature in any direction *n* at a point of the middlesurface can be calculated if we know at that point the curvatures

$$\frac{1}{r_x} = -\frac{\partial^2 w}{\partial x^2} \qquad \frac{1}{r_y} = -\frac{\partial^2 w}{\partial y^2}$$

and the quantity

$$\frac{1}{r_{xy}} = \frac{\partial^2 w}{\partial x \, \partial y} \qquad (h)$$

which is called the *twist of the surface* with respect to the *x* and *y* axes.

If instead of the direction *an* (Fig. 5b) we take the direction *at* perpendicular to *an*, the curvature in this new direction will be obtained from expression (g) by substituting $\pi/2 + \alpha$ for α. Thus we obtain

$$\frac{1}{r_t} = \frac{1}{r_x}\sin^2\alpha + \frac{1}{r_{xy}}\sin 2\alpha + \frac{1}{r_y}\cos^2\alpha \qquad (i)$$

Adding expressions (g) and (i), we find

$$\frac{1}{r_n} + \frac{1}{r_t} = \frac{1}{r_x} + \frac{1}{r_y} \qquad (5)$$

which shows that at any point of the middle surface the sum of thecurvatures in two perpendicular directions such as *n* and *t* is independentof the angle α. This sum is usually called the *average curvature* of thesurface at a point.

3.5. Determination of Bending and Twisting Moments on any plane

In the case of pure bending of prismatic bars arigorous solution for stress distribution is obtained by assuming thatcross sections of the bar remain plane during bending and rotate onlywith respect to their neutral axes so as to be always normal to the deflection curve. Combination of such bending in two perpendicular directionsbrings us to pure bending of plates.

Let us begin with pure bending of arectangular plate by moments that are uniformly distributed along theedges of the plate, as shown in Fig. 6. We take the *xy* plane to coincide with the middle plane of the plate before deflection and the *x*and *y* axesalong the edges of the plate as shown. The *z* axis, which is then perpendicular to the middle plane, is taken positive downward. We denote by M_x the bending moment per unit length acting on the edges parallel to the y axis and by M_y the moment per unit length acting on the edges parallel to the x axis. These moments we consider positive when they are directed as shown in the figure, i.e., when they produce compressionin the upper surface of the plate and tension in the lower. The thickness of the plate we denote, as before, by h and consider it small in comparison with other dimensions.

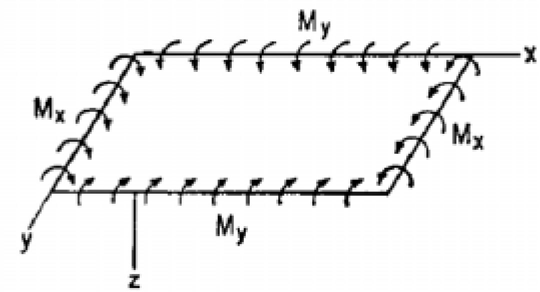


Fig. 6

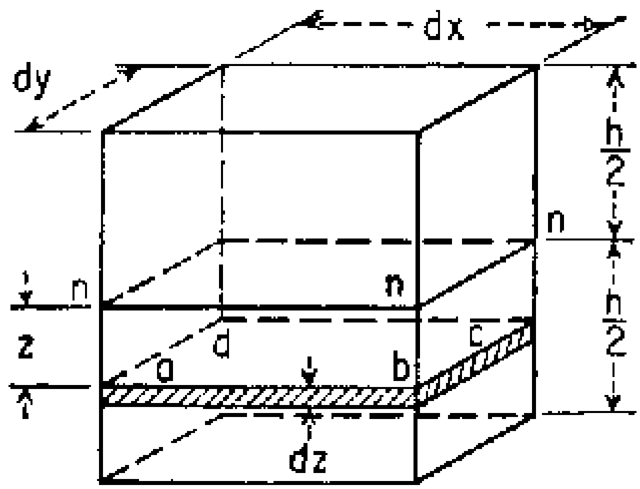


Fig. 7

Let us consider an element cut out of the plate by two pairs of planes parallel to the xz and yz planes, as shown in Fig. 7. Since the case shown in Fig. 6 represents the combination of two uniform bending, the stress conditions are identical in all elements, as shown in Fig. 7, and we havea uniform bonding of the plate. Assuming that during bending of the plate the lateral sides of the element remain plane and rotateabout the neutral axes *nn* so as to remain normal to the deflected middle surface of theplate, it can be concluded that the middleplane of the plate does not undergo any extension during this bending, and the middlesurface is therefore the *neutral surface*. Let$1/r_x$ and $1/r_y$ denote, as before, the curvatures of this neutral surface in sections parallel to the xz and yz planes, respectively. Then the unit elongations in the x and y directions of an elemental lamina *abcd* (Fig. 7), at a distance z from the neutral surface, are found, as in the case of a beam, and are equal to

$$\epsilon_x = \frac{z}{r_x} \qquad \epsilon_y = \frac{z}{r_y} \qquad (a)$$

Using Hooke's law

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$
$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} = 0$$

The corresponding stresses in lamina *abcd* are

$$\begin{aligned}\sigma_x &= \frac{Ez}{1-\nu^2}\left(\frac{1}{r_x} + \nu\frac{1}{r_y}\right) \\ \sigma_y &= \frac{Ez}{1-\nu^2}\left(\frac{1}{r_y} + \nu\frac{1}{r_x}\right)\end{aligned} \qquad (b)$$

These stresses are proportional to the distance *z* of the lamina *abcd* from the neutral surface and depend on the magnitude of the curvatures of the bent plate.

The normal stresses distributed over the lateral sides of the element in Fig. 7 can be reduced to couples, the magnitudes of which per unit length evidently must be equal to the external moments M_x and M_y. In this way we obtain the equations

$$\begin{aligned}\int_{-h/2}^{h/2} \sigma_x z\, dy\, dz &= M_x\, dy \\ \int_{-h/2}^{h/2} \sigma_y z\, dx\, dz &= M_y\, dx\end{aligned} \qquad (c)$$

Substituting expressions (b) for σ_x and σ_y, we obtain

$$M_x = D\left(\frac{1}{r_x} + \nu\frac{1}{r_y}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad (6)$$

$$M_y = D\left(\frac{1}{r_y} + \nu\frac{1}{r_x}\right) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \qquad (7)$$

where *D* is the flexural rigidity of the plate, and *w* denotes small deflections of the plate in the *z* direction.

Let us now consider the stresses acting on a section of the lamina *abcd* parallel to the *z* axis and inclined to the *x* and *y* axes. If *acd* (Fig. 8)

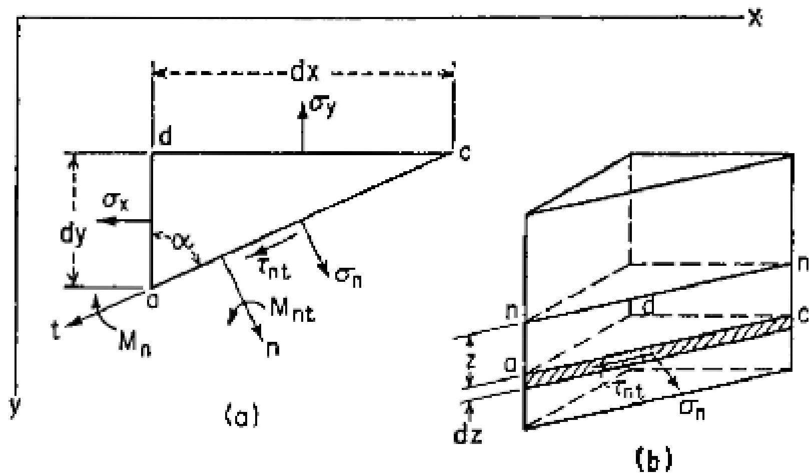


Fig. 8

represents a portion of the lamina cut by such a section, the stress acting on the side *ac* can be found by means of the equations of statics. Resolving this stress into a normal component σ_n and a shearing component τ_{nt}, the magnitudes of these components are obtained by projecting the forces acting on the element *acd* on the *n* and *t* directions respectively, which gives the known equations

$$\begin{aligned}\sigma_n &= \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha \\ \tau_{nt} &= \tfrac{1}{2}(\sigma_y - \sigma_x) \sin 2\alpha\end{aligned} \qquad (d)$$

in which α is the angle between the normal *n* and the *x* axis or between the direction *t* and the *y* axis (Fig. 8a). The angle is considered positive if measured in a clockwise direction.

Considering all laminas, such as *acd* in Fig. 8b, over the thickness of the plate, the normal stresses σ_n give the bending moment acting on the section *ac* of the plate, the magnitude of which per unit length along *ac* is

$$M_n = \int_{-h/2}^{h/2} \sigma_n z \, dz = M_x \cos^2 \alpha + M_y \sin^2 \alpha \qquad (8)$$

The shearing stresses τ_{nt} give the twisting moment acting on the section*ac* of the plate, the magnitude of which per unit length of *ac* is

$$M_{nt} = -\int_{-h/2}^{h/2} \tau_{nt} z \, dz = \tfrac{1}{2} \sin 2\alpha (M_x - M_y) \qquad (9)$$

The signs of M_n and M_{nt} are chosen in such a manner that the positive values of these moments are represented by vectors in the positive directions of n and t (Fig. 8a) if the rule of the right-hand screw is used.When α is zero or π,

Eq. (8) gives $M_n = M_x$. For $\alpha = \pi/2$ or $3\pi/2$, we obtain $M_n = M_y$. The moments M_{nt} become zero for these values of α. Thus we obtain the conditions shown in Fig. 6.

By using Eqs. (8) and (9) the bending andtwisting moments can be readily calculatedfor any value of α.

3.6. Membrane Analogy for Bending of a Plate

There are cases in which it is advantageous toreplace the differential equation of the fourth order developed fora plate by two equations of the second order which represent the deflections of a membrane.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{q}{D} \qquad (a)$$

and observe that by adding together the two expressions for bending moments

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

we have

$$M_x + M_y = -D(1+\nu)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad (b)$$

Introducing a new notation

$$M = \frac{M_x + M_y}{1+\nu} = -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

the two Eqs. (a) and (b) can be represented in the following form:

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = -q$$
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M}{D} \qquad (10)$$

Both these equations are of the same kind as that obtained for a uniformly stretched and laterally loaded membrane.

The solution of these equations is very much simplified in the case of a simply supported plate of polygonal shape, in which case along each rectilinear portion of the boundary we have $\partial^2 w/\partial s^2 = 0$ since $w = 0$ at the boundary. Observing that $M_n = 0$ at a simply supported edge, we conclude also that $\partial^2 w/\partial n^2 = 0$ at the boundary. Hence we have

$$\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial n^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M}{D} = 0 \qquad (c)$$

at the boundary. It is seen that the solution of the plate problem reduces in this case tothe integration of the two equations (10) in succession. We begin withthe first of these equations and find a solution satisfying the condition $M = 0$ at the boundary. Substituting this solution in the second equation and integrating it, we find the deflections w. Both problems are of the same kind as the problem of the deflection of a uniformly stretched and laterally loaded membrane having zero deflection at the boundary. This latter problem is much simpler than the plate problem, and it can always be solved with sufficient accuracy by using an approximate method of integration such as Ritz's or the method of finite differences.

3.7. Symmetrical Bending of a Circular Plate

Several problems of practicalinterest can be solved with the help of the foregoing solutions. Amongthese are various cases of the bending ofsymmetrically loaded circular plates (Fig. 9).

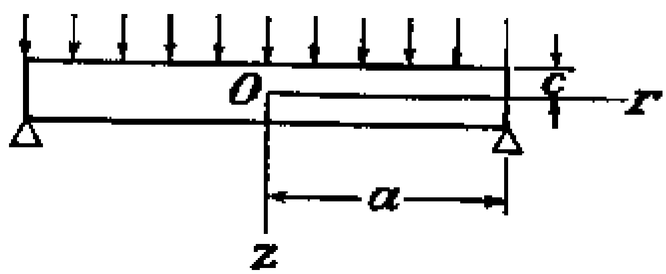


Fig. 9

Taking, for instance, thepolynomials of the third degree, from Eqs.

$$
\begin{aligned}
\phi_0 &= A_0 \\
\phi_1 &= A_1 z \\
\phi_2 &= A_2[z^2 - \tfrac{1}{3}(r^2 + z^2)] \\
\phi_3 &= A_3[z^3 - \tfrac{3}{5}z(r^2 + z^2)] \\
\phi_4 &= A_4[z^4 - \tfrac{6}{7}z^2(r^2 + z^2) + \tfrac{3}{35}(r^2 + z^2)^2] \\
\phi_5 &= A_5[z^5 - \tfrac{10}{9}z^3(r^2 + z^2) + \tfrac{5}{21}z(r^2 + z^2)^2] \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\phi_2 &= B_2(r^2 + z^2) \\
\phi_3 &= B_3 z(r^2 + z^2) \\
\phi_4 &= B_4(2z^2 - r^2)(r^2 + z^2) \\
\phi_5 &= B_5(2z^3 - 3r^2 z)(r^2 + z^2) \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{12}
$$

we obtain the stress function

$$
\phi = a_3(2z^3 - 3r^2 z) + b_3(r^2 z + z^3) \tag{a}
$$

Substituting in Eqs.

$$
\begin{aligned}
\sigma_r &= \frac{\partial}{\partial z}\left(\nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2}\right) \\
\sigma_\theta &= \frac{\partial}{\partial z}\left(\nu \nabla^2 \phi - \frac{1}{r}\frac{\partial \phi}{\partial r}\right) \\
\sigma_z &= \frac{\partial}{\partial z}\left[(2 - \nu)\nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2}\right] \\
\tau_{rz} &= \frac{\partial}{\partial r}\left[(1 - \nu)\nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2}\right]
\end{aligned} \tag{13}
$$

we find

$$
\begin{aligned}
\sigma_r &= 6a_3 + (10\nu - 2)b_3, \qquad \sigma_\theta = 6a_3 + (10\nu - 2)b_3 \\
\sigma_z &= -12a_3 + (14 - 10\nu)b_3, \qquad \tau_{rz} = 0
\end{aligned} \tag{14}
$$

The stress components are thus constant throughout the plate. By asuitable adjustment of constants a_3 and b_3 we can get the stresses in aplate when any constant values of σ_z and σ_r at the surface of the plateare given.

Let us take now the polynomials of the fourth degree from (11) and (12), which gives us

$$
\phi = a_4(8z^4 - 24r^2z^2 + 3r^4) + b_4(2z^4 + r^2z^2 - r^4) \tag{b}
$$

Substituting in Eqs. (13), we find

$$\begin{aligned} \sigma_r &= 96a_4z + 4b_4(14\nu - 1)z \\ \sigma_z &= -192a_4z + 4b_4(16 - 14\nu)z \\ \tau_{rz} &= 96a_4r - 2b_4(16 - 14\nu)r \end{aligned} \tag{15}$$

Taking

$$96a_4 - 2b_4(16 - 14\nu) = 0$$

we have

$$\sigma_z = \tau_{rz} = 0, \qquad \sigma_r = 28(1 + \nu)b_4z \tag{c}$$

If z is the distance from the middle plane of the plate, the solution (c)represents pure bending of the plate by moments uniformly distributedalong the boundary.

To get the solution for a circular plate uniformly loaded, we take the stress function in the form of a polynomial of the sixth power.

$$\phi = \tfrac{1}{3}a_6(16z^6 - 120z^4r^2 + 90z^2r^4 - 5r^6) + b_6(8z^6 - 16z^4r^2 - 21z^2r^4 + 3r^6)$$

Substituting in (13),

$$\begin{aligned} \sigma_r &= a_6(320z^3 - 720r^2z) + b_6[64(2 + 11\nu)z^3 + (504 - 48 \cdot 22\nu)r^2z] \\ \sigma_z &= a_6(-640z^3 + 960r^2z) + b_6\{[-960 + 32 \cdot 22(2 - \nu)]z^3 \\ &\qquad + [384 - 48 \cdot 22(2 - \nu)]r^2z\} \\ \tau_{rz} &= a_6(960rz^2 - 240r^3) \\ &\qquad + b_6[(-672 + 48 \cdot 22\nu)z^2r + (432 - 12 \cdot 22\nu)r^3] \end{aligned}$$

To these stresses we add the stresses

$$\sigma_r = 96a_4z, \qquad \sigma_z = -192a_4z, \qquad \tau_{rz} = 96a_4r$$

obtained from (15) by taking $b_4 = 0$, and a uniform tension in the z-direction $\sigma_z = b$, which can be obtained from (14). Thus we arriveat expressions for the stress components containing four constantsa_6, b_6, a_4, b. These constants can be adjusted so as to satisfy theboundary conditions on the upper and lower surfaces of the plate (Fig. 9). The conditions are

$$\begin{aligned} \sigma_z &= 0 && \text{for} \quad z = c \\ \sigma_z &= -q && \text{for} \quad z = -c \\ \tau_{rz} &= 0 && \text{for} \quad z = c \\ \tau_{rz} &= 0 && \text{for} \quad z = -c \end{aligned} \tag{d}$$

Here q denotes the intensity of the uniform load and $2c$ is the thickness of the plate. Substituting the expressions for the stress components in these equations, we determine the four constants a_6, b_6, a_4, b. Using these values, the expressions for the stress components satisfying conditions (d) are

$$\begin{aligned} \sigma_r &= q\left[\frac{2+\nu}{8}\frac{z^3}{c^3} - \frac{3(3+\nu)}{32}\frac{r^2 z}{c^3} - \frac{3}{8}\frac{z}{c}\right] \\ \sigma_z &= q\left(-\frac{z^3}{4c^3} + \frac{3}{4}\frac{z}{c} - \frac{1}{2}\right) \qquad (e) \\ \tau_{rz} &= \frac{3qr}{8c^3}(c^2 - z^2) \end{aligned}$$

It will be seen that the stresses σ_z and τ_{rz} are distributed in exactly the same manner as in the case of a uniformly loaded beam of narrow rectangular cross section. The radial stresses σ_r are represented by an odd function of z, and at the boundary of the plate they give bending moments uniformly distributed along the boundary. To get the solution for a simply supported plate (Fig. 10), we superpose a pure bending stress (c) and adjust the constant b_4 so as to obtain for the boundary ($r = a$)

$$\int_{-c}^{c} \sigma_r z \, dz = 0$$

Then the final expression for σ_r becomes

$$\sigma_r = q\left[\frac{2+\nu}{8}\frac{z^3}{c^3} - \frac{3(3+\nu)}{32}\frac{r^2 z}{c^3} - \frac{3}{8}\frac{2+\nu}{5}\frac{z}{c} + \frac{3(3+\nu)}{32}\frac{a^2 z}{c^3}\right] \qquad (16)$$

and at the center of the plate we have

$$(\sigma_r)_{r=0} = q\left[\frac{2+\nu}{8}\frac{z^3}{c^3} - \frac{3}{8}\frac{2+\nu}{5}\frac{z}{c} + \frac{3(3+\nu)}{32}\frac{a^2 z}{c^3}\right] \qquad (f)$$

The elementary theory of bending of plates, based on the assumptions that linear elements of the plate perpendicular to the *middle plane*($z = 0$) remain straight and normal to the deflection surface of theplate during bending, gives for the radial stresses at the center

$$\sigma_r = \frac{3(3+\nu)}{32}\frac{a^2 z}{c^3} q \qquad (g)$$

Comparing this with (f), we see that the additional terms of the exact solution are small if the thickness of the plate, *2c*, is small in comparison with the radius *a*.

It should be noted that by superposing pure bending we eliminated bending moments along the boundary of the plate, but the radial stresses are not zero at the boundary but are

$$(\sigma_r)_{r=a} = q\left(\frac{2+\nu}{8}\frac{z^3}{c^3} - \frac{3}{8}\frac{2+\nu}{5}\frac{z}{c}\right) \qquad (h)$$

The resultant of these stresses per unit length of the boundary line and their moment, however, are zero. Hence, on the basis of Saint-Venant's principle, we can say that the removal of these stresses does not affect the stress distribution in the plate at some distance from the edge.

3.8. Navier's Solution for Simply Supported Rectangular Plates

The deflections produced in a simply supported rectangular plate by any kind of loading is given by the equation

$$q = f(x,y) \qquad (a)$$

For this purpose we represent the function *f(x,y)* in the form of a double trigonometric series

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad (17)$$

To calculate any particular coefficient $a_{m'n'}$ of this series we multiply both sides of Eq. (17) by $\sin (n'\pi y/b)\, dy$ and integrate from *0* to *b*. Observingthat

$$\int_0^b \sin\frac{n\pi y}{b} \sin\frac{n'\pi y}{b}\, dy = 0 \qquad \text{when } n \neq n'$$

$$\int_0^b \sin\frac{n\pi y}{b} \sin\frac{n'\pi y}{b}\, dy = \frac{b}{2} \qquad \text{when } n = n'$$

we find in this way

$$\int_0^b f(x,y) \sin\frac{n'\pi y}{b}\, dy = \frac{b}{2}\sum_{m=1}^{\infty} a_{mn'} \sin\frac{m\pi x}{a} \qquad (b)$$

Multiplying both sides of Eq. (b) by *sin (m'πx/a) dx* and integrating from *0* to *a*, we obtain

$$\int_0^a \int_0^b f(x,y) \sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b} \, dx \, dy = \frac{ab}{4} a_{m'n'}$$

from which

$$a_{m'n'} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin \frac{m'\pi x}{a} \sin \frac{n'\pi y}{b} \, dx \, dy \tag{18}$$

Performing the integration indicated in expression (18) for a given load distribution, i.e., for a given *f(x,y)*, we find the coefficients of series (17) and represent in this way the given load as a sum of partial sinusoidal loadings. The deflection produced by each partial loading is

$$w = \frac{q_0}{\pi^4 D \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

and the total deflection will be obtained by summation of such terms. Hence we find

$$w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{19}$$

Take the case of a load uniformly distributed over the entire surface of the plate as an example of the application of the general solution (19). In such a case

$$f(x,y) = q_0$$

whereq_0 is the intensity of the uniformly distributed load. From formula (18) we obtain

$$a_{mn} = \frac{4q_0}{ab} \int_0^a \int_0^b \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy = \frac{16q_0}{\pi^2 mn} \tag{c}$$

where*m* and *n* are odd integers. If *m* or *n* or both of them are even numbers, $a_{mn} = 0$. Substituting in Eq. (19), we find

$$w = \frac{16q_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \tag{20}$$

where$m$ = 1, 3, 5, . . . and n = 1, 3, 5, ...

In the case of a uniform load we have a deflection surface symmetrical with respect to the axes $x = a/2$, $y = b/2$; and quite naturally all terms with even numbers for m or n in series (20) vanish, since they are unsymmetrical with respect to the above-mentioned axes. The maximum deflection of the plate is at its center and is found by substituting $x = a/2$, $y = b/2$ in formula (20), giving

$$w_{max} = \frac{16q_0}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{\frac{m+n}{2}-1}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \qquad (21)$$

This is a rapidly converging series, and a satisfactory approximation is obtained by taking only the first term of the series, which, for example, in the case of a square plate gives

$$w_{max} = \frac{4q_0a^4}{\pi^6 D} = 0.00416\frac{q_0a^4}{D}$$

3.9. Combined bending and stretching of rectangular plates

If the boundary conditions applied to a plate are such that the distances between opposite edgesare constrained then, in addition to bending effects, direct and shear forces may be induced in the plane of the plate and it may be necessary to consider the resulting stretching, Fig. 10 shows the in-plane

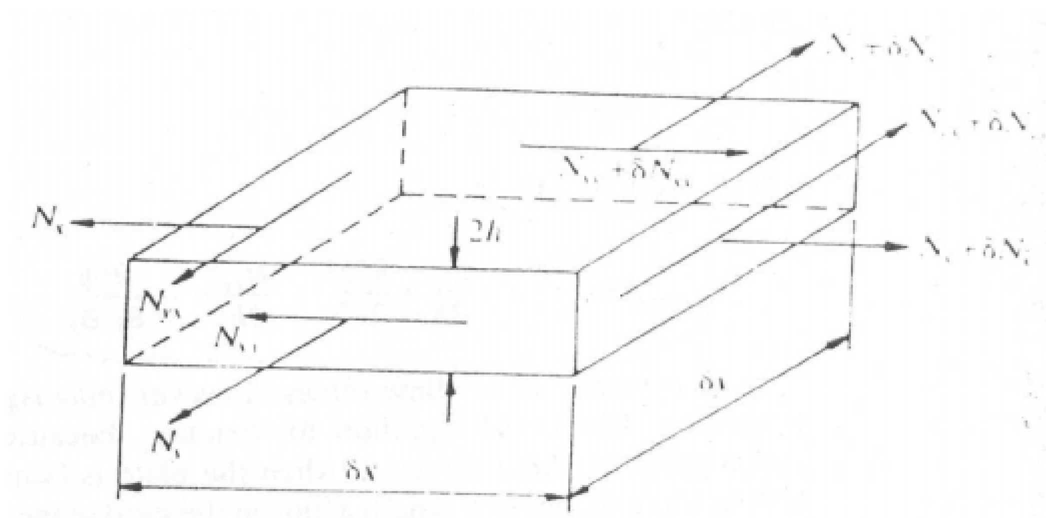


Fig. 10

tensile and shear forces per unit length, N_x,N_y and $N_{xy} = N_{yx}$, which act on the small element of a flat plate. Theseforces are in addition to the moments and out-of-plane shear forces already considered.

Previously, the only force equilibrium equation employed was for the z direction, normal to the plate. The equation of equilibrium in the x direction may be derived with the aid of Fig. 10 as

$$(N_x + \delta N_x)\,\delta y - N_x\,\delta y + (N_{xy} + \delta N_{xy})\,\delta x - N_{xy}\,\delta x = 0$$

which in view of the fact that

$$\delta N_x = \frac{\partial N_x}{\partial x}\delta x \qquad \delta N_{xy} = \frac{\partial N_{xy}}{\partial y}\delta y$$

reduces to

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

Similarly for equilibrium of forces in the y direction

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

The solution to the stretching problem can be superimposed on the bending behaviour already treated. Since pure stretching of a plate of constant thickness, *2h*, is a plane stress problem, it may be solved with the aid of

$$\nabla^4 \phi = 0$$

where

$$\frac{N_x}{2h} = \frac{\partial^2 \phi}{\partial y^2} \qquad \frac{N_y}{2h} = \frac{\partial^2 \phi}{\partial x^2} \qquad \frac{N_{xy}}{2h} = -\frac{\partial^2 \phi}{\partial x\,\partial y}$$

The presence of in-plane forces, however, may significantly affect the governing differential equation for bending, because these forces have components in the z direction when the plate is bent. For example, the tensile force N_x per unit length acting in the local plane of the plate over the left-hand edge, of length δy. of the element shown in Fig. 10, has a component in the z direction of

$$-N_x \delta y \frac{\partial w}{\partial x}$$

Similarly, the force $N_x + \delta N_x$, on the right-hand edge has a component

$$+(N_x + \delta N_x)\,\delta y\left[\frac{\partial w}{\partial x} + \delta\left(\frac{\partial w}{\partial x}\right)\right] =$$

$$= +\left(N_x + \frac{\partial N_x}{\partial x}\delta x\right)\delta y\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\delta x\right)$$

Neglecting terms involving products of more than two length increments, the resultant of these two forces per unit area of plate is given by

$$N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x}$$

which can he added to the applied pressure, p. In the same way, the force per unit area in the z direction due to N_y is

$$N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y}$$

and the sum of those due to the shearing forces is

$$2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial N}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial N_{xy}}{\partial y}\frac{\partial w}{\partial x}$$

The resulting differential equation becomes

$$\nabla^4 w = \frac{1}{D}\left(p + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x\,\partial y}\right)$$

3.10. Summary

In this unit we have studied

- Cylindrical Bending of Rectangular Plates
- Slope and Curvatures
- Determination of Bending and Twisting Moments on any plane
- Membrane Analogy for Bending of a Plate
- Symmetrical Bending of a Circular Plate
- Navier's Solution for simply supported Rectangular Plates
- Combined Bending and Stretching of Rectangular Plates

3.11. Keywords

Twisting Moments

Membrane Analogy

Rectangular Plates

Navier's Solution

3.12. Exercise

1. Explain bending of plates and find out moments in terms of Flexural Rigidity of the plate.
2. What do you mean by Cylindrical Bending of Rectangular Plates? Explain.
3. Show that at any point of the middle surface the sum of the curvatures in two perpendicular directions is independent of the angle α.
4. Derive expressions for Bending and Twisting Moments on any plane.
5. Explain Bending of plates with the help of Membrane Analogy.
6. Show that:

   $$(\sigma_r)_{r=a} = q\left(\frac{2+\nu}{8}\frac{z^3}{c^3} - \frac{3}{8}\frac{2+\nu}{5}\frac{z}{c}\right)$$

7. Derive an expression for maximum deflection in simply supported Rectangular Plates by Navier's Solution.

8. Explain combined Bending and Stretching of rectangular plates.

Unit 4
THIN SHELLS

Structure

4.1. Introduction

In the following discussion we denote the thickness of the shell by h, this quantity always being considered small in comparison with the other dimensions of the shell and with its radii of curvature. The surface that bisects the thickness of the plate is called the middle surface. By specifying the form of the middle surface and the thickness of the shell at each point, a shell is entirely defined geometrically.

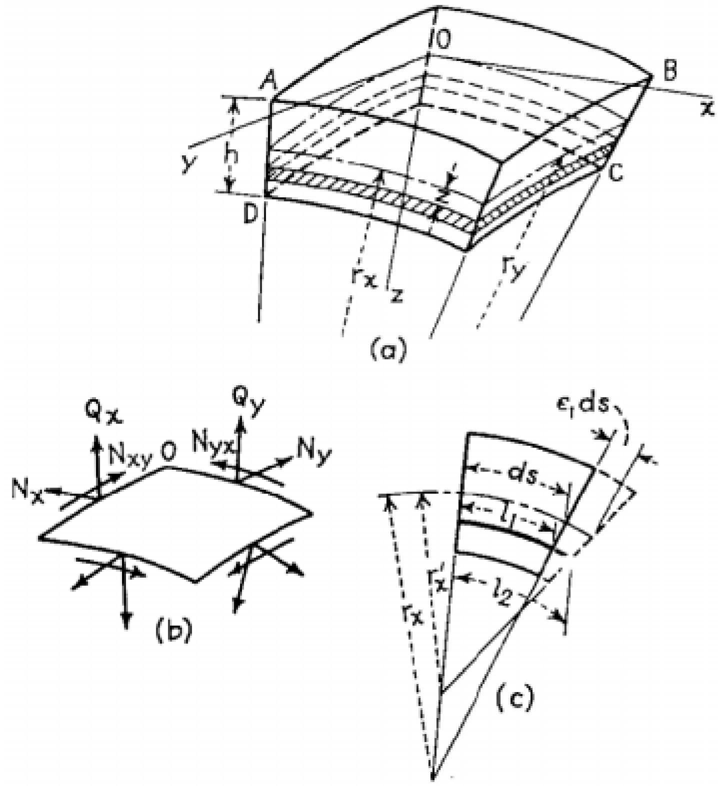


Fig. 1

To analyze the internal forces we cut from the shell an infinitely small element formed by two pairs of adjacent planes which are normal to the middle surface of the shell and which contain its principal curvatures (Fig. 1a). We take the coordinate axes *x* and *y* tangent at *O* to the lines of principal curvature and the axis *z* normal to the middle surface, as shown in the figure. The principal radii of curvature which lie in the *xz* and *yz* planes are denoted by r_x and r_y, respectively. The stresses acting on the plane faces of the element are resolved in the directions of the coordinate axes, and the stress components are denoted by our previous symbols σ_x, σ_y, $\tau_{xy} = \tau_{yx}$, τ_{xz}. With this notation the resultant forces per unit length of the normal sections shown in Fig. 1b are

$$N_x = \int_{-h/2}^{+h/2} \sigma_x \left(1 - \frac{z}{r_y}\right) dz \qquad N_y = \int_{-h/2}^{+h/2} \sigma_y \left(1 - \frac{z}{r_x}\right) dz \qquad (a)$$

$$N_{xy} = \int_{-h/2}^{+h/2} \tau_{xy} \left(1 - \frac{z}{r_y}\right) dz \qquad N_{yx} = \int_{-h/2}^{+h/2} \tau_{yx} \left(1 - \frac{z}{r_x}\right) dz \qquad (b)$$

$$Q_x = \int_{-h/2}^{+h/2} \tau_{xz} \left(1 - \frac{z}{r_y}\right) dz \qquad Q_y = \int_{-h/2}^{+h/2} \tau_{yz} \left(1 - \frac{z}{r_x}\right) dz \qquad (c)$$

The small quantities z/r_x and z/r_y appear in expressions (a), (b), (c), because the lateral sides of the element shown in Fig. 1a have a trapezoidal form due to the curvature of the shell. As a result of this, the shearing forces N_{xy} and N_{yx} are generally not equal to each other, although it still holds that $\tau_{xy} = \tau_{yx}$. In our further discussion we shall always assume that the thickness h is very small in comparison with the radii r_x, r_y and omit the terms z/r_x and z/r_y in expressions (a), (b), (c). Then $N_{xy} = N_{yx}$ and the resultant shearing forces are given by the same expressions as in the case of plates.

The bending and twisting moments per unit length of the normal sections are given by the expressions

$$M_x = \int_{-h/2}^{+h/2} \sigma_x z \left(1 - \frac{z}{r_y}\right) dz \qquad M_y = \int_{-h/2}^{+h/2} \sigma_y z \left(1 - \frac{z}{r_x}\right) dz \qquad (d)$$

$$M_{xy} = -\int_{-h/2}^{+h/2} \tau_{xy} z \left(1 - \frac{z}{r_y}\right) dz \qquad M_{yx} = \int_{-h/2}^{+h/2} \tau_{yx} z \left(1 - \frac{z}{r_x}\right) dz \qquad (e)$$

in which the rule used in determining the directions of the moments is the same as in the case of plates. In our further discussion we again neglect the small quantities z/r_x and z/r_y, due to the curvature of the shell, and use for the moments the same expressions as in the discussion of plates.

4.2. Objectives

After studying this unit we have studied

- Membrane Theory of Shells
- Geometry of Shells of Revolution

4.3. Membrane Theory of Cylindrical Shells

In discussing a cylindrical shell (Fig. 2a) we assume that the generator of the shell is horizontal and parallel to the x axis. An clement is cut from the shell bytwo adjacent generators and two cross sections perpendicular to the x axis, and its position is defined by the coordinate x and the angle ϕ . Theforces acting on the sides of the element are shown in Fig. 2b.

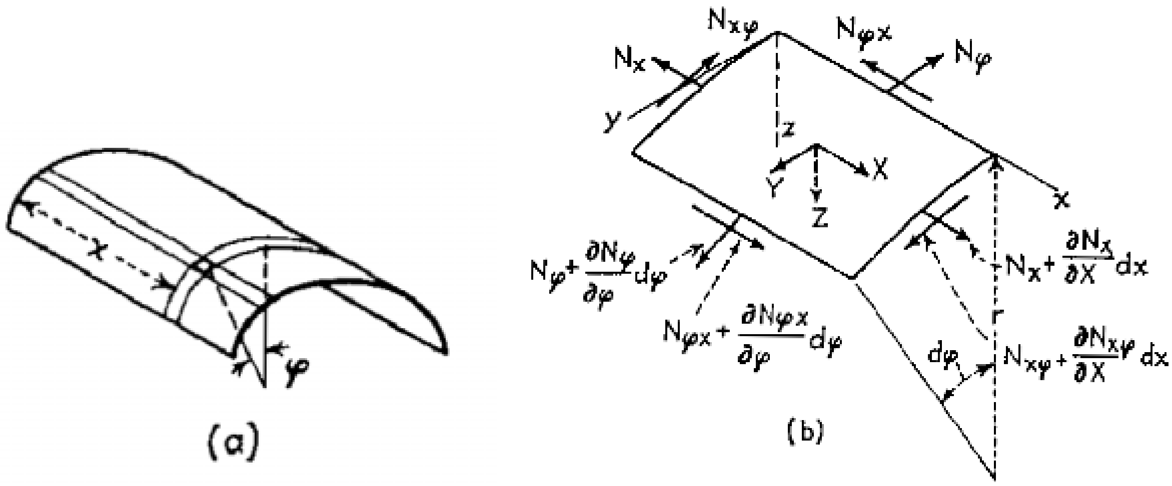


Fig. 2

Inaddition a load will be distributed over the surface of the element, the components of the intensity of this load being denoted, as before, by *X*, *Y*, and *Z*. Considering the equilibrium of the element and summing up the forces in the *x* direction, we obtain

$$\frac{\partial N_x}{\partial x} r\, d\varphi\, dx + \frac{\partial N_{\varphi x}}{\partial \varphi} d\varphi\, dx + Xr\, d\varphi\, dx = 0 \qquad (a)$$

Similarly, the forces in the direction of the tangent to the normal cross section, i.e., in the *y* direction, give as a corresponding equation of equilibrium

$$\frac{\partial N_{x\varphi}}{\partial x} r\, d\varphi\, dx + \frac{\partial N_\varphi}{\partial \varphi} d\varphi\, dx + Yr\, d\varphi\, dx = 0 \qquad (b)$$

The forces acting in the direction of the normal to the shell, i.e., in the*z* direction, give the equation

$$N_\varphi\, d\varphi\, dx + Zr\, d\varphi\, dx = 0 \qquad (c)$$

After simplification, the three equations of equilibrium can be represented in the following form:

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{x\varphi}}{\partial \varphi} = -X$$
$$\frac{\partial N_{x\varphi}}{\partial x} + \frac{1}{r}\frac{\partial N_\varphi}{\partial \varphi} = -Y$$
$$N_\varphi = -Zr$$

In each particular case we readily find the value of N_ϕ . Substituting this value in the second of the equations, we then obtain $N_{x\phi}$ by integration. Using the value of $N_{x\phi}$ thus obtained we find N_x by integrating the first equation.

4.4. Geometry of Shells of Revolution

Shells that have the form ofsurfaces of revolution find extensive application in various kinds of containers, tanks, and domes. A surface of revolution is obtained by rotation of a plane curve about an axis lying in the plane of the curve. Thiscurve is called the meridian, and its plane is a meridian plane. An element of a shell is cut out by two adjacent meridians and two parallel circles, as shown in Fig. 3a.

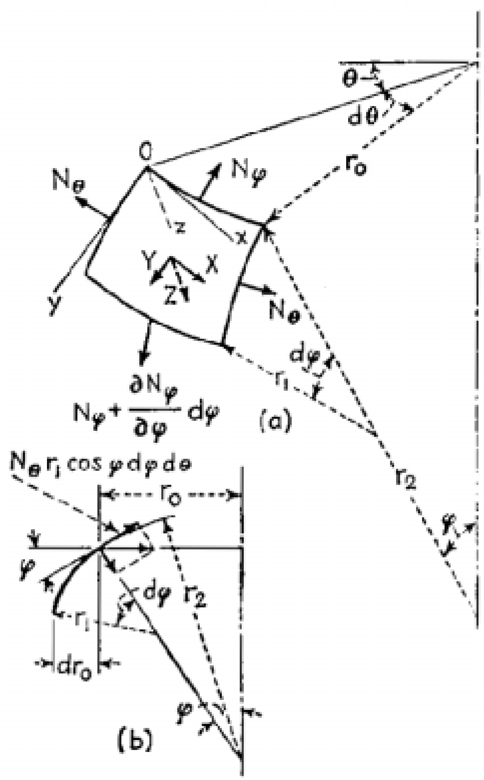


Fig. 3

The position of a meridian is defined by an angle θ, measured from some datum meridian plane; and the position of a parallel circle is defined by the angle ϕ , made by the normal to thesurface and the axis of rotation. The meridian plane and the plane perpendicular to the meridian are the planes of principal curvature at a point of a surface of revolution, and the corresponding radii of curvature are denoted by r_1 and r_2, respectively. The radius of the parallel circle is denoted by r_0 so that the length of the sides of the element meeting at *O*, as shown in the figure, are $r_1 d\phi$ and $r_0 d\theta = r_2 \sin \phi \, d\theta$. The surface area of the element is then $r_1 r_2 \sin \phi \, d\phi \, d\theta$.

From the assumed symmetry of loading and deformation it can be concluded that there will be no shearing forces acting on the sides of the element. The magnitudes of the normal forces per unit length are denoted by N_ϕ and N_θ as shown in the figure. The intensity of the external load, which acts in the meridian plane, in the case of symmetry is resolved in two components Y and Z parallel to the coordinate axes. Multiplying these components with the area $r_1 r_2 \sin \phi \, d\phi \, d\theta$, we obtain the components of the external load acting on the element.

In writing the equations of equilibrium of the element, let us begin with the forces in the direction of the tangent to the meridian. On the upper side of the element the force

$$N_\varphi r_0 \, d\theta = N_\varphi r_2 \sin \varphi \, d\theta \qquad (a)$$

is acting. The corresponding force on the lower side of the element is

$$\left(N_\varphi + \frac{dN_\varphi}{d\varphi} d\varphi\right)\left(r_0 + \frac{dr_0}{d\varphi} d\varphi\right) d\theta \qquad (b)$$

From expressions (a) and (b), by neglecting a small quantity of second order, we find the resultant in the y direction to be equal to

$$N_\varphi \frac{dr_0}{d\varphi} d\varphi \, d\theta + \frac{dN_\varphi}{d\varphi} r_0 \, d\varphi \, d\theta = \frac{d}{d\varphi} (N_\varphi r_0) \, d\varphi \, d\theta \qquad (c)$$

The component of the external force in the same direction is

$$Y r_1 r_0 \, d\varphi \, d\theta \qquad (d)$$

The forces acting on the lateral sides of the element are equal to $N_\theta r_1 d\phi$ and have a resultant in the direction of the radius of the parallel circle equal to $N_\theta r_1 d\phi \, d\theta$. The component of this force in the y direction (Fig. 3b) is

$$-N_\theta r_1 \cos \varphi \, d\varphi \, d\theta \qquad (e)$$

Summing up the forces (c), (d), and (e), the equation of equilibrium in the direction of the tangent to the meridian becomes

$$\frac{d}{d\varphi} (N_\varphi r_0) - N_\theta r_1 \cos \varphi + Y r_1 r_0 = 0 \qquad (f)$$

The second equation of equilibrium is obtained by summing up the projections of the forces in the z direction. The forces acting on the upper and lower sides of the element have a resultant in the z direction equal to

$$N_\varphi r_0 \, d\theta \, d\varphi \qquad (g)$$

The forces acting on the lateral sides of the element and having the resultant $N_\theta r_1 d\phi \, d\theta$ in the radial direction of the parallel circle give a component in the z direction of the magnitude

$$N_\theta r_1 \sin \varphi \, d\varphi \, d\theta \qquad (h)$$

The external load acting on the element has in the same direction a component

$$Z r_1 r_0 \, d\theta \, d\varphi \qquad (i)$$

Summing up the forces (g), (h), and (i), we obtain the second equation of equilibrium

$$N_\varphi r_0 + N_\theta r_1 \sin \varphi + Z r_1 r_0 = 0 \qquad (j)$$

From the two Eqs. (f) and (j) the forces N_ϕ and N_θ can be calculated in each particular case if the radii r_0 and r_1 andthe components Y and Z of the intensity of the external load are given.

Instead of the equilibrium of an element, the equilibrium of the portion of the shell above the parallel circle defined by the angle ϕ may be considered (Fig. 4).

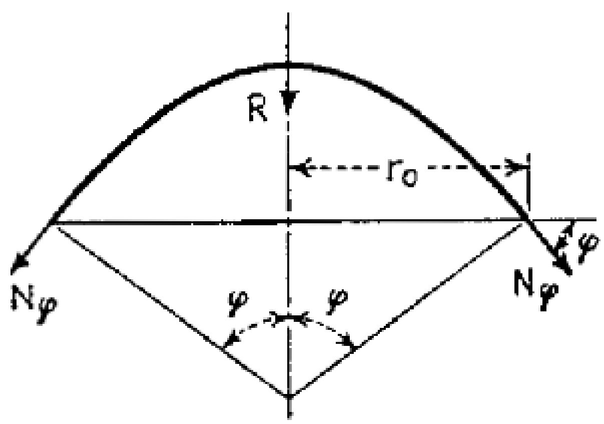


Fig. 4

If the resultant of the total load on that portion of the shell is denoted by R, the equation of equilibrium is

$$2\pi r_0 N_\varphi \sin \varphi + R = 0$$

This equation can be used instead of the differential equation (f), from which it can be obtained by integration. If Eq. (j) is divided by $r_1 r_0$, it can be written in the form

$$\frac{N_\varphi}{r_1} + \frac{N_\theta}{r_2} = -Z$$

4.5. Summary

In this unit we studied

- Membrane Theory of Shells
- Geometry of Shells of Revolution

4.6. Keywords

Membrane theory

Revolution

Cylindrical shell

4.7. Exercise

1. Write a short note on Membrane Theory of Cylindrical Shells.
2. Explain Geometry of Shells of Revolution.

Unit 1

NUMERICAL AND ENERGY METHODS

Structure

1.1. Introduction

1.2. Objectives

1.3. Rayleigh's Method

1.4. Rayleigh – Ritz Method

1.5. Finite Difference and Finite Element Method

1.6. Summary

1.7. Keywords

1.8. Exercise

1.1. Introduction

Through the use of numerical methods many problems can be solved that would otherwise be thought to be insoluble. In the past, solving problems numerically often meant a great deal of programming and numerical problems. Programming languages such as Fortran, Basic, Pascal and C have been used extensively by scientists and engineers, but they are often difficult to program and to debug. Modern commonly-available software has gone a long way to overcoming such difficulties. Matlab, Maple, Mathematical, and MathCAD for example, are rather more user-friendly, as many operations have been modularized, such that the programmer can see rather more clearly what is going on. However, spreadsheet programs provide engineers and scientists with very powerful tools. The two which will be referred to in these lectures are Microsoft Excel and OpenOffice.org Calc. Spreadsheets are much more intuitive than using high- level languages, and one can easily learn to use a spreadsheet to a certain level. Yet often users do not know how to translate powerful numerical procedures into spreadsheet calculations.

Dynamic systems can be characterized in terms of one or more natural frequencies. The natural frequency is the frequency at which the system would vibrate if it were given an initial disturbance and then allowed to vibrate freely.

There are many available methods for determining the natural frequency. Some examples are

- Newton' Law of Motion
- Rayleigh' Method
- Energy Method
- Lagrange' Equation

Not that the Rayleigh, Energy, and Lagrange methods are closely related.
Some of these methods directly yield the natural frequency. Others yield a governing equation of motion, from which the natural frequency may be determined. The energy method, which is an example of a method which yields an equation of motion.

Definition of the Energy Method

The total energy of a conservative system is constant. Thus,

$$\frac{d}{dt}(KE + PE) = 0$$

where

KE = kinetic energy

PE = potential energy

Kinetic energy is the energy of motion, as calculated from the velocity.

Potential energy has several forms. One is strain energy. Another is the work done

1.2. Objectives

After studying this unit we are able to understand

- Rayleigh's Method
- Rayleigh – Ritz Method
- Finite Difference Method
- Finite Element Method

1.3. Rayleigh's Method

Consider now two sets of applied forces and reactions: P'_k (k = 1,2,..., m), set1;P'_j (j = 1,2,...,n), set 2. If only the first set is applied, the strain energy is, from Eq.

$$U = W = \frac{1}{2}\sum_{k=1}^{m} P_k \delta_k$$

This gives,

$$U_1 = \frac{1}{2}\sum_{k=1}^{m} P'_k \delta'_k \qquad \textbf{(a)}$$

where δ'_k are the displacements corresponding to the set P'_k. Application of only set 2 results in the strain energy

$$U_2 = \frac{1}{2}\sum_{j=1}^{n} P''_j \delta''_j \qquad \textbf{(b)}$$

in which δ''_j corresponds to the set P''_j.

Suppose that the first force system P'_k is applied, followed by the second force system P''_j. The total strain energy is

$$U = U_1 + U_2 + U_{1,2} \qquad \text{(c)}$$

where $U_{1,2}$ is the strain energy attributable to the work done by the first force system as a result of deformations associated with the application of the second forcesystem. Because the forces comprising the first set are unaffected by the action ofthe second set, we may write

$$U_{1,2} = \sum_{k=1}^{m} P'_k \delta''_k \qquad \text{(d)}$$

Here δ''_k represents the displacements caused by the forces of the second set at the points of application of P'_k, the first set. If now the forces are applied in reverse order, we have

$$U = U_2 + U_1 + U_{2,1} \qquad \text{(e)}$$

where

$$U_{2,1} = \sum_{j=1}^{n} P''_j \delta'_j \qquad \text{(f)}$$

Here δ'_j represents the displacements caused by the forces of set 1 at the points of application of the forces P''_j, set 2.

The loading processes described must, according to the principle of superposition, cause identical stresses within the body. The strain energy must therefore beindependent of the order of loading, and it is concluded from Eqs. (c) and (e) that $U_{1,2} = U_{2,1}$. We thus have

$$\sum_{k=1}^{m} P'_k \delta''_k = \sum_{j=1}^{n} P''_j \delta'_j$$

The above expression is the *reciprocity* or *reciprocal theorem* due to E. Betti and Lord Rayleigh: the work done by one set of forces owing to displacements due to a second set is equal to the work done by the second system of forces owing to displacements due to the first.

1.4. Rayleigh- Ritz Method

The analytical minimization via the calculus of variations, is a very powerful tool for deriving fundamental mathematical laws governing the behaviour of elastic systems, and is also used in many other areas of physics. Of more direct practical importance in engineering arethe various approximate computationalmethods which seek the minimum of the internal energy via computational methods.One of these methods is known as the *Rayleigh-Ritz* method and it is this method whichwe will explore here. This is just one of a whole class of approximate methods whichalso includes the methods used in finite element structures.

The Rayleigh-Ritz method assumes that the solution to the problem can be expressedin terms of some series, often a polynomial or a series of sin and cos functions (a Fourier series). The series is manipulated so as to make it satisfy the boundary conditions. Thecoefficients in this series can be determined so as to make the potential energy *W* forthe system a minimum. This is best illustrated by an example. Let us consider again a beam, and assume that the displacement *w* can bewritten as

$$w(x, C_1, C_2) = C_1 \sin \frac{\pi x}{L} + C_2 \sin \frac{3\pi x}{L} \qquad (1)$$

This expression has the property that $w(x = 0) = w(x = L) = 0$ and that $w''(x = 0) = w''(x = L) = 0$, i.e. it satisfies the boundary conditions. Then we obtain an estimate for Π. We can attempt to minimize this estimate by adjusting the coefficients C_1,C_2 in (1). In fact the minimum valueof Π is given by the values of C_1,….., that make the derivatives

$$\frac{\partial \Pi}{\partial C_1} = 0, \qquad \frac{\partial \Pi}{\partial C_2} = 0 \qquad (2)$$

Equations (2) provide a set of equations that can be solved to find values of C_1 and C_2. The analysis is done, giving values

$$C_1 = \frac{4qL^4}{\pi^5 EI}, \qquad C_2 = \frac{4qL^4}{243\pi^5 EI}$$

Equation (1) with these values for the co-efficient, is then an approximation to the real solution for the problem. If necessary, further terms in the Fourier series can be taken to provide higher accuracy.

1.5. Finite Difference and Finite Element Method

The analytical solutions to elasticity problems are normally accomplished for regions and loadings with relatively simple geometry. Forexample, many solutions can be developed for two-dimensional problems, while only a limited number exist for three dimensions. Solutions are commonly available for problems with simpleshapes such as those having boundaries coinciding with Cartesian, cylindrical, and sphericalcoordinate surfaces. Unfortunately, however, problems with more general boundary shape andloading are commonly intractable or require very extensive mathematical analysis and numerical evaluation. Because most real-world problems involve structures with complicated shapeand loading, a gap exists between what is needed in applications and what can be solved by analytical closed-form methods.

Over the years, this need to determine deformation and stresses in complex problems has lead to the development of many approximate and numerical solution methods. Approximate methods based on energy techniques have limited success in developingsolutions for problems of complex shape. Methods of numerical stress analysis normally recastthe mathematical elasticity boundary value problem into a direct numerical routine. One suchearly scheme is the finite differencemethod (FDM) in which, derivatives of the governing field equations are replaced by algebraic difference equations. This method generates a system of algebraic equations at various computational grid points in the body, and solution to the systemdetermines the unknown variable at each grid point. Although simple in concept, FDM has notbeen able to provide a useful and accurate scheme to handle general problems with geometricand loading complexity. Over the past few decades, two methods have emerged that providenecessary accuracy, general applicability, and ease of use. This has lead to their acceptance by the stress analysis community and has resulted in the development of many private andcommercial computer codes implementing each numerical scheme.

The first of these techniques is known as the finite element method (FEM) and involvesdividing the body under study into a number of pieces or subdomains called elements. Thesolution is then approximated over each element and is quantified in terms of values at speciallocations within the element called the nodes. The discretization process establishes analgebraic system of equations for the unknown nodal values, which approximate the continuous solution. Because element size, shape, and approximating scheme can be varied to suit the problem, the method can accurately simulate solutions to problems of complex geometry andloading. FEM has thus become a primary tool for practical stress analysis and is also usedextensively in many other fields of engineering and science.

The second numerical scheme, called the boundary element method (BEM), is based on anintegral statement of elasticity. This statement may be cast into a form with unknowns only over the boundary of the domain under study. The boundary integralequation is then solved using finite element concepts where the boundary is divided intoelements and the solution is approximated over each element using appropriate interpolationfunctions. This method again produces an algebraic system of equations to solve for unknownnodal values that approximate the solution. Similar to FEM techniques, BEM also allowsvariation in element size, shape, and approximating scheme to suit the application, and thus the method can accurately solve a large variety of problems.

Typical basic steps in a linear, static finite element analysis include the following:

1. Discretize the body into a finite number of element subdomains
2. Develop approximate solution over each element in terms of nodal values
3. Based on system connectivity, assemble elements and apply all continuity and boundary conditions to develop an algebraic system of equations among nodal values
4. Solve assembled system for nodal values; post process solution to determine additional variables of interest if necessary.

1.6. Summary

In this unit we have studied

- Rayleigh's Method
- Rayleigh – Ritz Method
- Finite Difference and Finite Element Method

1.7. Keywords

Rayleigh

Ritz

Finite difference

Finite element

1.8. Exercise

1. Write a short note on Finite Difference and Finite Element Method.
2. Derive an expression for Reciprocal Theorem.
3. Explain Rayleigh- Ritz Method.

Unit 2
Hertz's Contact Stresses

Structure

2.1.Introduction

Application of a load over a small area of contact results in unusually high stresses. Situations of this nature are found on a microscopic scale whenever force is transmitted through bodies in contact. There are important practical cases when thegeometry of the contacting bodies results in large stresses, disregarding the stresses associated with the asperities found on any nominally smooth surface. The Hertzproblem relates to the stresses owing to the contact of a sphere on a plane, a sphereon a sphere, a cylinder on a cylinder,and the like. The practical implications with respect to ball and roller bearings, locomotive wheels, valve tappets, and numerousmachine components are apparent.

Consider, in this regard, the contact without deformation of two bodies havingspherical surfaces of radii r_1 andr_2, in the vicinity of contact. If now a collinear pairof forces P acts to press the bodies together, as in Fig. 1,

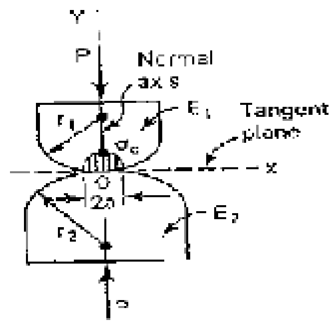


Fig 1.

deformation will occur,and the point of contact *O* will be replaced by a small area of contact. A commontangent plane and common normal axis are denoted *Ox* and *Oy*, respectively. Thefirst steps taken toward the solution of this problem are the determination of thesize and shape of the contact area as well as the distribution of normal pressure acting on the area. The stresses and deformations resulting from the interfacial pressure are then evaluated.

The following assumptions are generally made in the solution of the contactproblem:

1. The contacting bodies are isotropic and elastic.
2. The contact areas are essentially flat and small relative to the radii of curvature of the undeformed bodies in the vicinity of the interface.
3. The contacting bodies are perfectly smooth, and therefore only normal pressures need be taken into account.

The foregoing set of assumptions enables an elastic analysis to be conducted. It is important to note that, in all instances, the contact pressure varies from zero at the side of the contact area to a maximum value σ_c at its center.

2.2.Objectives

After studying this unit we are able to understand

- Pressure between Two-Bodies in contact
- Pressure between two-Spherical Bodies in contact
- Contact Pressure between two parallel cylinders
- Stresses for two Bodies in line contact

2.3.Pressure between Two Bodies in Contact

The general case of compression of elastic bodies in contact may be treated in the same manner as the case of spherical bodies. Consider the tangent plane at thepoint of contact *O* as the *xy*-plane (Fig. 2).

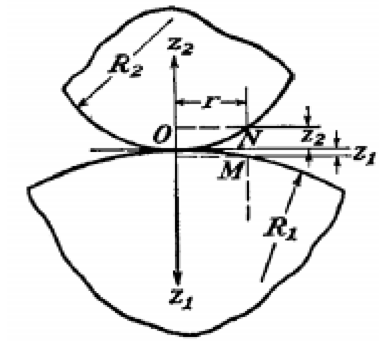


Fig. 2

The surfaces of the bodies near the point of contact, by neglecting small quantities ofhigher order, can be represented by the equations

$$\begin{aligned} z_1 &= A_1x^2 + A_2xy + A_3y^2 \\ z_2 &= B_1x^2 + B_2xy + B_3y^2 \end{aligned} \qquad (a)$$

The distance between two points such as *M* and *N* is then

$$z_1 + z_2 = (A_1 + B_1)x^2 + (A_2 + B_2)xy + (A_3 + B_3)y^2 \qquad (b)$$

We can always take for *x* and *y* such directions as to make the term containing the product *xy* disappear. Then

$$z_1 + z_2 = Ax^2 + By^2 \qquad (c)$$

in which *A* and *B* are constants depending on the magnitudes of the principal curvatures of the surfaces in contact and on the angle n between the planes of principal curvatures of the two surfaces. If R_1 and R_1' denote the principal radii of curvature at the point of contact of one of the bodies, and R_2 and R_2'those of the other, and ψ the angle between the normal planes containing the curvatures $1/R_1$and $1/R_2$, then the constants *A* and *B* are determined from the equations

$$\begin{aligned} A + B &= \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_1'} + \frac{1}{R_2} + \frac{1}{R_2'}\right) \\ B - A &= \frac{1}{2}\left[\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)^2 + \left(\frac{1}{R_2} - \frac{1}{R_2'}\right)^2 \right. \\ &\qquad \left. + 2\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)\left(\frac{1}{R_2} - \frac{1}{R_2'}\right)\cos 2\psi\right]^{\frac{1}{2}} \end{aligned} \qquad (d)$$

It can be shown that A and B in Eq. (c) both have the same sign, and it can therefore be concluded that all points with the same mutual distance $z_1 + z_2$ lie on one ellipse. Hence, if we press the bodies together in the direction of the normal to the tangent plane at O, the surface of contact will have an elliptical boundary.

Then, for points on the surface of contact, we have

$$\begin{aligned} w_1 + w_2 + z_1 + z_2 &= \alpha \\ w_1 + w_2 &= \alpha - Ax^2 - By^2 \end{aligned} \qquad (e)$$

This is obtained from geometrical considerations. Consider now the local deformation at the surface of contact. Assuming that this surface is very small and applying Eq.

$$(u)_{z=0} = -\frac{(1-2\nu)(1+\nu)P}{2\pi E r}, \qquad (w)_{z=0} = \frac{P(1-\nu^2)}{\pi E r}$$

obtained for semi-infinitebodies, the sum of the displacements w_1 and $w_2$for points of the surfaceof contact is

$$w_1 + w_2 = \left(\frac{1-\nu_1^2}{\pi E_1} + \frac{1-\nu_2^2}{\pi E_2}\right) \int\int \frac{q\,dA}{r} \qquad (f)$$

Where qdA is the pressure acting on an infinitely small element of the surface of contact, and r is the distance of this element from the point under consideration. The integration must be extended over theentire surface of contact. Using notations

$$k_1 = \frac{1-\nu_1^2}{\pi E_1}, \qquad k_2 = \frac{1-\nu_2^2}{\pi E_2}$$

we obtain, from (e) and (f),

$$(k_1 + k_2) \int\int \frac{q\,dA}{r} = \alpha - Ax^2 - By^2 \qquad (g)$$

The problem now is to find a distribution of pressures q to satisfy Eq. (g). H. Hertz showed that this requirement is satisfied by assuming that the intensity of pressures q over the surface of contact is represented by the ordinates of a semi-ellipsoid constructed on the surface of contact. The maximum pressure is then clearly at the center of the surface of contact. Denoting it by q_0 and denoting by a and b the semiaxes of the elliptic boundary of the surface of contact the magnitude of the maximum pressure is obtained from the equation

$$P = \iint q\, dA = \tfrac{2}{3}\pi ab q_0$$

from which

$$q_0 = \frac{3}{2}\frac{P}{\pi ab}$$

We see that the maximum pressure is 1.5 times the average pressure on the surface of contact. To calculate this pressure we must know the magnitudes of the semiaxes *a* and *b*. From an analysis analogous to that used for spherical bodies we find that

$$a = m\sqrt[3]{\frac{3\pi}{4}\frac{P(k_1 + k_2)}{(A + B)}}$$
$$b = n\sqrt[3]{\frac{3\pi}{4}\frac{P(k_1 + k_2)}{(A + B)}}$$

in which *A* + *B* is determined from Eqs. (d) and the coefficients *m* and *n* are numbers depending on the ratio (*B* - *A*):(*A* + *B*). Using thenotation

$$\cos\theta = \frac{B - A}{A + B} \qquad (h)$$

the values of *m* and *n* for various values of *θ* are given below.

$\theta =$	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
$m =$	2.731	2.397	2.136	1.926	1.754	1.611	1.486	1.378	1.284	1.202	1.128	1.061	1.000
$n =$	0.493	0.530	0.567	0.604	0.641	0.678	0.717	0.759	0.802	0.846	0.893	0.944	1.000

2.4.Pressure between two-Spherical Bodies in contact

Because of forces *P* (Fig. 1), the contact pressure is distributed over a small *circle*of radius *a* given by

$$a = 0.88\left[\frac{P(E_1 + E_2)r_1r_2}{E_1E_2(r_1 + r_2)}\right]^{1/3} \qquad (1)$$

Where E_1 and E_2 (r_1 and r_2) are the respective moduli of elasticity (radii) of the spheres. The force *P* causing the contact pressure acts in the direction of the normalaxis, perpendicular to the tangent plane passing through the contact area. The*maximum contact pressure*is found to be

$$\sigma_c = 1.5\,\frac{P}{\pi a^2} \tag{2}$$

This is the maximum principal stress owing to the fact that, at the center of the contact area, material is compressed not only in the normal direction but also in the lateral directions. The relationship between the force of contact P, and the relative displacement of the centers of the two elastic spheres, owing to local deformation, is

$$\delta = 0.77\left[P^2\left(\frac{1}{E_1}+\frac{1}{E_2}\right)^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]^{1/3} \tag{3}$$

In the special case of a *sphere* of radius r contacting a body of the same material but having a *flat surface* (Fig. 3a), substitution of $r_1 = r$, $r_2 = \infty$, and$E_1 = E_2 = E$ into Eqs. (1)through (3) leads to

$$a = 0.88\left(\frac{2Pr}{E}\right)^{1/3}, \qquad \sigma_c = 0.62\left(\frac{PE^2}{4r^2}\right)^{1/3}, \qquad \delta = 1.54\left(\frac{P^2}{2E^2 r}\right)^{1/3} \tag{4}$$

For the case of a *sphere* in a *spherical seat* of the same material (Fig. 3b) substituting $r_2 = -\, r_2$and $E_1 = E_2 = E$ in Eqs.(1) through (3), we obtain

$$a = 0.88\left[\frac{2Pr_1r_2}{E(r_2 - r_1)}\right]^{1/3}, \qquad \sigma_c = 0.62\left[PE^2\left(\frac{r_2 - r_1}{2r_1r_2}\right)^2\right]^{1/3}$$
$$\delta = 1.54\left[\frac{P^2(r_2 - r_1)}{2E^2r_1r_2}\right]^{1/3} \tag{5}$$

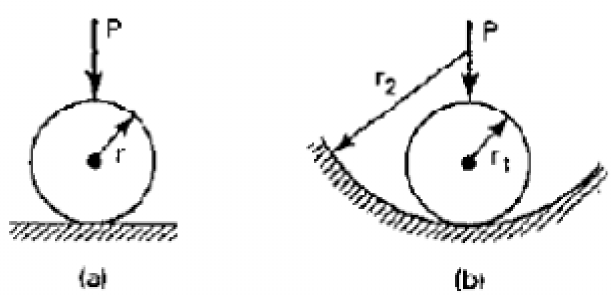


Fig. 3

2.5.Contact Pressure between two parallel cylinders

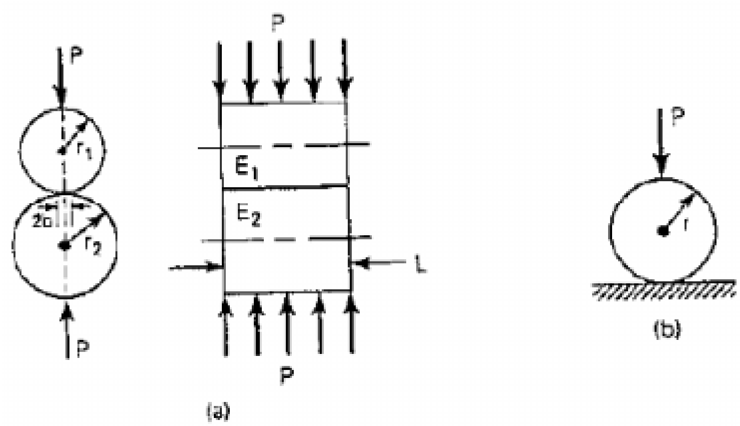


Fig. 4

Here the contact area is a *narrow rectangle* of width *2b* and length *L* (Fig.4a). The *maximum contact pressure* is given by

$$\sigma_c = \frac{2}{\pi}\frac{P}{bL} \qquad (6)$$

where

$$b = \left[\frac{4Pr_1r_2}{\pi L(r_1 + r_2)}\left(\frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}\right)\right]^{1/2} \qquad (7)$$

In this expression $E_i(\nu_i)$ and r_i, with i = 1, 2, are the moduli of elasticity (Poisson's ratio) of the two rollers and the corresponding radii, respectively. If the cylindershave the same elastic modulus E and Poisson's ratio $\nu = 0.3$, these expressions reduce to

$$\sigma_c = 0.418\sqrt{\frac{PE}{L}\frac{r_1 + r_2}{r_1r_2}}, \qquad b = 1.52\sqrt{\frac{P}{EL}\frac{r_1r_2}{r_1 + r_2}} \qquad (8)$$

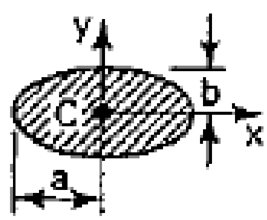
Figure 4b shows the special case of contact between a circular*cylinder* at radius *r* and a *flat surface*, both bodies of the same material. After rearranging the termsand taking $r_1 = r$, $r_2 = \infty$in Eqs. (8), we have

$$\sigma_c = 0.418\sqrt{\frac{PE}{Lr}}, \qquad b = 1.52\sqrt{\frac{Pr}{EL}} \tag{9}$$

2.6.Stresses for two Bodies in line contact

Consider now two rigid bodies of equal elastic moduli *E*, compressed by force *P*(Fig. 5). The load lies along the axis passing through the centers of the bodiesand through the point of contact and is perpendicular to the plane tangent to bothbodies at the point of contact. The minimum and maximum radii of curvature of thesurface of the upper body are r_1 and r'_1; those of the lower body are r_2 and r'_2 at thepoint of contact. Thus, $1/r_1$,$1/r'_1$, $1/r_2$,and $1/r'_2$ are the principal curvatures. The *sign convention* of the *curvature* is such that it is *positive* if the corresponding center ofcurvature is inside the body. If the center of the curvature is *outside* the body, thecurvature is *negative*. (For example, in Fig. 6a, r_1 and $r'_1$are positive, while $r_2$andr'_2 arenegative.)

Let θ be the angle between the normal planes in which radii r_1 and r_2 lie. Subsequent to loading, the area of contact will be an *ellipse* with semiaxes *a* and *b*



The *maximum contact pressure* is

$$\sigma_c = 1.5\frac{P}{\pi ab} \tag{10}$$

In this expression the semiaxes are given by

$$a = c_a\sqrt[3]{\frac{Pm}{n}}, \qquad b = c_b\sqrt[3]{\frac{Pm}{n}} \tag{11}$$

Here

$$m = \frac{4}{\frac{1}{r_1} + \frac{1}{r'_1} + \frac{1}{r_2} + \frac{1}{r'_2}}, \qquad n = \frac{4E}{3(1 - \nu^2)} \tag{12}$$

The constants c_a and c_b are read in Table 1. The first column of the table lists values of α, calculated from

$$\cos\alpha = \frac{B}{A} \tag{13}$$

where

α (degrees)	c_a	c_b
20	3.778	0.408
30	2.731	0.493
35	2.397	0.530
40	2.136	0.567
45	1.926	0.604
50	1.754	0.641
55	1.611	0.678
60	1.486	0.717
65	1.378	0.759
70	1.284	0.802
75	1.202	0.846
80	1.128	0.893
85	1.061	0.944
90	1.000	1.000

Table 1

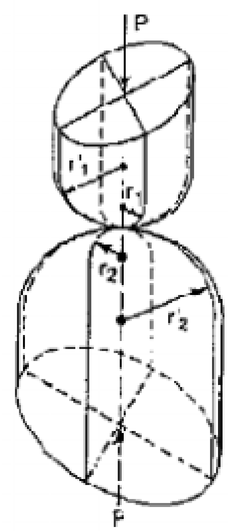
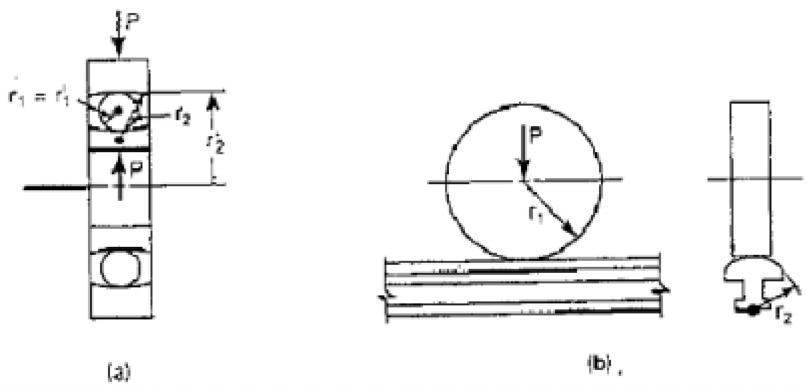


Fig. 5



Contact load: (a) in a single row ball bearing; (b) in a cylindrical wheel and rail.

Fig. 6

2.7.Summary

In this unit we have studied

- Pressure between Two-Bodies in contact
- Pressure between two-Spherical Bodies in contact
- Contact Pressure between two parallel cylinders
- Stresses for two Bodies in line contact

2.8.Keywords

Hertz contact

Parallel cylinders

2.9.Exercise

1. Write a short note on Hertz Contact Stresses.
2. Derive an expression for max. pressure between two bodies in contact.
3. Explain stresses for two bodies in line contact.

Unit 3

STRESS CONCENTRATION PROBLEMS

Structure

3.1. Introduction

It is very important for the engineer to be aware of the effects of stress raisers such as notches, holes or sharp corners in his/her design work. Stress concentration effects in machine parts and structures can arise from internal holes or voids created in the casting or forging process, from excessively sharp corners or fillets at the shoulders of stepped shafts, or even from punch or stamp marks left during layout work or during inspection of parts.

3.2. Objectives

After studying this unit we are able to understand

- Stress-Concentration Factor
- Fatigue Stress-Concentration Factors

3.3. Stress Concentration Factors

For situations in which the cross section of aload-carrying member varies gradually, reasonably accurate results can be expectedif we apply equations derived on the basis of constant section. On the other hand, where abrupt changes in the cross section exist, the mechanics of materials approach cannot predict the high values of stress that actually exist. The condition referred to occurs in such frequently encountered configurations as holes, notches,and fillets. While the stresses in these regions can in some cases be analyzed by applying the theory of elasticity, it is more usual to rely on experimental techniques and, in particular, photoelastic methods. The finite clementmethod is very efficient for this purpose.

It is to be noted that irregularities in stress distribution associated with abruptchanges in cross section are of practical importance in the design of machine elements subject to variable external forces and stress reversal. Under the action of stress reversal, progressive cracks are likely to start at certain points at which the stress is far above the average value.The majority of fractures in machineelements in service can be attributed to such progressive cracks.

It is usual to specify the high local stresses owing to geometrical irregularitiesin terms of a *stress concentration factor*, *k*. That is,

$$k = \frac{\text{maximum stress}}{\text{nominal stress}}$$

Clearly, the nominal stress is the stress that would exist in the section in question inthe absence of the geometric feature causing the stress concentration.

3.4. Fatigue Stress Concentration Factor

Recall that a stress concentration factor need not be used with ductile materials when they are subjected to only static loads, because (local) yielding will relieve the stress concentration. However under fatigue loading, the response of material may not be adequate to nullify the effect and hence has to be accounted. The factor k_f commonly called a fatigue stress concentration factor is used for this. Normally, this factor is used to indicate the increase in the stress; hence this factor is defined in the following manner. Fatigue stress concentration factor can be defined as

$$k_f = \frac{\text{fatigue strength (limit) of un-notched specimen}}{\text{fatigue strength (limit) of notched free specimen}}$$

3.5. Summary

In this unit we have studied

- Stress-Concentration Factor
- Fatigue Stress-Concentration Factors

3.6. Keywords

Stress

Fatigue stress

3.7. Exercise

1. Write short note on Stress Concentration Factor
2. Write short note on Fatigue Stress Concentration Factor

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Assignment 1

The due date for submitting this assignment has passed. **Due on 2018-08-15, 23:59 IST.**

Submitted assignment (Submitted on 2018-08-14, 10:39)

Which of the following statements is appropriate for perfectly elastic material? ***1 point***

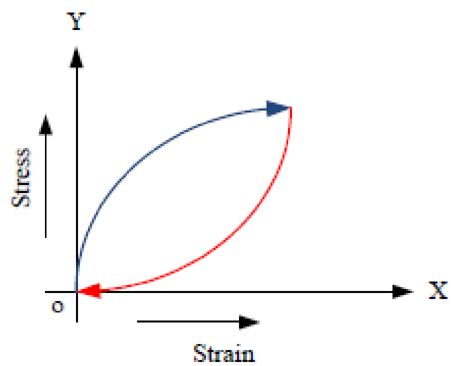
- (a) The state of stress is independent of previous history of stresses
- (b) Stress is an unique function of strain
- (c) Both (a) and (b)
- (d) None of the above

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) Both (a) and (b)

1 point

The following diagram shows a stress strain diagram of any material. Which kind of material is it?



- (a) Plastic
- (b) Linear Elastic
- (c) Non-linear Elastic
- (d) Visco-elastic

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d) Visco-elastic

1 point

Consider the following matrix [A]. What is the value of A_{kk}?

$$A_{ij} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 3 & 1 & 6 \end{pmatrix}$$

- (a) 11
- (b) 1
- (c) -7
- (d) 21

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) 11

1 point

Consider the following matrix [A] and vector b. What is the value of $A_{ji}b_i$?

$$A_{ij} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 3 & 1 & 6 \end{pmatrix} b_i = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$$

- (a) $[22 \quad 10 \quad 43]^T$
- (b) $[14 \quad 14 \quad 44]^T$
- (c) $[6 \quad 7 \quad 60]^T$
- (d) $[12 \quad 13 \quad 8]^T$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) $[22 \quad 10 \quad 43]^T$

1 point

Consider the following matrix [A]. What are the eigenvalues of [A]?

$$A_{ij} = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 0 & 4 \\ 2 & 4 & 3 \end{pmatrix}$$

- (a) -2.786, 7.637, 3.149
- (b) 2.785, 7.637, 3.149
- (c) -2.785, -7.637, 3.149
- (d) 2.785, -7.637,-3.149

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) -2.786, 7.637, 3.149

Choose the correct indicial notation of the cross product of two vectors u and v ? ***1 point***

- (a) $\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u_j v_i e_k$
- (b) $\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u_i v_j e_k$
- (c) $\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u_k v_j e_i$
- (d) $\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u_j v_k e_i$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d) $\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u_j v_k e_i$

1 point

Let us consider a vector field $\mathbf{u} = -6x^2\mathbf{e}_1 + 3xy\mathbf{e}_2 - 5xyz\mathbf{e}_3$. Calculate $\nabla \times \mathbf{u}$?

(a) $-6xe_1 + 3ye_2 - 5xze_3$

(b) $-5xze_1 + 5yze_2 + 3ye_3$

(c) $5yze_1 - 5xze_2 + 3ze_3$

(d) $+6xe_1 - 3ye_2 - 5ze_3$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) $-5xze_1 + 5yze_2 + 3ye_3$

1 point

Let us consider a vector field $\mathbf{u} = -6x^3\mathbf{e}_1 + 3xy^2\mathbf{e}_2 - 5xyz\mathbf{e}_3$. Calculate $\nabla . \mathbf{u}$?

(a) -12

(b) $-18x^2 + xy$

(c) -36x + 6y

(d) $-18x^2 + 6xy$

No, the answer is incorrect.
Score: 0

Accepted Answers:
(b) $-18x^2 + xy$

1 point

Let us consider a scalar field $\phi = x^3 - xy^2z$. Calculate $\nabla^2 \phi$?

(a) 0

(b) 6x - 2xz

(c) $3x^2 - 2xyz$

(d) 4x

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) 6x - 2xz

Choose the correct option among the following statements regarding Divergence and Stokes theorem. *1 point*

I. Divergence theorem relates volume integral to surface integral.
II. Stokes theorem relates contour integral to volume integral.

(a) Only I is correct but II is incorrect

(b) Both are wrong

(c) Only II is correct but I is incorrect

(d) Both are correct

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) Only I is correct but II is incorrect

Week 2 : Assignment 2

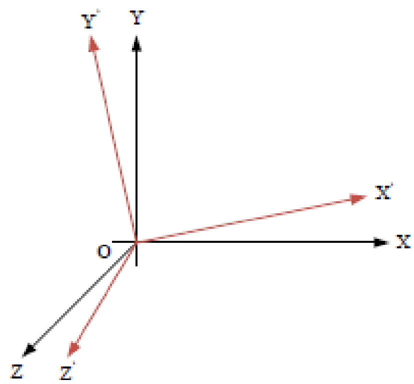
The due date for submitting this assignment has passed.

Due on 2018-08-15, 23:59 IST.

Submitted assignment (Submitted on 2018-08-14, 11:22)

1 point

Consider the following figure. If the stress tensor in XYZ coordinate system is σ what is value of stress tensor σ' in $X'Y'Z'$ coordinate system. $\mathbf{Q}$ is the orthogonal rotation matrix between $X'Y'Z'$ and XYZ?



- (a) $\sigma' = \mathbf{Q}\sigma$
- (b) $\sigma' = \mathbf{Q}\sigma\mathbf{Q}^T$
- (c) $\sigma' = \mathbf{Q}^T\sigma$
- (d) $\sigma' = \mathbf{Q}^{-1}\sigma\mathbf{Q}^T$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) $\sigma' = \mathbf{Q}\sigma\mathbf{Q}^T$

Which of the following quantity is a 2nd order tensor? ***1 point***

- (a) Strain
- (b) Constitutive matrix
- (c) Velocity
- (d) Potential energy

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) Strain

What does the notation σ_{xz} mean? ***1 point***

- (a) Stress acting normally to the y plane
- (b) Stress acting tangentially to the y plane
- (c) Stress acting on the z plane and in the x direction
- (d) Stress acting on the x plane and in the z direction

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d) Stress acting on the x plane and in the z direction

Consider the following state of stress σ at any point. Calculate the principle stresses. ***1 point***

$$\sigma_{ij} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{pmatrix}$$

- (a) 5, -2, -5
- (b) 5, 2, -5
- (c) 3, 2, -7
- (d) 3, -2, -7

No, the answer is incorrect.
Score: 0

Accepted Answers:
(b) 5, 2, -5

1 point

Consider the following state of stress σ at any point. Calculate the stress invarients I_1, I_2, I_3?

$$\sigma_{ij} = \begin{pmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$$

- (a) 0, -14, -7
- (b) 0, -1, -14
- (c) 13, -1, -20
- (d) 0, -33, 16

No, the answer is incorrect.
Score: 0

Accepted Answers:
(d) 0, -33, 16

1 point

What is tensorial representation of strain at a point with displacement field $u = [u_1, u_2, u_3]^T$?

- (a) $\epsilon_{ij} = [\frac{\partial u_i}{\partial x_j}]$
- (b) $\epsilon_{ij} = \frac{1}{2}[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}]$
- (c) $\epsilon_{ij} = \frac{1}{2}[\frac{\partial u_i}{\partial x_j}]$
- (d) $\epsilon_{ij} = [\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}]$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) $\epsilon_{ij} = \frac{1}{2}[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}]$

How many elements are required for the constitutive matrix in case of a general 3D infinite stress block ? ***1 point***

- (a) 2
- (b) 36
- (c) 81
- (d) 9

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) 81

Consider the following state of stress σ at any point. Calculate the deviatoric stress. ***1 point***

$$\sigma_{ij} = \begin{pmatrix} 6 & 5 & 7 \\ 5 & 3 & 4 \\ 7 & 4 & -3 \end{pmatrix}$$

(a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 3 & 5 \\ 3 & 3 & 2 \\ 5 & 2 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 5 & 7 \\ 5 & 1 & 4 \\ 7 & 4 & -5 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & 3 & 5 \\ 3 & 1 & 2 \\ 5 & 2 & -5 \end{pmatrix}$

No, the answer is incorrect.
Score: 0

Accepted Answers:

(c) $\begin{pmatrix} 4 & 5 & 7 \\ 5 & 1 & 4 \\ 7 & 4 & -5 \end{pmatrix}$

1 point

Let us consider a infinitesimal stress block with stress σ. Find out the traction vector on a plane whose normal is defined by $n = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0]^T$?

$$\sigma_{ij} = \begin{pmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$$

(a) $[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 3]^T$

(b) $[-\frac{4}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]^T$

(c) $[-\frac{6}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]^T$

(d) $[-\frac{4}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]^T$

No, the answer is incorrect.
Score: 0

Accepted Answers:

(c) $[-\frac{6}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]^T$

1 point

Let us consider a infinitesimal stress block with stress σ. Find out the magnitude of the normal stress on a plane whose normal is defined by $n = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0]^T$?

$$\sigma_{ij} = \begin{pmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$$

- (a) 23
- (b) -5/2
- (c) $\frac{6}{\sqrt{2}}$
- (d) $\frac{3}{\sqrt{2}}$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) -5/2

Week 3 : Assignment 3

The due date for submitting this assignment has passed.

Due on 2018-09-05, 23:59 IST.

Submitted assignment (Submitted on 2018-08-22, 06:26)

What is the order of constitutive tensor C_{ijkl} ? ***1 point***

(a) 1^{st} order

(b) 2^{nd} order

(c) 3^{rd} order

(d) 4^{th} order

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d) 4^{th} order

Number of independent element in constitutive tensor for an isotropic material is ***1 point***

(a) 2
(b) 9
(c) 21
(d) 81

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) 2

Number of independent element in constitutive tensor for an anisotropic material is ***1 point***

(a) 2
(b) 9
(c) 21
(d) 81

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) 21

Number of independent element in constitutive tensor for an orthotropic material is ***1 point***

(a) 2
(b) 9
(c) 21
(d) 81

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) 9

1 point

Find out the Lame's constant (λ & μ) for an isotropic material having modulus of elasticity (E) and Poisson's ratio (ν) as 200 GPa and 0.2, respectively .

- (a) 80 GPa, 80 GPa
- (b) 35.71 GPa, 166.6 GPa
- (c) 55.55 GPa, 83.33 GPa
- (d) 73.33 GPa, 66.66 GPa

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) 55.55 GPa, 83.33 GPa

Find out the bulk modulus (K) for an isotropic material having modulus of elasticity (E) and Poisson's ratio (ν) as 210 GPa and 0.3, respectively . ***1 point***

- (a) 80 GPa
- (b) 116.67 GPa
- (c) 65.25 GPa
- (d) 175 GPa

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d) 175 GPa

1 point

Consider the state of stress at any point as σ_{xx}= 250 MPa, σ_{yy}= -350 MPa, σ_{zz}= 0. The Young's modulus and Poison's ratio of the material is considered as 2 GPa and 0.18, respectively. Determine the ϵ_{zz} at the point.

- (a)5.4×10^{-3}
- (b) 0
- (c)9×10^{-3}
- (d)-9×10^{-3}

No, the answer is incorrect.
Score: 0

Accepted Answers:
(c)9×10^{-3}

1 point

Consider the state of strain at any point as ϵ_{xx} = $0.5\text{x}10^{-3}$, ϵ_{yy} = $-0.4\text{x}10^{-3}$, ϵ_{zz} = $0.7\text{x}10^{-3}$. The Young's modulus and Poison's ratio of the material is considered as 2 GPa and 0.18, respectively. Determine the $\sigma_{\text{hydrostatic}}$ at the point.

- (a) $1.6\text{x}10^{6}$ Pa
- (b) $1.6\text{x}10^{6}$ Pa
- (c) 0 Pa
- (d) $0.833\text{x}10^{6}$ Pa

No, the answer is incorrect.
Score: 0

Accepted Answers:
(d) $0.833\text{x}10^{6}$ Pa

1 point

Let us consider three strain rossete in xy plane as a, b and c. The measured strains in these rossete are $e_a = 0.5\text{x}10^{-3}$, $e_b = 0.4\text{x}10^{-3}$, $e_c = 0.3\text{x}10^{-3}$, respectively. The angles of the rossete with respect to the positive x axis as $\theta_a = 45°, \theta_b = 90°$, and $\theta_c = 135°$ respectively. If $\lambda = 140.6$ GPa and $\mu = 75.0$ GPa calculate σ_{xy}.

- (a) 562.4 MPa
- (b) -562.4 MPa
- (c) 15.0 MPa
- (d) 7.5 MPa

No, the answer is incorrect.
Score: 0

Accepted Answers:
(c) 15.0 MPa

1 point

Let us consider the following displacement field. Calculate the σ_{xx} at point (5, 0, 1). λ and μ are the Lame's constant.

$u = \frac{M(1-\mu^2)}{EI}xyz, v = \frac{M(1-\mu^2)}{EI}\left(x^2 - \frac{yz}{3}\right), w = \frac{M(1-\mu^2)}{EI}\left(x^2 - z^2\right)$

- (a) $-\frac{7}{3}\lambda\frac{M(1-\mu^2)}{EI}$
- (b) $-\frac{M(1-\mu^2)}{3EI}(7\lambda - 4\nu)$
- (c) $\frac{M(1-\mu^2)}{3EI}(7\lambda + 2\nu)$
- (d) $-\frac{M(1-\mu^2)}{EI}(7\lambda - 2\nu)$

No, the answer is incorrect.
Score: 0

Accepted Answers:
(a) $-\frac{7}{3}\lambda\frac{M(1-\mu^2)}{EI}$

Week 4 : Assignment 4

The due date for submitting this assignment has passed.

Due on 2018-09-05, 23:59 IST.

Submitted assignment (Submitted on 2018-08-22, 09:01)

Number of independent elements in the constitutive matrix of a monoclinic material is?

1 point

- (a) 13
- (b) 21
- (c) 9
- (d) 36

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) 13

Number of independent elements in the constitutive matrix of a triclinic material is

1 point

- (a) 2
- (b) 21
- (c) 13
- (d) 81

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) 21

Number of independent elements in constitutive matrix of an transverse isotropic material is

1 point

- (a) 21
- (b) 9
- (c) 5
- (d) 13

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) 5

The correct relationship for an orthotropic material is

1 point

- (a) $\frac{\nu_{ij}}{\nu_{ji}} = \frac{E_j}{E_i}$
- (b) $\frac{\nu_{ij}}{\nu_{ji}} = \frac{E_i}{E_j}$
- (c) $\nu_{ij} = \nu_{ji}$
- (d) $E_j = E_i$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) $\frac{\nu_{ij}}{\nu_{ji}} = \frac{E_i}{E_j}$

1 point

If the constitutive matrix is given as $[C]$ in XY coordinate system, what will be the transformation matrix $[C_1]$ in X_1Y_1 coordinate system? $[T_\epsilon]$ is the strain transformation matrix between the coordinate systems.

- (a) $[C_1] = [T_\epsilon][C][T_\epsilon]^T$
- (b) $[C_1] = [T_\epsilon][C]$
- (c) $[C_1] = [C][T_\epsilon]^T$
- (d) $[C_1] = [C]$

Yes, the answer is correct.
Score: 1
Accepted Answers:
(a) $[C_1] = [T_\epsilon][C][T_\epsilon]^T$

1 point

Consider the following comments on the matrix [A]. Which of the following options is correct?

$$[A] = \begin{pmatrix} 5 & 1 & 2 \\ 1 & -3 & 3 \\ 2 & 3 & 7 \end{pmatrix}$$

1. [A] is positive definite matrix
2. All the eigenvalues are not positive

- (a) Only statement 1 is correct
- (b) Only statement 2 is correct
- (c) Only statement 1 is correct but statement 2 is wrong
- (d) Only statement 2 is correct but statement 1 is wrong

No, the answer is incorrect.
Score: 0
Accepted Answers:
(d) Only statement 2 is correct but statement 1 is wrong

In case of any orthotropic material which of the following relations are correct? ***1 point***

- (a) $|\nu_{ij}| < \sqrt{\frac{E_i}{E_j}}$
- (b) $|\nu_{ij}| < \frac{E_i}{E_j}$
- (c) $|\nu_{ij}| > \sqrt{\frac{E_i}{E_j}}$
- (d) $|\nu_{ij}| > \frac{E_i}{E_j}$

Yes, the answer is correct.
Score: 1
Accepted Answers:
(a) $|\nu_{ij}| < \sqrt{\frac{E_i}{E_j}}$

1 point

If ν_{12},ν_{13},and ν_{23} are the Poison's ratios of any orthotropic material, which of the following relations holds true?

- (a) $\nu_{12} + \nu_{13} + \nu_{23} < 0.5$
- (b) $\nu_{12} < 0.5, \nu_{13} < 0.5, \nu_{23} < 0.5$
- (c) $\nu_{12}^2 + \nu_{13}^2 + \nu_{23}^2 < 0.5$
- (d) $\nu_{12}\nu_{13}\nu_{23} < 0.5$

Yes, the answer is correct.
Score: 1
Accepted Answers:
(d) $\nu_{12}\nu_{13}\nu_{23} < 0.5$

1 point

For a isotropic material if the Poison ration is ν. Which is the proper range of values for ν?

- (a) $0 < \nu < 1$
- (b) $-1 < \nu < 0.5$
- (c) $0 < \nu < 0.5$
- (d) $-0.5 < \nu < 0.5$

Yes, the answer is correct.
Score: 1
Accepted Answers:
(b) $-1 < \nu < 0.5$

1 point

Let us consider two coordinate systems XY and X_1Y_1. The stress and strain tensors in these coordinate systems are σ, ϵ and σ_1, ϵ_1 respectively. If the transformation matrix for stress and strain are respectively T_σ and T_ϵ, what is the realtion between these two transformation matrices.

- (a) $T_\sigma = T_\epsilon$
- (b) $T_\sigma = T_\epsilon^{-1}$
- (c) $T_\sigma^T = T_\epsilon^{-1}$
- (d) $T_\sigma^T = T_\epsilon$

Yes, the answer is correct.
Score: 1
Accepted Answers:
(c) $T_\sigma^T = T_\epsilon^{-1}$

Assignment 5

The due date for submitting this assignment has passed.

Due on 2018-09-12, 23:59 IST.

Assignment submitted on 2018-09-12, 15:03 IST

Number of independent strain compatibility equations for 3D systems? ***1 point***

- (a) 81
- (b) 9
- (c) 6
- (d) 3

No, the answer is incorrect.
Score: 0

Accepted Answers:
(d) 3

Saint-Venant compatibility equations are written in terms of ***1 point***

- (a) Stress
- (b) Strain
- (c) Displacement
- (d) none of the above

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) Strain

Beltrami-Michell compatibility equations are written in terms of ***1 point***

- (a) Stress
- (b) Strain
- (c) Displacement
- (d) none of the above

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) Stress

Choose the correct option regarding a continuum system form the following ***1 point***
1. The matter is continuously distributed over the body.
2. The field variable can be continuously defined over the body.

- (a) Both the statements are to be true in case of a continuum body
- (b) Only the statement 1 is to be true in case of a continuum body
- (c) None of the statements are to be true in case of a continuum body
- (d) Only the statement 2 is to be true in case of a continuum body

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) Both the statements are to be true in case of a continuum body

Number of independent equations in stress formulation of a 3D elasticity problem is ***1 point***

- (a) 15
- (b) 9
- (c) 6
- (d) 3

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) 6

Number of independent equations in displacement formulation of a 3D elasticity problem is ***1 point***

- (a) 15
- (b) 9
- (c) 6
- (d) 3

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d) 3

Number of independent equilibrium equations in a 3D elasticity problem is ***1 point***

- (a) 9
- (b) 6
- (c) 3
- (d) 2

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) 3

Choose the correct option form the following ***1 point***
1. strain compatibility condition is a necessary and sufficient criteria to get a single valued
displacement field for multiply connected domains.
2. strain compatibility condition is a necessary and sufficient criteria to get a single valued
displacement field for simply connected domains.

- (a) Only statement 1 is correct
- (b) Only statement 2 is correct
- (c) Both the statements are wrong
- (d) Both the statements are correct

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) Only statement 2 is correct

Let us consider a 2D system where we find the stress distribution is independent of the material properties. What condition we can arrive in from the fact ***1 point***

- (a) The system is subjected to no body force
- (b) The system is subjected to constant body force
- (c) We can not arrive in any conclusion from the observation
- (d) The system is subjected to either constant or zero body force.

No, the answer is incorrect.
Score: 0

Accepted Answers:
(d) The system is subjected to either constant or zero body force.

1 point

What kind of boundary condition is to be applied at the fixed edge of the cantilever beam shown in the figure?

- (a) Traction boundary condition
- (b) Displacement boundary condition
- (c) Mixed boundary condition
- (d) Initial conditions

No, the answer is incorrect.
Score: 0

Accepted Answers:
(b) Displacement boundary condition

Assignment 6

The due date for submitting this assignment has passed.

Due on 2018-09-12, 23:59 IST.

Assignment submitted on 2018-09-12, 14:44 IST

What is the number of non zero strain components for a plane stress problem? ***1 point***

- (a) 6
- (b) 4
- (c) 3
- (d) 2

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) 4

What is the number of non zero stress components for a plane stress problem? ***1 point***

- (a) 4
- (b) 6
- (c) 2
- (d) 3

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d) 3

What is the number of non zero strain components for a plane strain problem? ***1 point***

- (a) 3
- (b) 4
- (c) 2
- (d) 6

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) 3

What is the number of non zero stress components for a plane strain problem? ***1 point***

- (a) 2
- (b) 3
- (c) 4
- (d) 6

No, the answer is incorrect.
Score: 0

Accepted Answers:
(c) 4

Choose the correct option regarding a 2D continuum system from the following ***1 point***
1. Airy's stress function automatically satisfies the equilibrium equations.
2. In absence of body forces, Airy's stress function converts Beltrami-Michell equation to a Bi-harmonic equation.

- (a) Both the statements are true
- (b) Only the statement 1 is true

(c) None of the statements are true

(d) Only the statement 2 is true

No, the answer is incorrect.
Score: 0

Accepted Answers:
(a) Both the statements are true

1 point

Let us consider a plane stress problem without any body forces. The Airy's stress function (ϕ) is defined as; $\phi = 6x^2y^3$. Determine $\sigma_{xx}, \sigma_{yy}, and\ \sigma_{xy}$

(a) $\sigma_{xx} = 36x^2y, \sigma_{yy} = 12y^3, and\ \sigma_{xy} = -36xy^2$

(b) $\sigma_{xx} = 12y^3, \sigma_{yy} = 36x^2y, and\ \sigma_{xy} = -36xy^2$

(c) $\sigma_{xx} = -36x^2y, \sigma_{yy} = -12y^3, and\ \sigma_{xy} = 36xy^2$

(d) $\sigma_{xx} = 36xy, \sigma_{yy} = 12y^2, and\ \sigma_{xy} = 36xy^2$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) $\sigma_{xx} = 36x^2y, \sigma_{yy} = 12y^3, and\ \sigma_{xy} = -36xy^2$

1 point

In a plane stress problem $\sigma_{xx} = 5MPa$, $\sigma_{yy} = -10MPa$, $\sigma_{xy} = 7.5MPa$. Calculate ϵ_{zz} if the Young's modulus is 2 GPa and Poison ratio is 0.15.

(a) $-3.75x10^{-4}$

(b) 0

(c) $7.5x10^{-4}$

(d) $3.75x10^{-4}$

No, the answer is incorrect.
Score: 0

Accepted Answers:
(d) $3.75x10^{-4}$

1 point

In a plane strain problem $\epsilon_{xx} = 0.005$, $\epsilon_{yy} = -0.001$, $\epsilon_{xy} = 0.006$. Calculate σ_{xz} if the Young's modulus is 2 GPa and Poison ratio is 0.25.

(a) 7.2 MPa

(b) 0 MPa

(c) 4.8 MPa

(d) 2.4 MPa

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) 0 MPa

Choose the correct statement regarding generalised plane stress problem ***1 point***

1. The out of plane displacement is zero
2. The average out of plane displacement is zero

(a) Only statement 1 is correct
(b) Only statement 2 is correct
(c) Both of them are correct
(d) None of them are correct

No, the answer is incorrect.
Score: 0

Accepted Answers:
(b) Only statement 2 is correct

1 point

Let us consider a thin cylinder with wall thickness t and average radius r_0. The cylinder is acted upon by a uniform pressure of P. What is the hoop stress (σ_θ) generated?

(a) $\frac{Pr_0}{t}$
(b) $\frac{Pr_0}{2t}$
(c) $\frac{Pr_0}{3t}$
(d) $\frac{Pr_0}{4t}$

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a) $\frac{Pr_0}{t}$

Assignment 7

The due date for submitting this assignment has passed.

Due on 2018-09-19, 23:59 IST.

Assignment submitted on 2018-09-19, 15:45 IST

1 point

Consider the following second order stress function $\phi = \frac{m}{2}x^2 - nxy + \frac{p}{2}y^2$. For which of the below combinations of the values of m, n and p the problem represent a pure shear condition?

(a) $m = 0, n = 0, p \neq 0$

(b) $n \neq 0$

(c) $m = 0, n \neq 0, p = 0$

(d) $m = 0, n = 0$

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c)

1 point

Which of the following is a form of compatibility equation for plane stress problem ?

(a) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\sigma_{xx} + \sigma_{yy}) = -(1 - \nu)(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y})$

(b) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\sigma_{xx} + \sigma_{yy}) = (1 + \nu)(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y})$

(c) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\sigma_{xx} + \sigma_{yy}) = -(1 + \nu)(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y})$

(d) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\sigma_{xx} + \sigma_{yy}) = \frac{1}{1-\nu}(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y})$

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c)

1 point

Let us consider a 2D continuum body subjected to body forces due to self weight $b_x = 0, b_y = \rho g$. If the stress function is considered as ϕ what will be the expression of σ_{xx}?

(a) $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$

(b) $\sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2} - \rho g y$

(c) $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} - \rho g x$

(d) $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} - \rho g y$

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d)

1 point

Let us consider a 2D continuum body subjected to body forces due to self weight $b_x = 0, b_y = \rho g$. If the stress function is considered as ϕ what will be the expression of σ_{xy}

(a) $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

(b) $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x^2}$

(c) $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial y^2}$

(d) $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} - \rho g y$

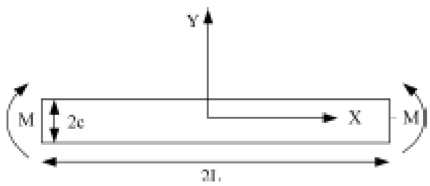
- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

Consider a straight beam is subjected to end moments as shown in the figure. If the stress function is considered as $\phi = A_0 y^3$, determine the value of the constant A_0.



(a) $A_0 = -\frac{M}{4c}$

(b) $A_0 = -\frac{M}{4c^3}$

(c) $A_0 = -\frac{M}{2c}$

(d) $A_0 = -\frac{M}{4c^2}$

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

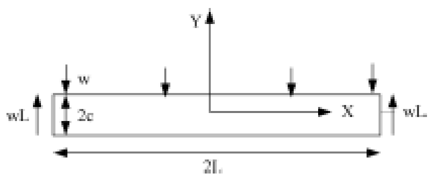
Accepted Answers:
(b)

1 point

Consider a straight beam is subjected to uniform transverse loading as shown in figure. Consider the following boundary conditions of the problem?

1. $\sigma_{yy}(x, \pm c) = 0$

2. $\int_{-c}^{+c} \tau_{xy}(\pm l, y)dy = \mp wl$



(a) Only 1st condition is true

(b) Only 2nd condition is true

(c) Both the conditions are true

(d) None of the conditions are true

- (a)
- (b)
- (c)
- (d)

No, the answer is incorrect.
Score: 0

Accepted Answers:
(b)

1 point

If ϕ is stress function of any 2D continuum problem in polar coordinate, which of the following is the correct expression of the biharmonic equation in polar coordinate system?

(a) $\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}\right)\phi = 0$

(b) $\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}\right)^2\phi = 0$

(c) $\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} + \frac{\partial^2}{r\partial\theta^2}\right)^2\phi = 0$

(d) $\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial\theta^2}\right)\phi = 0$

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b)

1 point

If ϕ is stress function of any 2D continuum problem in polar coordinate. Determine σ_{rr}.

(a) $\sigma_{rr} = \left(\frac{\partial^2}{\partial r^2}\right)\phi$

(b) $\sigma_{rr} = \left(\frac{\partial^2}{r^2\partial\theta^2}\right)\phi$

(c) $\sigma_{rr} = \frac{\partial}{\partial r}\left(\frac{\partial}{r\partial\theta}\right)\phi$

(d) $\sigma_{rr} = \left(\frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}\right)\phi$

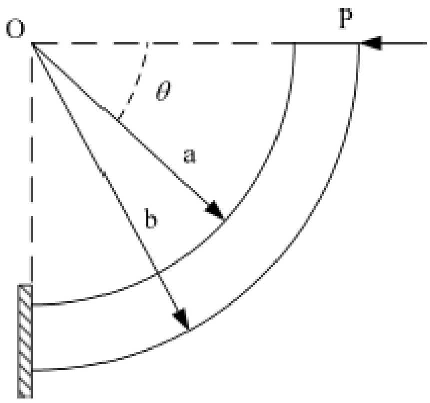
- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d)

1 point

Let us consider a 2D curved beam subjected to a point load at the tip of it as shown in the figure. Choose the proper boundary conditions of the problem



1. $\sigma_{rr}|_{r=a,r=b} = 0$
2. $\sigma_{r\theta}|_{r=a,r=b} = 0$
3. $\int_a^b \sigma_{r\theta} dr = P$

(a) Only statement 1 and 2 are correct

(b) Only statement 1 and 3 are correct

(c) All of them are correct

(d) None of them are correct

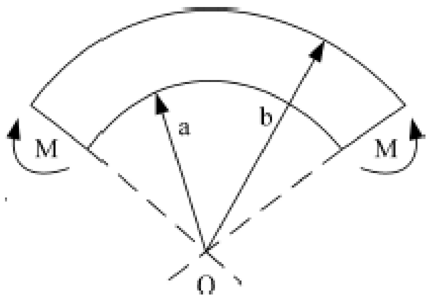
- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

Let us consider a 2D curved beam subjected to bending as shown in the figure. Which of the following condition should hold true to ensure that the concave and concave edges are free from the normal force?



(a) $\sigma_{rr}|_{r=a,r=b} = 0$

(b) $\sigma_{r\theta}|_{r=a,r=b} = 0$

(c) $\sigma_{\theta\theta}|_{r=a,r=b} = 0$

(d) $\int_a^b \sigma_{\theta\theta} r dr = -M$

- (a)
- (b)
- (c)
- (d)

No, the answer is incorrect.
Score: 0

Accepted Answers:
(a)

Assignment 8

The due date for submitting this assignment has passed.

Due on 2018-09-26, 23:59 IST.

Assignment submitted on 2018-09-26, 13:35 IST

1 point

In case of a torsional problem the assumption - "*Plane sections perpendicular to longitudinal axis before deformation remain plane and perpendicular to the longitudinal axis after deformation*" holds true for a shaft having

(a) circular cross section

(b) elliptical cross section

(c) square cross section

(d) triangular cross section

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

What is the number of non zero stress components for a torsional problem where the out of plane *i.e.* warping displacement (w) is a function of only the in-plane coordinates (x,y)? Consider the stress tensor is symmetric.

(a) 2

(b) 3

(c) 4

(d) 6

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

Compatibility equation in stress formulation of Torsional problem is given by [α is the angle of twist per unit length, μ is shear modulus]

(a) $\frac{\partial \sigma_{xz}}{\partial y} - \frac{\partial \sigma_{yz}}{\partial x} = -2\mu\alpha$

(b) $\frac{\partial \sigma_{xz}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial x} = -2\mu\alpha$

(c) $\frac{\partial \sigma_{xz}}{\partial x} - \frac{\partial \sigma_{yz}}{\partial y} = -2\mu\alpha$

(d) $\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = -2\mu\alpha$

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

For a torsional problem, Prandtl stress function (ψ) is given by $\psi(x) = ax^2 + by^2 - c^2$. Calculate σ_{xz} and σ_{yz}

(a) $\sigma_{xz} = 2by, \sigma_{yz} = -2ax$

(b) $\sigma_{xz} = 2ax, \sigma_{yz} = -2by$

(c) $\sigma_{xz} = -2by, \sigma_{yz} = 2ax$

(d) $\sigma_{xz} = -2ax, \sigma_{yz} = 2by$

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

Choose the correct option

1. In stress formulation of a torsional problem use of Prandtl stress function converts the compatibility equation to a Poisson equation $\nabla^2\psi = -2\mu\alpha$

2. On a traction free boundary Prandtl stress function becomes constant

(a) 1st statement is correct

(b) 2nd statement is correct

(c) Both the statements are correct

(d) None of the statements are correct

(a)
(b)
(c)
(d)

No, the answer is incorrect.
Score: 0

Accepted Answers:
(c)

1 point

Consider a elliptical shaft in x-y plane is subjected to a Torque T. If the Prandle stress function is ψ, What is the correct relationship between T and ψ?

(a) $T = \iint_R \frac{\partial \psi}{dx} dxdy$

(b) $T = \iint_R \frac{\partial \psi}{dy} dxdy$

(c) $T = \iint_R \psi dxdy$

(d) $T = \int_V \psi dxdydz$

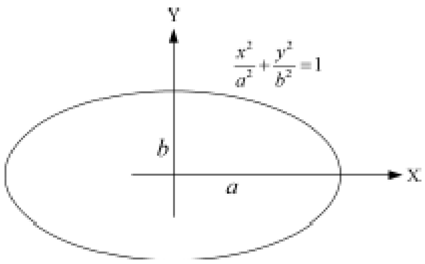
(a)
(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c)

1 point

For a shaft having elliptical cross section subjected to 100 kN-m torsion at one end and the other is fixed. Prandtl stress function is considered as $\psi = K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$, where $a = 0.4$ m and $b = 0.2$ m. Calculate the value of K in terms of shear modulus μ and angle of twist per unit length α



(a) $K = +\frac{18}{17}\mu\alpha x10^{-2}$

(b) $K = -\frac{26}{13}\mu\alpha x10^{-2}$

(c) $K = -\frac{13}{36}\mu\alpha x10^{-2}$

(d) $K = -\frac{16}{5}\mu\alpha x10^{-2}$

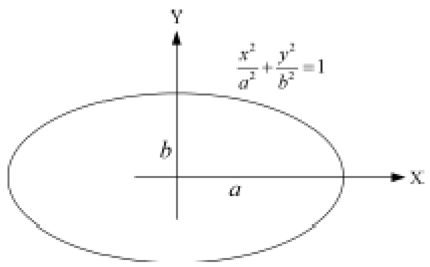
(a)
(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d)

1 point

For a shaft having elliptical cross section subjected to 10 kN-m torsion at one end and the other is fixed. Prandtl stress function is considered as $\psi = K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$, where $a = 40$ mm and $b = 20$ mm. Determine α for $\mu = 80$ GPa.



(a) $\alpha = 0.244$ rad/m

(b) $\alpha = 0.155$ rad/m

(c) $\alpha = 0$ rad/m

(d) $\alpha = 0.391$ rad/m

(a)
(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b)

1 point

If a circular shaft or radius 50 mm is subjected to an external torque of 50 kNm. Determine the maximum shear stress in the shaft.

(a) 25.46 MPa

(b) 12.83 MPa

(c) 50.92 MPa

(d) 0 MPa

(a)

(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

If a circular shaft in x-y plane of radius 50 mm is subjected to an external torque of 50 kNm. Determine the warping displacement at a point (25,0) mm in the shaft. The shear modulus is 80 GPa.

(a) 0.005 mm

(b) –0.005 mm

(c) 0 mm

(d) 0.012 mm

(a)
(b)
(c)
(d)

No, the answer is incorrect.
Score: 0

Accepted Answers:
(c)

Assignment 9

The due date for submitting this assignment has passed. **Due on 2018-10-03, 23:59 IST.**

Assignment submitted on 2018-10-02, 22:52 IST

1 point

Let us consider two complex numbers as $z_1 = 2 + 3i$ and $z_2 = 1 - 5i$. Determine $z_1 \times z_2$

(a) 10 + 6i

(b) -13 + 7i

(c) 17 - 7i

(d) 10 - 7i

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
c

1 point

Let us consider two complex numbers as $z_1 = 2 + 3i$ and $z_2 = 1 - 5i$. Determine $\frac{z_1}{z_2}$

(a) $\frac{-13+13i}{\sqrt{26}}$

(b) $\frac{10-13i}{\sqrt{13}}$

(c) $\frac{-17-7i}{\sqrt{26}}$

(d) $\frac{-1+i}{\sqrt{2}}$

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
a

1 point

Let us consider a complex function as $f(z) = (x^2 - y^2) + v(x, y)i$. If the function is analytic in nature what is the value of v(x,y)?

(a) $2x^2y^2$

(b) $-2xy$

(c) $x^2 + y^2$

(d) $2xy$

- a
- b
- c
- d

No, the answer is incorrect.
Score: 0

Accepted Answers:
d

1 point

Which of the following conditions are to be satisfied for the complex function $f(z) = u(r,\theta) + v(r,\theta)i$ to be analytic in polar coordinate?

1. $\frac{\partial u}{\partial r} = \frac{\partial v}{r\partial \theta}$
2. $\frac{\partial u}{r\partial \theta} = -\frac{\partial v}{\partial r}$

(a) Only condition 1 is to be satisfied

(b) Only condition 2 is to be satisfied

(c) Both the conditions are to be satisfied

(d) None of the conditions is to be satisfied

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
c

1 point

Let us consider a complex function as $f(z) = (y^3 - 3x^2y) + v(x,y)i$. If the function is analytic in nature what is the value of v(x,y)?

(a) $2x^2y^2$

(b) $-3xy^2$

(c) $x^3 - 3xy^2$

(d) $2xy$

- a
- b
- c
- d

No, the answer is incorrect.
Score: 0

Accepted Answers:
c

1 point

f(z) is an analytic function in a simply connected domain A. C is a closed curve in side the domain A. C_1 is any arbitrary curve in the domain A as shown in the figure. which of the following conditions hold true?

1. $\oint_C f(z)\,dz = 0$
2. $\oint_{C_1} f(z)\,dz$ is path dependent
3. $f(z)$ has an anti-derivative.

(a) Only 1 and 2 are correct

(b) Only 1 and 3 are correct

(c) Only 2 and 3 are correct

(d) All of the above are correct

- a
- b
- c
- d

No, the answer is incorrect.
Score: 0

Accepted Answers:
b

1 point

Evaluate the integral $\int_1^3 (z-2)^3\,dz$, where the path is an arbitrary contour between the limits of integration?

(a) 0

(b) $e + \frac{1}{e}$

(c) $\frac{1+i}{\pi}$

(d) 1.5

- a
- b
- c
- d

No, the answer is incorrect.
Score: 0

Accepted Answers:
a

1 point

Evaluate the integral $\oint_C \cos z\, dz$, where C is the unit circle $|Z| = 1$

(a) 0

(b) $e + \frac{1}{e}$

(c) $\frac{1+i}{\pi}$

(d) 1.5

- a
- b
- c
- d

No, the answer is incorrect.
Score: 0

Accepted Answers:
a

1 point

let us consider a complex number $z = x + iy$ and a complex function $f(z) = a/z + bz^2$.

What is the correct expression of $\overline{f(z)}$ in terms of x and y?

(a) $(ax + bx^2 + by^2) + i(ay + 2bxy)$

(b) $(ax/\sqrt{(x^2 + y^2)} + bx^2 - by^2) + i(ay/\sqrt{(x^2 + y^2)} - 2bxy)$

(c) $(ax/\sqrt{(x^2 + y^2)} + bx^2 - by^2) + i(ay/\sqrt{(x^2 + y^2)} + 2bxy)$

(d) $(ax + bx^2 - by^2) - i(ay + 2bxy)$

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
b

1 point

let us consider a complex number $z = x + iy$ and a complex function $f(z) = az + bz^2$.

What is the correct expression of $\overline{f(z)}$ in terms of x and y?

(a) $(ax + bx^2 + by^2) + i(ay + 2bxy)$

(b) $(ax + bx^2 + by^2) + i(ay - 2bxy)$

(c) $(ax + bx^2 - by^2) + i(ay + 2bxy)$

(d) $(ax + bx^2 - by^2) - i(ay + 2bxy)$

- a
- b

- c
- d

No, the answer is incorrect.
Score: 0

Accepted Answers:
d

Week 10 Assignment 10

The due date for submitting this assignment has passed.

Due on 2018-10-10, 23:59 IST.

Assignment submitted on 2018-10-10, 18:01 IST

1 point

The specific heat (c_p) of gold is 129 J/kg K. What is the quantity of heat energy required to raise the temperature of 100 g fold by 50 K?

(a) 215 J

(b) 1290 J

(c) 645 J

(d) 345 J

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c)

1 point

A pot of water is heated by transferring 1676 kJ of heat energy to the water. If there is 5 kg of water in the pot and temperature is raised by 80 K. What is the specific heat (c_p) of water?

(a) 4190 J/kg K

(b) 1190 J/kg K

(c) 2190 J/kg K

(d) 3190 J/kg K

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

If 1500 J of heat is applied to a copper ball with mass 45 g what will be the change in temperature? Specific heat (c_p) of copper is 0.39 J/g K.

(a) 45.87 K

(b) 56.12 K

(c) 23.84 K

(d) 85.47 K

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(d)

1 point

Calculate the thermal diffusivity of a material having density (ρ) 2700 kg/m^3, thermal conductivity (k) 155 W/mK and specific heat (c_p) 900 J/kg K.

(a) $6.37 \times 10^{-5} m^2/s$

(b) $2.37 \times 10^{-5} m^2/s$

(c) $4.52 \times 10^{-5} m^2/s$

(d) $0.64 \times 10^{-5} m^2/s$

- (a)
- (b)
- (c)
- (d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(a)

1 point

Which of the following statement is correct?

1. In conduction, heat transfer takes place through physical contact

2. In convection, heat transfer takes place by emission of electromagnetic radiation

(a) Both of them are correct

(b) Only 2 is correct

(c) Only 1 is correct

(d) None of them are correct

- (a)

(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c)

1 point

The correct expression of Duhamel-Neumann constitutive relationship of an isotropic material is [λ and μ are Lame's Constants]

(a) $\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\,\epsilon_{kk}\delta_{ij}$

(b) $\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\,\epsilon_{kk}\delta_{ij} - (3\lambda + 2\mu)\alpha\,(T - T_0)\,\delta_{ij}$

(c) $\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\,\epsilon_{kk}\delta_{ij} - (2\lambda + 3\mu)\alpha\,(T - T_0)\,\delta_{ij}$

(d) None of the above

(a)
(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b)

1 point

For plane strain formulation of uncoupled thermo-elasticity problem, the compatibility equation is given by

(a) $\nabla^2\left(\sigma_{xx} + \sigma_{yy}\right) = 0$

(b) $\nabla^2\left(\sigma_{xx} + \sigma_{yy}\right) + E\alpha\,\nabla^2\,T = 0$

(c) $\nabla^2\left(\sigma_{xx} + \sigma_{yy}\right) + \frac{E\alpha}{(1-\nu)}\,\nabla^2\,T = 0$

(d) None of the above

(a)
(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c)

1 point

A mild steel straight bar is clamped between two wall at 300 K. Determine the thermal stress induced in the bar when it is heated upto 375 K. E = 200 GPa and $\alpha = 11.2 \text{ x } 10^{-6}$.

(a) 54 MPa

(b) 168 MPa

(c) 112 MPa

(d) 224 MPa

(a)
(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b)

1 point

Wall of an industrial furnace is constructed from 0.20 m thick fire-clay brick having a thermal conductivity of 1.5 W/m K. The temperature inside and outside of the furnaces are 800 K and 400 K respectively. Calculate the rate of heat loss through the wall having a cross sectional area of 0.6 m^2.

(a) 3400 W

(b) 1530 W

(c) 1800 W

(d) 3600 W

(a)
(b)
(c)
(d)

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c)

1 point

A mild steel straight bar is free at both ends at 300 K. Determine the thermal stress induced in the bar when it is heated upto 400 K. E = 200 GPa and $\alpha = 11.2 \times 10^{-6}$.

(a) 0 MPa

(b) 168 MPa

(c) 112 MPa

(d) 224 MPa

- (a)
- (b)
- (c)
- (d)

No, the answer is incorrect.
Score: 0

Accepted Answers:
(a)

Assignment 11

The due date for submitting this assignment has passed.

Due on 2018-10-17, 23:59 IST.

Assignment submitted on 2018-10-16, 16:37 IST

Optically anisotropic materials differ from optically isotropic materials by ***1 point***

- (a) having high critical angles
- (b) having low critical angles
- (c) being able to polarize light
- (d) none of the above

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) being able to polarize light

In experimental stress analysis technique under which category photo elasticity lies in? ***1 point***

- (a) Point by point technique
- (b) Full field technique
- (c) Special technique
- (d) None of these

Yes, the answer is correct.
Score: 1

Accepted Answers:
(b) Full field technique

Which of the following statements are true? ***1 point***
1. Temporary double refraction criterion persists in a material when the loads are maintained.
2.Sometransparentnoncrystallinematerialsthatareareopticallyisotropicinstressfree state behaves like an optically anisotropic material when subjected to load.

- (a) Only statement 1 is correct
- (b) Only statement 1 is correct
- (c) Both of them are correct
- (d) Both of them are wrong

Yes, the answer is correct.
Score: 1

Accepted Answers:
(c) Both of them are correct

1 point

What is the correct relationship between wave number (ξ) and frequency or number of oscillation per second (f)?

(a) $\xi = \frac{2\pi}{\lambda}$

(b) $\xi = \frac{\lambda}{c}$

(c) $\xi = \frac{f}{T}$

(d) $\xi = 2\pi f$

- a
- b
- c

d

No, the answer is incorrect.
Score: 0

Accepted Answers:
d

1 point

what is the relationship between incident light intensity (I_i), reflected light intensity (I_r) and reflection coefficient (R)?

(a) $I_i = RI_r$

(b) $I_r = RI_i$

(c) $\ln I_i = R \ln I_r$

(d) $\ln I_r = R \ln I_i$

a
b
c
d

Yes, the answer is correct.
Score: 1

Accepted Answers:
b

1 point

Consider two simple wave fronts $E_1 = a_1 \, cos(\omega_1 \, t - \phi_1)$ and $E_2 = a_2 \, cos(\omega_2 \, t - \phi_2)$ in two mutually orthogonal planes. When these two wave fronts are superimposed a new wave front E is formed. if $a_1 = a_2 = a$ and $\delta = \frac{\lambda}{2\pi}(\phi_2 - \phi_1) = (2n+1)\pi/4$, what is the shape of trace of the tip of the polarised light ?

(a) An ellipse

(b) A straight line

(c) A circle

(d) A hyperbola

a
b
c
d

Yes, the answer is correct.
Score: 1

Accepted Answers:
c

1 point

Consider two simple wave fronts $E_1 = a_1 cos(\omega_1 t - \phi_1)$ and $E_2 = a_2 cos(\omega_2 t - \phi_2)$ in same plane. When these two wave fronts are superimposed a new wave front E is formed. Which of the following is correct?

(a) $E = \sqrt{{E_1}^2 + {E_2}^2}$

(b) $E = E_1 + E_2$

(c) $E = {E_1}^2 + {E_2}^2$

(d) $E = \frac{E_1}{E_2}$

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
b

1 point

Consider two simple wave fronts $E_1 = a_1 cos(\omega_1 t - \phi_1)$ and $E_2 = a_2 cos(\omega_2 t - \phi_2)$ in two mutually orthogonal planes. When these two wave fronts are superimposed a new wave front E is formed. if $a_1 = a_2 = a$ and $\delta = \frac{\lambda}{2\pi}(\phi_2 - \phi_1) = n\lambda/2$, what is the shape of trace of the tip of the polarised light ?

(a) An ellipse

(b) A straight line

(c) A circle

(d) A hyperbola

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
b

1 point

Consider two simple wave fronts $E_1 = a_1 \, cos(\omega_1 t - \phi_1)$ and $E_2 = a_2 \, cos(\omega_2 t - \phi_2)$ in two mutually orthogonal planes. When these two wave fronts are superimposed a new wave front E is formed. Which of the following is correct?

(a) $E = \sqrt{{E_1}^2 + {E_2}^2}$

(b) $E = E_1 + E_2$

(c) $E = {E_1}^2 + {E_2}^2$

(d) $E = \frac{E_1}{E_2}$

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
a

A polariscope tests for ***1 point***

- (a) Diffraction
- (b) Refractive index
- (c) Dispersion
- (d) none of the above

No, the answer is incorrect.
Score: 0

Accepted Answers:
(d) none of the above

ssignment 12

The due date for submitting this assignment has passed.

Due on 2018-10-24, 23:59 IST.

Assignment submitted on 2018-10-18, 08:50 IST

1 point

A function f(ax+by) is said to be linear function, where a and b are constants, if

(a) f(ax+by) = a f(x) + b f(y)

(b) f(ax+by) = f(x) + b f(y)

(c) f(ax+by) = a f(x) + f(y)

(d) none of the above

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
a

1 point

Which of the following is a linear ordinary differential function?

(a) $\left(\frac{d^4y}{dx^4}\right)^2 = 2$

(b) $\frac{d^4y}{dx^4} = 2\left(\frac{dy}{dx}\right)^{1.5} + 1$

(c) $\frac{d^4y}{dx^4} = \frac{d^2y}{dx^2} + \frac{dy}{dx}$

(d) none of the above

- a
- b
- c
- d

No, the answer is incorrect.
Score: 0

Accepted Answers:
c

1 point

If stress in a system is a nonlinear function of strain what kind of nonlinearity can we expect in the material response?

(a) Geometric nonlinearity

(b) Material nonlinearity

(c) Both option (a) and (b)

(d) None of the above

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
b

1 point

If strain displacement relationship in a system is nonlinear, what kind of nonlinearity can we expect in the material response?

(a) Material nonlinearity

(b) Geometric nonlinearity

(c) Both option (a) and (b)

(d) None of the above

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
b

1 point

Choose the correct option among the following statements

1. In case of nonlinear elasticity there is a residual strain retained in the material after the external load is removed.

2. The loading and unloading path of the stress strain curve is same in case of nonlinear elastic materials.

(a) Only statement 1 is correct

(b) Only statement 2 is correct

(c) None of the statements are correct

(d) Both the statements are correct

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
b

1 point

If the higher order terms are not neglected the correct expression of the curvature is

(a) $\kappa = \frac{d^2y}{dx^2}$

(b) $\kappa = \frac{\frac{d^2y}{dx^2}}{1+(\frac{dy}{dx})^2}$

(c) $\kappa = \frac{\frac{d^2y}{dx^2}}{[1+(\frac{dy}{dx})^2]^{1.5}}$

(d) None of the above

- a
- b
- c
- d

Yes, the answer is correct.
Score: 1

Accepted Answers:
c

1 point

Let us say X is the undeformed and x is the deformed configuration of any system. The two configurations are related by a mapping ϕ such that $x = \phi(X)$. Which of the following is true for the characteristics of the mapping ϕ

(a) ϕ is an one to one mapping

(b) ϕ is a many to one mapping

(c) Both of these

(d) None of these

- a
- b
- c
- d

No, the answer is incorrect.
Score: 0

Accepted Answers:
a

1 point

In linear elasticity approach the stress strain relationship is defined in the

(a) Deformed configuration

(b) Undeformed configuration

(c) Both deformed and undeformed configuration

(d) None of these

- a
- b

c
d

No, the answer is incorrect.
Score: 0

Accepted Answers:
b

1 point

In nonlinear elasticity approach the stress strain relationship is defined in the

(a) Deformed configuration

(b) Undeformed configuration

(c) Both deformed and undeformed configuration

(d) None of these

a
b
c
d

No, the answer is incorrect.
Score: 0

Accepted Answers:
c

1 point

Nonlinear elasticity problem encompasses

(a) Large deformation problems

(b) Small deformation but large rotation / displacement problems

(c) Both of the above

(d) None of the above

a
b
c
d

Yes, the answer is correct.
Score: 1

Accepted Answers:
c

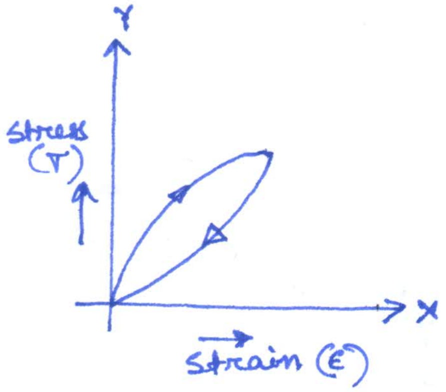
Assignment Solution

Assignment-1

1. In case of a perfectly elastic material, the state of stress at any instant is independent of the previous history of stresses. The stress induced in the material can be uniquely defined as a function of strains.

   Both the statements are correct.

2.



The stress-strain diagram shown in the figure is typical for visco-elastic material.

3.

$$A_{ij} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 3 & 1 & 6 \end{bmatrix}$$

$$A_{KK} = A_{11} + A_{22} + A_{33}$$

$$A_{KK} = 1 + 4 + 6$$

$$A_{KK} = 11$$

4.

$$A_{ij} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 3 & 1 & 6 \end{bmatrix} \qquad b_i = \begin{Bmatrix} 2 \\ 1 \\ 6 \end{Bmatrix}$$

$A_{ji} b_i$

~~$= A_{1i}b_i + A_{2i}b_i + A_{3i}b_i$.~~

$$= A_{j1} b_1 + A_{j2} b_2 + A_{j3} b_3$$

$$= 2\begin{Bmatrix} 1 \\ 0 \\ 3 \end{Bmatrix} + 1\begin{Bmatrix} 2 \\ 4 \\ 1 \end{Bmatrix} + 6\begin{Bmatrix} 3 \\ 1 \\ 6 \end{Bmatrix}$$

$$= \begin{Bmatrix} 2 \\ 0 \\ 6 \end{Bmatrix} + \begin{Bmatrix} 2 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 18 \\ 6 \\ 36 \end{Bmatrix}$$

$$= \begin{Bmatrix} 22 \\ 10 \\ 42 \end{Bmatrix} = \begin{Bmatrix} 22 & 10 & 42 \end{Bmatrix}^T$$

5.

$$A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 0 & 4 \\ 2 & 4 & 3 \end{bmatrix}$$

Eigen value problem

$$AX = \lambda X.$$

~~or, $(A-\lambda)I = 0$.~~

or, $[A - \lambda I] X = 0$.

$$\det |A - \lambda I| = 0$$

$$\det \begin{vmatrix} 5-\lambda & 1 & 2 \\ 1 & 0-\lambda & 4 \\ 2 & 4 & 3-\lambda \end{vmatrix}$$

or, $(5-\lambda)\left[-\lambda(3-\lambda) - 16\right] - 1\left[3 - \lambda - 8\right] + 2\left[4 + 2\lambda\right] = 0$

or, $\lambda^3 - 8\lambda^2 - 6\lambda + 67 = 0$

$$\lambda = \{-2.788, 7.637, 3.149\}$$

6. The correct indicial notation of vector cross product

$u \times v = \varepsilon_{ijk} u_j v_i e_k$

7. $u = -6x^2 e_1 + 3xy e_2 - 5xyz e_3$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -6x^2 & 3xy & -5xyz \end{vmatrix}$$

$$= e_1\left[\frac{\partial}{\partial y}(-5xyz) - \frac{\partial}{\partial z}(3xy)\right] - e_2\left[\frac{\partial}{\partial x}(-5xyz) + \frac{\partial}{\partial z}(6x^2)\right]$$

$$+ e_3\left[\frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial y}(6x^2)\right]$$

$$= e_1[-5xz] - e_2[-5yz] + e_3[3y]$$

$$\nabla \times u = -5xz e_1 + 5yz e_2 + 3y e_3$$

8. $u = -6x^3 e_1 + 3xy^2 e_2 - 5xyz e_3$

$$\nabla \cdot u = \left(\frac{d}{\partial x} e_1 + \frac{\partial}{\partial y} e_2 + \frac{\partial}{\partial z} e_3\right) \cdot \left(-6x^3 e_1 + 3xy^2 e_2 - 5xyz e_3\right)$$

$$\nabla \cdot u. = (-18x^2 + 6xy - 5xy.)$$

$$\nabla \cdot u = (-18x^2 + xy)$$

9.

$\phi = x^3 - xy^2z$

$\nabla^2\phi$

$= \nabla.(\nabla\phi)$

$= \left(\frac{\partial}{\partial x}e_1 + \frac{\partial}{\partial y}e_2 + \frac{\partial}{\partial z}e_3\right) \cdot \left[(3x^2 - y^2z)e_1 + (-2xyz)e_2 - xy^2 e_3\right]$

$= 6x - 2x\,(6x - 0) + (-2xz) + 0$

$= 6x - 2xz$

$$\boxed{\nabla^2\phi = 6x - 2xz}$$

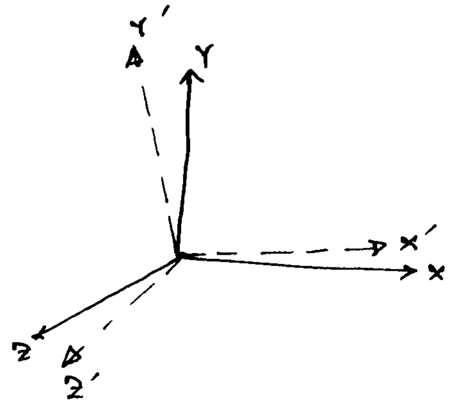
10. Divergence theorem relates volume integral to surface integral.
Stokes theorem relates contour/line integral to surface integral.

Hence; only statement I is correct and statement II is incorrect.

Assignment-Solution

Week 2 Assignment-1

1.



State of stress in XYZ coordinate = σ.
State of stress in X'Y'Z' coordinate = σ'
Rotation matrix = Q.

$$\boxed{\sigma' = Q \sigma Q^T}$$

2. Strain matrix is a 2^{nd} order tensor.
ε_{xx} is the strain on a plane whose normal is along x-x axis and in the direction of x-x axis.
Hence Strain is expressed with two directions (direction and plane). Hence it is a 2^{nd} order tensor.

3. The notation σ_{xz} means the stress is acting on a plane whose normal is along x-x axis and the direction of stress is along z axis.
The correct answer is option (d)

.

$$\sigma_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

Let the principle stresses are $\lambda_1, \lambda_2, \lambda_3$.

or. $\sigma\{x\} = \lambda_i\{x\}$ $i = 1, 2, 3.$

or, $[\sigma - \lambda I]\{x\} = \phi.$

$$[\sigma - \lambda I] = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & -3-\lambda \end{bmatrix}$$

$\det|\sigma - \lambda I| = 0.$

The characteristic equation is,

or, $(2-\lambda)\left[(3-\lambda)(-3-\lambda)-16\right] = 0$

or, $(2-\lambda)\left[-(3-\lambda)(3+\lambda)-16\right] = 0$

or, $-(2-\lambda)\left[9-\lambda^2+16\right] = 0$

or, $(\lambda-2)(25-\lambda^2) = 0$

or, $(\lambda-2)(5-\lambda)(5+\lambda) = 0.$

or, $\boxed{\lambda = \{5, 2, -5\}}$

The principle stresses are $\{5, 2, -5\}$.

5. $\sigma = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}.$

$I_1 = -5+2+3 = 0$

$\boxed{I_1 = 0}$

$I_2 = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} -5 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} -5 & 1 \\ 1 & 2 \end{vmatrix}$

$I_2 = (6-9) + (-15-4) + (-10-1)$

$I_2 = -3 - 19 - 11$

$\boxed{I_2 = -33}$

$I_3 = \det|\sigma| = 16.$

$\boxed{I_3 = 16}$

6. The tensorial representation of strain(ε_{ij}) at a point with displacement field $u = \{u_i, u_j, u_k\}$.

$$\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

7.

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

C is the constitutive matrix.

$i,j = 1,2,3$, $kl = 1,2,3$

$[\sigma_{ij}]$ contains 3^2 elements
= 9 elements.

$[\varepsilon_{kl}]$ contains 3^2 elements
= 9 elements.

C_{ijkl} contains 3^4 elements. as $i,j,k,l = 1,2,3$.
= 81 Elements.

81 elements are required to define constitutive matrix

8.

$$\sigma = \begin{bmatrix} 6 & 5 & 7 \\ 5 & 3 & 4 \\ 7 & 4 & -3 \end{bmatrix}$$

$\sigma_m = \frac{1}{3}\sigma_{ii}$

$\sigma_m = \frac{1}{3}[6+3-3] = 2$

Deviatoric stress $\sigma_D = \begin{bmatrix} 6-\sigma_m & 5 & 7 \\ 5 & 3-\sigma_m & 4 \\ 7 & 4 & -3-\sigma_m \end{bmatrix}$

$$\sigma_D = \begin{bmatrix} 4 & 5 & 7 \\ 5 & 1 & 4 \\ 7 & 4 & -5 \end{bmatrix}$$

9. $\sigma_{ij} = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$ $\quad n = \left\{ \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\}^T$

Traction $(T_i) = \sigma_{ij} n_j$

$$\hat{T}_i = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix} \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{Bmatrix}$$

$$\hat{T}_i = \begin{Bmatrix} -\frac{6}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{Bmatrix}$$ option (c)

10. $\sigma_{ij} = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$ $\quad \hat{n} = \left\{ \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\}^T$

Normal stress $= \hat{T}_i \cdot \hat{n}_i$

$= (\sigma_{ij} \hat{n}_j) \cdot \hat{n}_i$

$$T_i = \begin{bmatrix} -5 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix} \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\frac{6}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{Bmatrix}$$

Normal stress $= \left\{ -\frac{6}{\sqrt{2}} e_1 - \frac{1}{\sqrt{2}} e_2 - \frac{1}{\sqrt{2}} e_3 \right\} \cdot \left\{ \frac{1}{\sqrt{2}} e_1 - \frac{1}{\sqrt{2}} e_2 \right\}$

$= -\frac{6}{2} + \frac{1}{2} = -\frac{5}{2}$ (Answer)

Assignment Solution

Week 3 Assignment-1

1. constitutive tensor C_{ijkl} is a 4th order tensor.

2. For isotropic material the number of independent element in constitutive tensor is 2.

3. For anisotropic material the number of independent element in constitutive tensor is 21.

4. For orthotropic material the number of independent element in constitutive tensor is 9.

5. $E = 200\ GPa$, $\nu = 0.2$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{200 \times 0.2}{(1+0.2)(1-0.4)} = 55.55\ GPa$$

$$\mu = \frac{E}{2(1+\nu)} = \frac{200}{2(1+0.2)} = 83.33\ GPa.$$

(c) 55.55 GPa , 83.33 GPa.

6. $E = 210\ GPa$, $\nu = 0.3$

$$\text{bulk modulus } K = \frac{E}{3(1-2\nu)} = \frac{210}{3(1-0.6)} = 175\ GPa.$$

(d) 175 GPa.

7. $\sigma_{xx} = 250\ MPa$, $\sigma_{yy} = -350\ MPa$, $\sigma_{zz} = 0.$

$E = 2\ GPa$

$\nu = 0.18$

$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]$$

$$= \frac{1}{2 \times 10^3}\left[0 - 0.18(250-350)\right]$$

$$= +\frac{0.18 \times 100}{2 \times 10^3} = +9 \times 10^{-3}$$

(c) $\epsilon_{zz} = 9 \times 10^{-3}$

8. $\epsilon_{xx} = 0.5\times10^{-3}$, $\epsilon_{yy} = -0.4\times10^{-3}$, $\epsilon_{zz} = 0.7\times10^{-3}$

$E = 2\,GPa$, $\nu = 0.18$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 476.6\ MPa$$

$$\sigma_{hydrostatic} = \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3}$$

$$\mu = \frac{E}{2(1+\nu)} = 847.45\ MPa.$$

$$\sigma_{xx} = \lambda(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}) + 2\mu\,\epsilon_{xx}$$
$$\sigma_{yy} = \lambda(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}) + 2\mu\,\epsilon_{yy}$$
$$\sigma_{zz} = \lambda(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}) + 2\mu\,\epsilon_{zz}$$

$$\therefore\ \sigma_{xx}+\sigma_{yy}+\sigma_{zz} = 3\lambda(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}) + 2\mu(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})$$

$$\sigma_{hydrostatic} = \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3}$$
$$= \frac{3\lambda+2\mu}{3}(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})$$
$$= \frac{3\times476.6+2\times847.45}{3}(0.5-0.4+0.7)\times10^{-3}$$
$$= 0.833\ MPa = 0.833\times10^{6}\ Pa$$

9. $\theta_a = 45^\circ$, $\theta_b = 90^\circ$, $\theta_c = 135^\circ$, $e_a = 0.5\times10^{-3}$, $e_b = 0.4\times10^{-3}$, $e_c = 0.3\times10^{-3}$, $\lambda = 140.6\ GPa$, $\mu = 75\ GPa$

$$e_x' = e_x\cos^2\theta + e_y\sin^2\theta + 2e_{xy}\sin\theta\cos\theta.$$

$$e_a = e_x\cos^2 45^\circ + e_y\sin^2 45^\circ + 2e_{xy}\sin45^\circ\cos45^\circ$$
$$\Rightarrow 0.5\times10^{-3} = \frac{e_x}{2} + \frac{e_y}{2} + e_{xy}$$
$$\Rightarrow e_x + e_y + 2e_{xy} = 1\times10^{-3} \quad \text{--- (1)}$$

$$e_b = e_x\cos^2 90^\circ + e_y\sin^2 90^\circ + 2e_{xy}\cos90^\circ\sin90^\circ$$
$$e_y = 0.4\times10^{-3} \quad \text{--- (2)}$$

$$e_c = e_x\cos^2 135^\circ + e_y\sin^2 135^\circ + 2e_{xy}\sin135^\circ\cos135^\circ$$
$$\Rightarrow 0.3\times10^{-3} = \frac{e_x}{2} + \frac{e_y}{2} - e_{xy}$$
$$\Rightarrow e_x + e_y - 2e_{xy} = 0.6\times10^{-3} \quad \text{--- (3)}$$

$\boxed{\sigma_{xy} = 15\ MPa}$

(1) − (3) ⇒

$$4e_{xy} = 0.4\times10^{-3} \Rightarrow e_{xy} = 0.1\times10^{-3}$$
$$\sigma_{xy} = 2\mu e_{xy} = 2\times75\times0.1\times10^{-3}\times10^{3} = 15\ MPa$$

10.

$$u = \frac{M(1-\mu^2)}{EI} xyz$$

$$v = \frac{M(1-\mu^2)}{EI} \left(x^2 - \frac{yz}{3}\right)$$

$$w = \frac{M(1-\mu^2)}{EI} (x^2 - z^2)$$

at the point

$x = 5$

$y = 0$

$z = 1$

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{M(1-\mu^2)}{EI} yz$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = -\frac{M(1-\mu^2)}{EI} \frac{z}{3}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = -2 \frac{M(1-\mu^2)}{EI} z.$$

$$\sigma_{xx} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\, \epsilon_{xx}$$

$$\sigma_{xx} = \lambda \frac{M(1-\mu^2)}{EI}\left[yz - \frac{z}{3} - 2z\right] + 2\mu \frac{M(1-\mu^2)}{EI} yz$$

$$\sigma_{xx} = \lambda \frac{M(1-\mu^2)}{EI}\left[0 - \frac{1}{3} - 2\right] + 0$$

$$\boxed{\sigma_{xx} = -\frac{7}{3}\lambda \frac{M(1-\mu^2)}{EI}}$$

Assignment Solution

Assignment - Week-4

1. Number of independent elements in the constitutive relationship matrix of a monoclinic material is 13.

2. Number of independent elements in the constitutive matrix of a triclinic material or general anisotrophy is 21.

3. Number of independent elements in the constitutive matrix of a transversely isotropic material is 5.

4. The stress strain relationship of an orthotropic material is

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Since $S_{ij} = S_{ji}$;

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1} ; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1} ; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$$

or

$$\boxed{\frac{\nu_{21}}{\nu_{12}} = \frac{E_2}{E_1}} ; \quad \boxed{\frac{\nu_{31}}{\nu_{13}} = \frac{E_3}{E_1}} ; \quad \boxed{\frac{\nu_{32}}{\nu_{23}} = \frac{E_3}{E_2}}$$

or,

$$\boxed{\frac{\nu_{ij}}{\nu_{ji}} = \frac{E_i}{E_j}}$$

5. $\{\sigma\} = [C]\{\varepsilon\}$; $\{\sigma_1\} = [C_1]\{\varepsilon_1\}$

σ, ε are stress and strain in XY coordinate

σ_1, ε_1 are stress and strain at $X_1 Y_1$ coordinate

T_ε strain transformation matrix

T_σ stress transformation matrix.

$\{\sigma_1\} = [T_\sigma]\{\sigma\} = [T_\varepsilon]^{-T}\{\sigma\}$

because $\boxed{T_\sigma = [T_\varepsilon]^{-T}}$ $\{\varepsilon_1\}$

$\{\sigma\} = [T_\varepsilon^T]\{\sigma_1\}$

$\{\sigma\} = [T_\varepsilon^T][C_1]\{\varepsilon_1\}$

$\{\sigma\} = \underbrace{[T_\varepsilon]^T[C_1][T_\varepsilon]}_{[C]}\{\varepsilon\}$

or,

$[C] = [T_\varepsilon]^T[C_1][T_\varepsilon]$

$\boxed{[C_1] = [T_\varepsilon][C][T_\varepsilon]^T}$

6. $[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & -3 & 3 \\ 2 & 3 & 7 \end{bmatrix}$

The Eigen values are

$\lambda_1 = -3.85$
$\lambda_2 = 3.82$
$\lambda_3 = 9.03$

all the eigen values are not positive.

[A] is a positive definite matrix if.

(i) All the diagonal members are positive (without row exchange)
(ii) $\det(A) \geq 0$.
(iii) for any $\{x\}$; $\{x\}^T A \{x\} > 0$.

Now for the matrix $A_{22} = -3 < 0$

[A] is not a positive definite matrix.

Only statement 2 is correct ; Statement 1 is wrong.

7. The constitutive matrix is a positive definite matrix.

det (c) ≥ 0

while determining the det(c) the following conditions have to be satisfied.

$1 - \nu_{12}\nu_{21} > 0$; $1 - \nu_{13}\nu_{31} > 0$; $1 - \nu_{23}\nu_{32} > 0$

or,

$1 - \nu_{12}\nu_{12}\frac{E_2}{E_1} > 0.$

or, $|\nu_{12}| < \left(\frac{E_1}{E_2}\right)^{1/2}$

Similar results come in other cases.

$|\nu_{ij}| < \left(\frac{E_i}{E_j}\right)^{1/2}$

8. for orthotropic material;

det (C) > 0 ⇒ $1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{13}\nu_{21}\nu_{32} > 0.$

or, $\nu_{13}\nu_{12}\nu_{23} < \frac{1 - \nu_{21}^2(E_1/E_2) - \nu_{32}^2(E_2/E_3) - \nu_{13}^2(E_3/E_1)}{2}.$

or, $\nu_{13}\nu_{12}\nu_{23} < \frac{1}{2}$

9. For a ~~transversely~~ isotropic material.

$-1 < \nu < \frac{1}{2}$

~~$\nu_{12}\ \nu_{13} < \left(\frac{1}{2}\right)$~~

$\nu < \frac{1}{2}$

$\nu_{12} = \nu_{13} = \nu_{23} = \nu$

$E_1 = E_2 = E_2 = E.$

$|\nu_{12}| < \left(\frac{E_1}{E_2}\right)^{1/2}.$

or, $|\nu| < (1)^{1/2}$

~~$|\nu|$~~ $-1 < \nu$

10. $T_\sigma = [T_\varepsilon]^{-T}$

or $[T_\sigma]^T = [T_\varepsilon]^{-1}$

$\{\sigma\} = c\,\{\varepsilon\}.$

$\{\sigma_1\} = T_\sigma \{\sigma\}$

$\{\varepsilon_1\} = T_\varepsilon\,\{\varepsilon\}.$

Assignment Solution

Week-5 Assignment 1

1. Number of independent strain compatibility equations for a 3D system is 3.

2. Saint-Venant compatibility equation is.

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

The compatibility equation is expressed in terms of strain.

3. Beltrami-Michell compatibility equations is

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu}\delta_{ij} b_{k,k} - b_{i,j} - b_{j,i}$$

The compatibily equation is expressed in terms of stress.

4. The conditions to be satisfied for a continuum body is

1. The matter is continuously distributed over the body.
2. The field variable is continuously distributed over the body.

Hence both the statements are to be true in case of any continuum body.

5. Stress formulation expression for a 3D system

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1\pm\nu}\delta_{ij} b_{k,k} - b_{i,j} - b_{j,i}$$

Hence there are 6 independent equations and 6 unknowns.

6. Displacement formulation for any 3D system.

$$\mu u_{i,kk} + (\lambda + \mu) u_{k,ki} + b_i = 0$$

Hence there are 3 independent equations and 3 unknowns.

7. Equilibrium equations for a 3D continuum system is

$$\sigma_{ij,j} + b_i = 0$$

There are 3 independent equations.

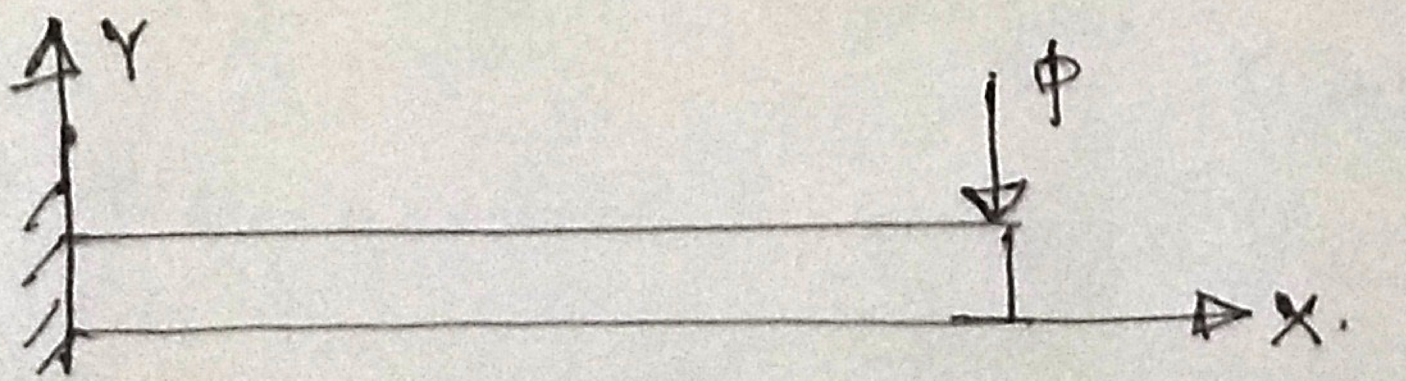
8. Strain compatibility condition is a necessary condition for a unique solution in continuum mechanics.
But, strain compatibility is not a sufficient condition for multiply connected domain. It is a necessary and sufficient condition only for simply-connected domains for unique displacement field.

* only statement 2 is correct.

9. For any 2D system, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)$$

for no body force or constant body force (option - d)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = 0.$$ [Stress distribution is independent of material property].

10. 

The boundary condition at the fixed edge is $u_x = 0,\ v = 0,\ \theta_z = 0$

Hence displacement boundary conditions are to be applied at the fixed edge.

Assignment Solution

Week-6 Assignment-1

1. In case of plane stress problem, the out of plane stress components are considered to be zero.

The non zero stress components are 3 ($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$)

As per generalised Hook's Law,

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu(\sigma_{xx} + \sigma_{yy})}{E}$$

$$\varepsilon_{xz} = \frac{\sigma_{xz}}{G}, \quad \varepsilon_{yz} = \frac{\sigma_{yz}}{G}.$$

As, $\sigma_{xz}, \sigma_{yz}, \sigma_{zz} = 0$ for plane stress problem.

$\varepsilon_{xz} = 0, \quad \varepsilon_{yz} = 0.$

but $$\boxed{\varepsilon_{zz} = -\frac{\nu(\sigma_{xx} + \sigma_{yy})}{E}}$$

Hence, non zero strain components are 4. ($\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}$)

2. For plane stress problem, non-zero stress components are 3.

($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$).

3. For plane strain problem, non-zero strain components are 3.

($\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$)

4. For plane strain problem, $\varepsilon_{zz} = 0$, $\varepsilon_{xz} = 0$, $\varepsilon_{yz} = 0$ (out of plane strain components).

Now, as per generalised Hook's law,

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu(\sigma_{xx} + \sigma_{yy})}{E}$$

or, $$0 = \frac{\sigma_{zz}}{E} - \frac{\nu(\sigma_{xx} + \sigma_{yy})}{E}$$

or, $$\boxed{\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})}$$

Hence non zero stress components are 4.

($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{zz}$).

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5. Airy's stress function (ϕ) automatically satisfies the equilibrium conditions.

In absence of body forces, Airy's stress function (ϕ) converts Beltrami-Michell equation to a Bi-harmonic equation.

- Both the statements are true.

6. $\phi = 6x^2y^3$.

$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$, $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$, $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

$\boxed{\sigma_{xx} = 36x^2y}$ $\boxed{\sigma_{yy} = 12y^3}$ $\boxed{\tau_{xy} = -36xy^2}$

7. $\sigma_{xx} = 5$ MPa, $\sigma_{yy} = -10$ MPa, $\tau_{xy} = 7.5$ MPa.

$\sigma_{zz} = 0$ [Plane-stress problem] $E = 2\times10^9$ Pa., $\nu = 0.15$

Now,

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu(\sigma_{xx}+\sigma_{yy})}{E}$$

$$\epsilon_{zz} = -\frac{0.15\,[5-10]\times10^6}{2\times10^9} = \frac{5\times0.15}{2}\times10^{-3}$$

$$\boxed{\epsilon_{zz} = 3.75\times10^{-4}}$$

8. $\epsilon_{xx} = 0.005$, $\epsilon_{yy} = -0.001$, $\epsilon_{xy} = 0.006$

$\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0$ [Plane strain problem]

$\boxed{\tau_{xz} = G.\,\epsilon_{xz}}$

$\therefore \boxed{\tau_{xz} = 0}$

9. For a generalised plane stress problem, the average out of plane displacement is zero.

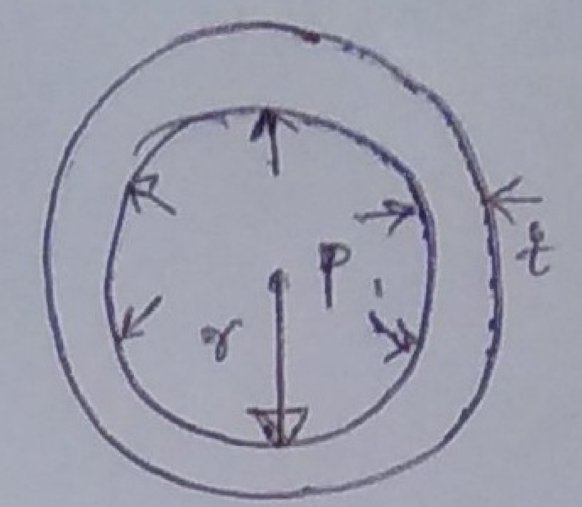
Only statement 2 is correct.

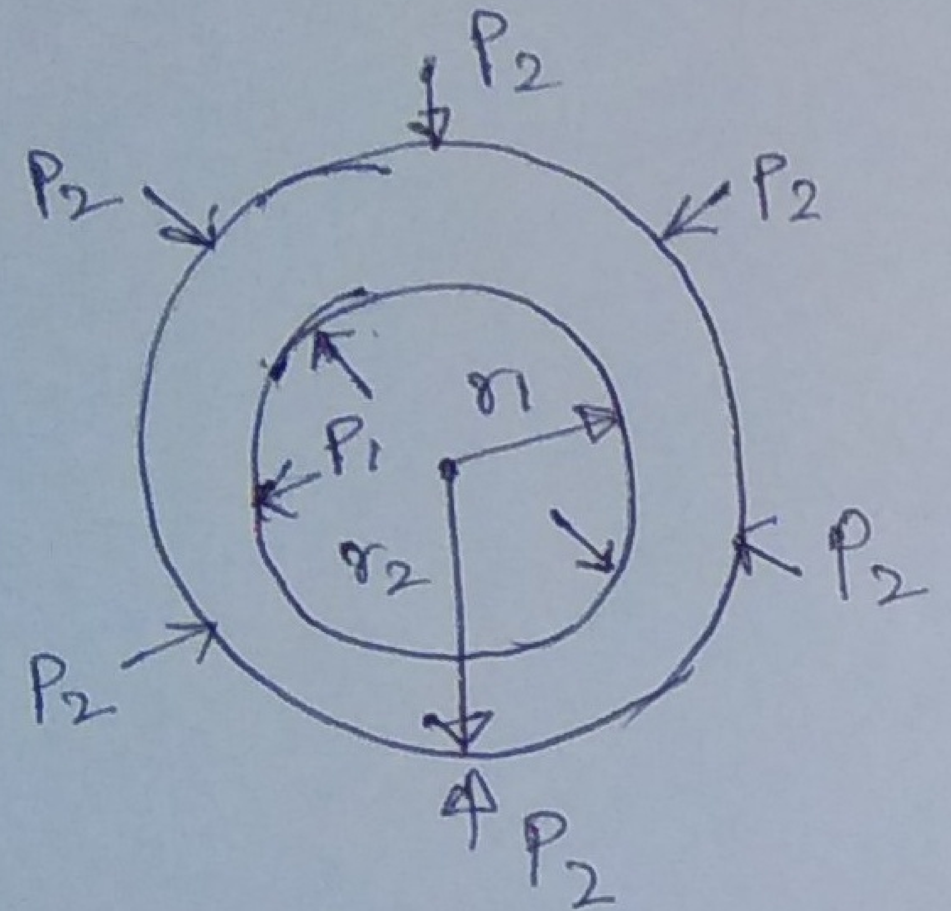
Average.

10. Radius of the cylinder = r

Wall thickness = t.

uniform pressure = p.





$$\sigma_{\theta\theta} = -\frac{r_1^2 r_2^2 (P_2 - P_1)}{r_2^2 - r_1^2}\cdot\frac{1}{r^2} + \frac{r_1^2 P_1 - r_2^2 P_2}{r_2^2 - r_1^2}$$

Now, $P_2 = 0$; $P_1 = P$

$t = (r_2 - r_1)$.

$r_0 = \frac{(r_1 + r_2)}{2}$

$r_1 = r_0 - t/2$
$r_2 = r_0 + t/2$
$r_1^2 r_2^2 \approx r_0^4$

$$\sigma_{\theta\theta} = -\frac{r_0^4(-P)}{t\times 2r_0}\times\frac{1}{r_0^2} + \frac{r_0^2 P_1}{t\cdot 2r_0}$$

$$\sigma_{\theta\theta} = \frac{P r_0}{2t} + \frac{P r_0}{2t}.$$

$$\boxed{\sigma_{\theta\theta} = \frac{P r_0}{t}.}$$

Assignment Solution

Week-7 Assignment-1

1. $\phi = \frac{m}{2}x^2 - nxy + \frac{p}{2}y^2$.

$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = m$

$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = p$

$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = n$

For pure shear condition;

$\sigma_{xx} = 0$; $m = 0$

$\sigma_{yy} = 0$; $p = 0$

$\sigma_{xy} \neq 0$; $n \neq 0$.

2. The compatibility equation in plane stress problem,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right)$$

3. $b_x = 0$; $b_y = \rho g$.

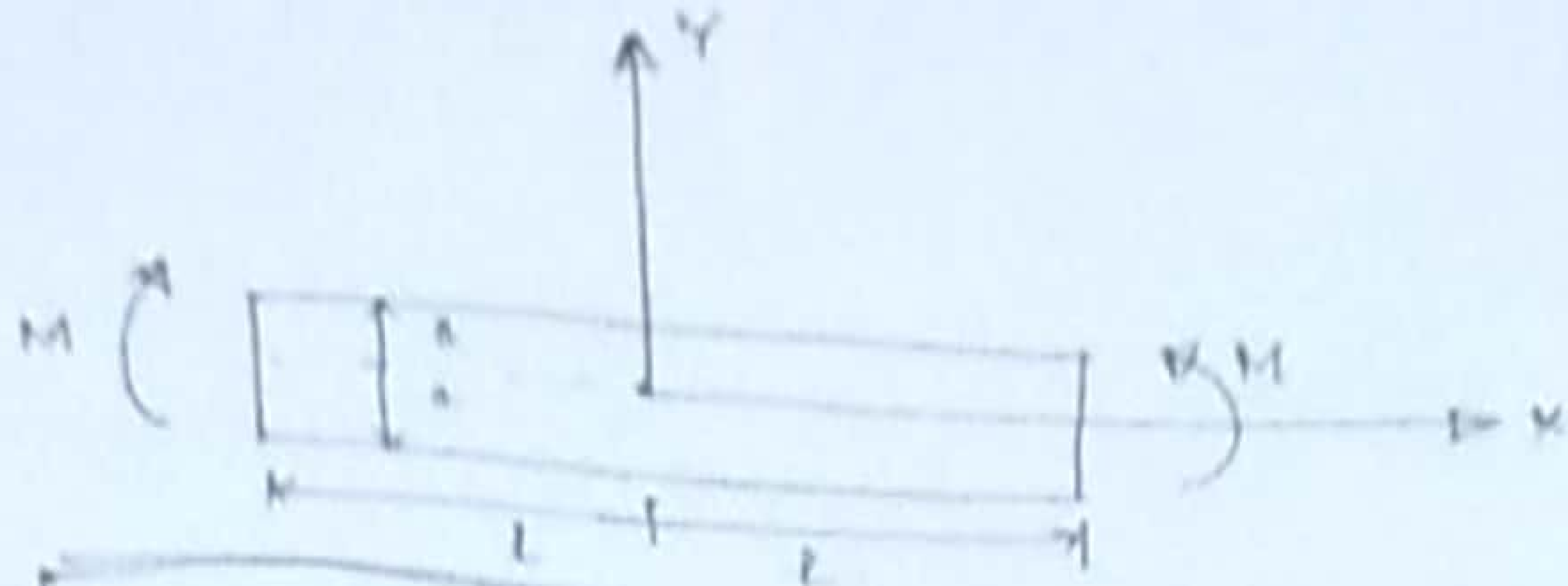
ϕ is the stress function;

$$\boxed{\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} - \rho g y} \qquad \boxed{\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} - \rho g y}$$

$$\boxed{\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}}$$

4. The expression of shear stress $\boxed{\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}}$.

5.



$$\phi = A_0 y^3$$

The boundary conditions are;

$\sigma_y (x, \pm c) = 0$

$\tau_{xy} (x, \pm c) = 0$

$\tau_{xy} (\pm L, y) = 0$

$\int_{-c}^{+c} \sigma_{xx} (\pm L, y)\, dy = 0$

$\int_{-c}^{+c} \sigma_{xx} (\pm L, y)\, y\, dy = -M.$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 6A_0 y$$

putting the value in the boundary condition criteria;

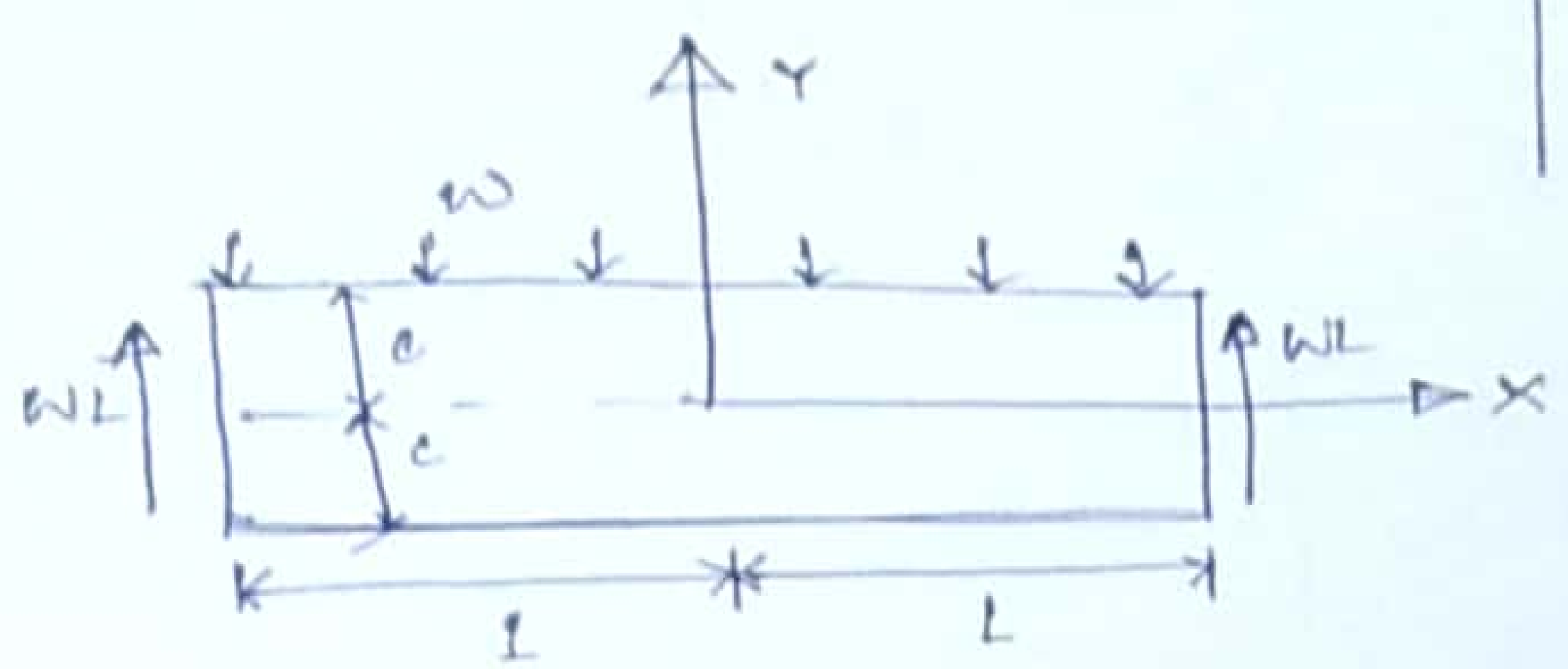
$$\int_{-c}^{+c} 6A_0 y \cdot y \cdot dy = -M$$

or, $6A_0 \left[\frac{y^3}{3}\right]_{-c}^{+c} = -M.$

or, $4A_0 c^3 = -M.$

or, $A_0 = \frac{-M}{4c^3}$

8.



The boundary conditions for this problem;

$\tau_{xy} (x, \pm c) = 0.$

$\sigma_{yy}(x, c) = -w.$

$\sigma_{yy}(x, -c) = 0.$

$\int_{-c}^{+c} \sigma_x (\pm L, y)\, dy = 0$

$\int_{-c}^{+c} \sigma_x (\pm L, y)\, y\, dy = 0.$

$\int_{-c}^{+c} \tau_{xy} (\pm L, y)\, dy = \mp WL$

Hence only 2nd statement is correct

7. ϕ is the stress function in polar coordinate system.
The biharmonic equation in polar coordinate is;

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)=0.$$

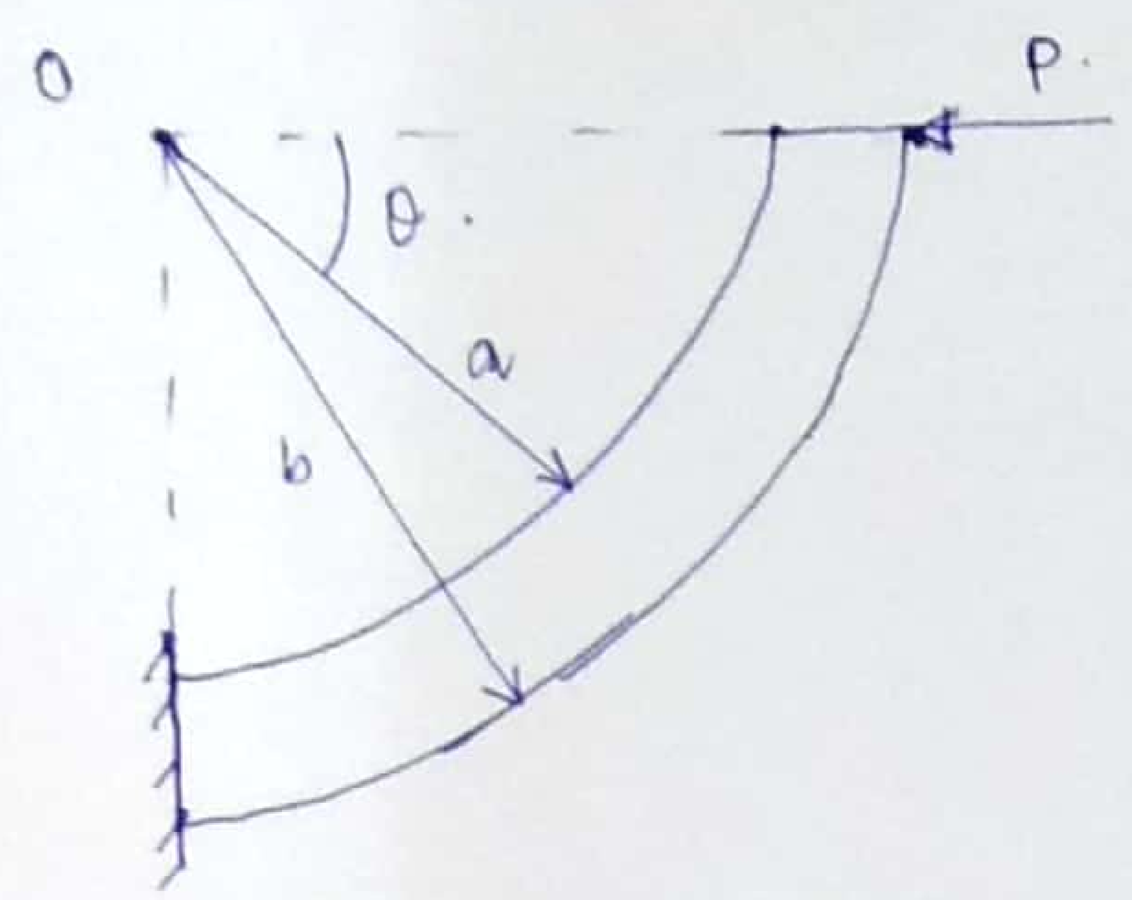
or,
$$\boxed{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi = 0.}$$

8. In polar coordinate system;

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

or,
$$\boxed{\sigma_{rr} = \left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi.}$$

9.



The boundary conditions for the problem is;

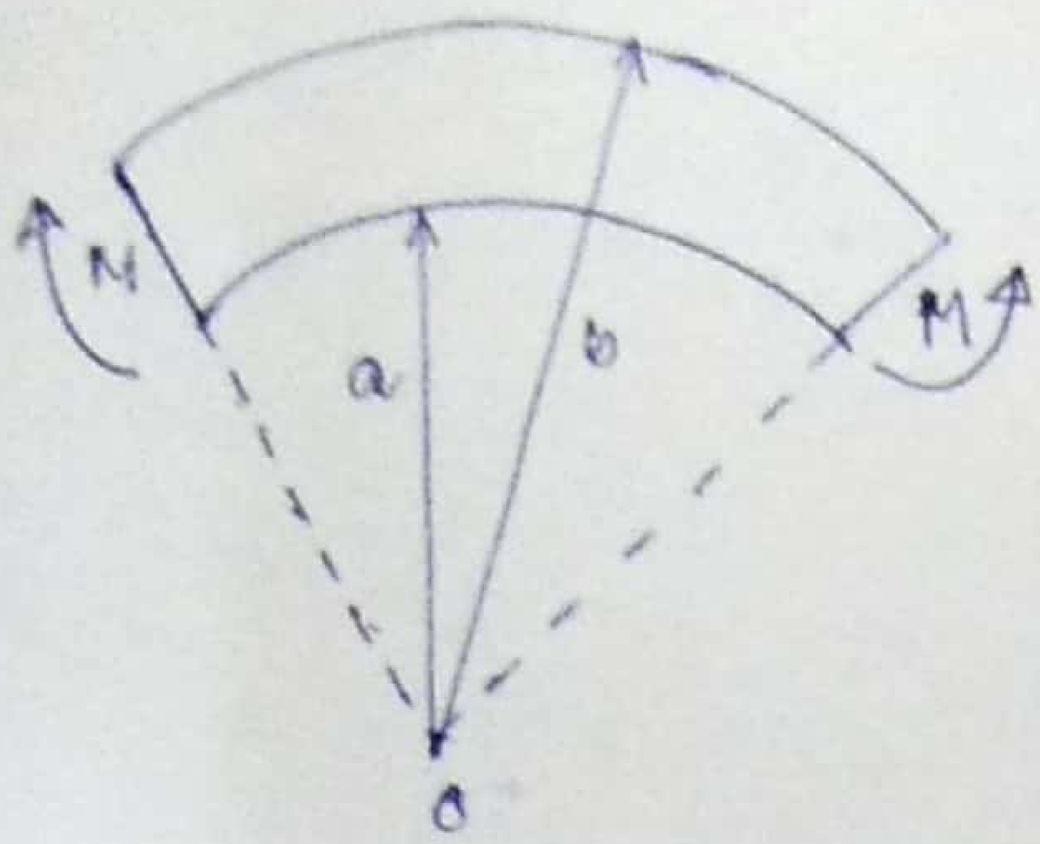
$$\sigma_{r\theta}\Big|_{r=a,\, r=b} = 0.$$

$$\sigma_{rr}\Big|_{r=a,\, r=b} = 0.$$

$$P = \int_a^b \sigma_{r\theta}\, dr. \quad \text{for } \theta = 0$$

Hence only statements 1 and 2 are correct.

10.



For the concave and convex edges free from normal force.

$$\sigma_{rr}\Big|_{r=a,\, r=b} = 0$$

Assignment Solution

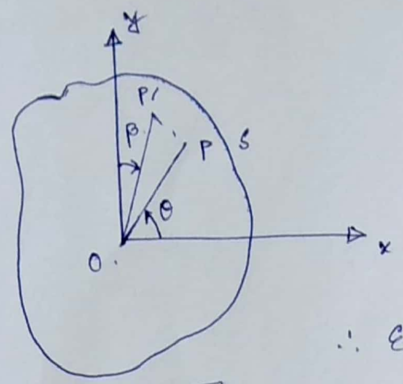
Week-8. Assignment-1

1. The assumption –

"Plane sections perpendicular to the longitudinal axis before deformation remain plane (and perpendicular to the longitudinal axis) after deformation"

– holds true for torsion of shafts having "circular cross section".

2.



$$\boxed{\alpha = \frac{\beta}{z}}$$

$u = -\alpha y z$

$v = \alpha x z$

$w = w(x,y)$ ← out of plane i.e. warping displacement.

$\therefore\ \varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0.$

$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = 0.$

$\varepsilon_{xz} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] = \frac{1}{2}\left[\frac{\partial w}{\partial x} - \alpha y\right].$

$\varepsilon_{yz} = \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right] = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \alpha x\right].$

$\sigma_{ij} = \lambda\, \varepsilon_{mm}\delta_{ij} + 2\mu\, \varepsilon_{ij}$

or, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0.$

$\sigma_{xz} = \mu\left[\frac{\partial w}{\partial x} - \alpha y\right]$

$\sigma_{yz} = \mu\left[\frac{\partial w}{\partial y} + \alpha x\right]$

} Number of non zero stress component = 2

3. We get the expressions of σ_{xz} & σ_{yz}

$$\frac{\partial \sigma_{xz}}{\partial y} = \mu\left[\frac{\partial^2 w}{\partial x \partial y} - \alpha\right] \qquad \frac{\partial \sigma_{yz}}{\partial x} = \mu\left[\frac{\partial^2 w}{\partial x \partial y} + \alpha\right].$$

Hence we get the compatibility equation.

$$-\frac{\partial \sigma_{yz}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial y} = -2\mu\alpha \quad \text{or,} \quad \boxed{\frac{\partial \sigma_{xz}}{\partial y} - \frac{\partial \sigma_{yz}}{\partial x} = -2\mu\alpha}$$

4. $\tau_{xz} = \frac{\partial \psi}{\partial y}$, $\tau_{yz} = -\frac{\partial \psi}{\partial x}$ ψ is the Prandtl Stress function

$$\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -2\mu\alpha.$$

or, $$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2\mu\alpha}$$

The given Prandtl Stress function given $\psi = ax^2 + by^2 - c^2$

$$\boxed{\begin{aligned} \tau_{xz} &= \frac{\partial \psi}{\partial y} = 2by. \\ \tau_{yz} &= -\frac{\partial \psi}{\partial x} = -2ax \end{aligned}}$$

5. If ψ is the Prandtl stress function.

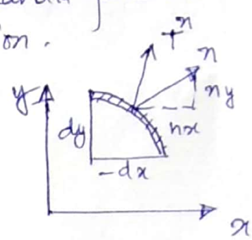
$$\tau_{xz} = \frac{\partial \psi}{\partial y} \qquad \tau_{yz} = -\frac{\partial \psi}{\partial x}$$

The compatibility Equation becomes,

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2\mu\alpha} \Rightarrow \boxed{\nabla^2 \psi = -2\mu\alpha}$$

This is Poisson Equation

① In stress formulation of a torsional problem use of Prandtl function converts the compatibility equation to Poisson equation.



② In case of Traction free boundary condition

$$\boxed{\overset{n}{T} = \sigma \cdot n = 0}.$$

$\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{xy} = 0.$

$$T_z = \tau_{xz} n_x + \tau_{yz} n_y = 0.$$

$n_x = \frac{dy}{ds}$ $n_y = (-)\frac{dx}{ds}$.

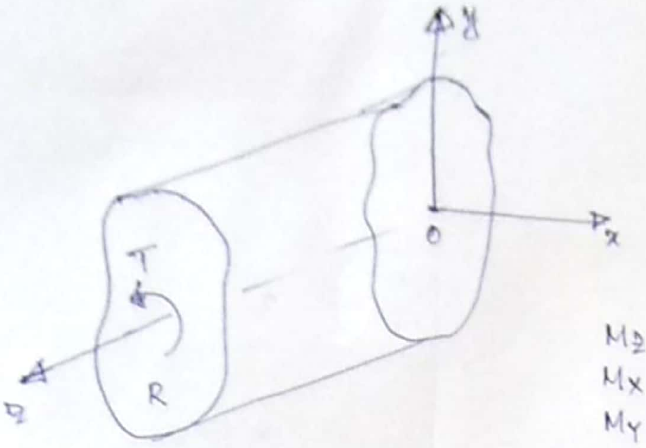
or, $$T_z = \frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\cdot\frac{dy}{ds} = 0.$$

or, $$\boxed{\frac{d\psi}{ds} = 0 \text{ on } S}$$

The Prandtl Stress function is constant over the surface.

Hence Both the statements are true.

8.



$M_z = T$
$M_x = 0$
$M_y = 0.$

ψ = Prandtl stress function

Now, $M_z = \iint_R \left(x T_y^n - y T_x^n \right) dx\,dy = T.$

$$T = -\iint_R \left(x \frac{\partial \psi}{\partial x} + y \frac{\partial \psi}{\partial y} \right) dx\,dy.$$

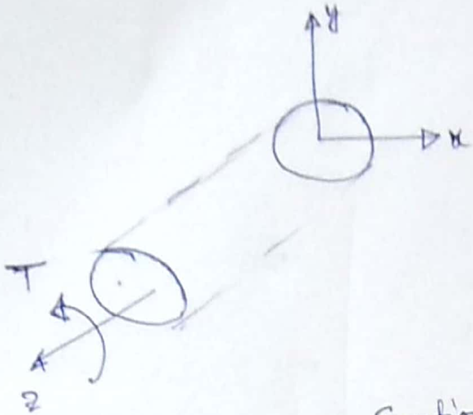
Now, $\iint_R x \frac{\partial \psi}{\partial x} dx\,dy = \iint_R \frac{\partial}{\partial x}(x\psi)\, dx\,dy - \iint_R \psi\, dx\,dy.$

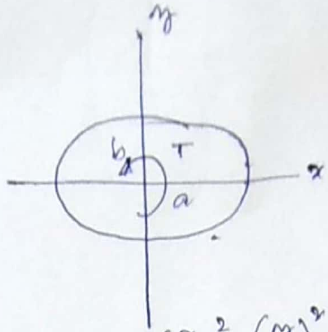
$$= \oint_S \underbrace{x \psi n_x}_{0}\, ds - \iint_R \psi\, dx\,dy$$

Similarly, $\iint_R y \frac{\partial \psi}{\partial y} dx\,dy = \oint_S \underbrace{y \psi n_y}_{0}\, ds - \iint_R \psi\, dx\,dy$

$$\therefore \boxed{T = 2 \iint_R \psi\, dx\,dy}$$

7.





Equation of Ellipse ⇒ $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-1=0$.

Prandtl Stress function; $\psi = K\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right]=0$.

Applied Torque (T) = 100 kN-m.

Now, $T = 2\iint_R \psi \, dx\, dy$.

$$T = 2K\left[\iint_R \frac{x^2}{a^2}dx\,dy + \iint_R \frac{y^2}{b^2}dx\,dy - \iint_R dx\,dy\right].$$

$$T = 2K\left[\frac{I_{yy}}{a^2}+\frac{I_{xx}}{b^2}-A\right].$$

A → Area of Ellipse
I_{xx} → 2nd Moment of Area about xx.
I_{yy} → 2nd Moment of Area about yy.

$A = \pi ab$

$I_{xx} = \frac{\pi}{4}ab^3$

$I_{yy} = \frac{\pi}{4}ba^3$

or, $T = 2K\left[\frac{\pi}{4}ab + \frac{\pi}{4}ab - \pi ab\right]$.

$$\boxed{T = \frac{\pi a^3 b^3 \mu\alpha}{a^2+b^2}}$$ putting the value of K.

$\psi = K\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right]$

$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = -2\mu\alpha$.

or, $2K\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = -2\mu\alpha$

or, $K = -\frac{a^2b^2\mu\alpha}{a^2+b^2}$

For the given problem

$a = 0.4\,m$, $b = 0.2\,m$.

$K = -\frac{16\times 4\,\mu\alpha}{(16+4)} = -\frac{64\,\mu\alpha}{20}$.

$K = -\frac{16}{5}\mu\alpha \times 10^{-2}$.

8. We, Know, $T = \frac{\pi a^3 b^3 \mu \alpha}{a^2+b^2}$. $K = -\frac{a^2 b^2 \mu \alpha}{a^2+b^2}$.

For the problem, $\mu = 80$ GPa $= 80 \times 10^9$ N/m² | $T = 100$ kN-m
$a = 40$ mm, $b = 20$ mm. | ~~= 100 N-m.~~

$$\alpha = \frac{T(a^2+b^2)}{\mu (a^3 b^3)\pi} = \frac{100 \times 10^3 \times (1600+400) \times 10^{-6}}{\pi \times 80 \times 10^9 \times 64000 \times 8000 \times 10^{-9} \times 10^{-9}}$$

$= $ ~~1.554~~ 0.155 rad/m.

9. Radius of shaft (r) = 50 mm
Torque (T) = 5 kN-m.

$$\tau_{max} = \frac{2T}{\pi r^3} = \frac{2 \times 5 \times 10^3}{\pi \times (50 \times 10^{-3})^3} = 25.46 \times 10^6 \text{ N/m}^2$$

$= 25.46$ MPa.

10. warping displacement $(w) = \frac{T(b^2-a^2)}{\pi a^3 b^3 \mu} xy$. at any point (x, y) under external torque T for elliptical shaft.

a = major axis radius
b = minor axis radius.

For a circular shaft $a = b = r$

warping displacement $(w) = 0$ (

Solution

Week-9 - Assignment-1

1. $z_1 = 2+3i \qquad z_2 = 1-5i$

$z_1 \times z_2 = (2+3i) \times (1-5i)$

$= 2 - 10i + 3i - 15i^2$

$= (2+15) - 7i$

$$\boxed{z_1 \times z_2 = 17 - 7i}$$

2. $z_1 = 2+3i \qquad z_2 = 1-5i$

$\frac{z_1}{z_2} = \frac{2+3i}{1-5i} = \frac{(2+3i)(1+5i)}{\sqrt{1^2 - 25i^2}}$

$\frac{z_1}{z_2} = \frac{(2+3i)(1+5i)}{\sqrt{26}} = \frac{2+15i^2 + 3i + 10i}{\sqrt{26}}$

$$\boxed{\frac{z_1}{z_2} = \frac{-13+13i}{\sqrt{26}}}$$

3. $f(z) = (x^2 - y^2) + v(x,y)i$

$f(z) = u(x,y) + v(x,y)i$

$f(z)$ is an analytic function if

$\frac{du}{dx} = +\frac{dv}{dy}$

$\frac{du}{dy} = -\frac{dv}{dx}$

$\frac{du}{dx} = 2x$

$\frac{du}{dy} = -2y$

$\frac{dv}{dy} = 2x$

$v = \int 2x\, dy = 2xy + c(x)$

$\frac{dv}{dx} = -(-2y) = 2y$

$v = \int 2y\, dx = 2xy + c.$

$$\boxed{v(x,y) = 2xy}$$

$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 2 - 2 = 0.$

$\frac{d^2v}{dx^2} + \frac{d^2v}{dx^2} = 0$

The function $u(x,y)$ & $v(x,y)$ ~~are~~ satisfy Laplace equation.

4. A complex function $f(z) = u(r,\theta) + v(r,\theta)i$ is analytic only when

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} &= -\frac{\partial v}{\partial r} \end{aligned}}$$

Both the conditions are true

5. $f(z) = y^3 - 3x^2y + v(x,y)i$

$f(z) = u(x,y) + v(x,y)i$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

→ These conditions are to be satisfied for the function $f(z)$ to be an analytic function.

$\frac{\partial v}{\partial y} = -6xy.$

$v = -\int 6xy\,dy = -3xy^2 + c(x)$

$\frac{\partial v}{\partial x} = -(+3y^2 - 3x^2).$

$\frac{\partial v}{\partial x} = -3y^2 + \frac{\partial c(x)}{\partial x}$

$-3y^2 + \frac{\partial c(x)}{\partial x} = -3y^2 + 3x^2$

or, $\frac{\partial c(x)}{\partial x} = 3x^2$

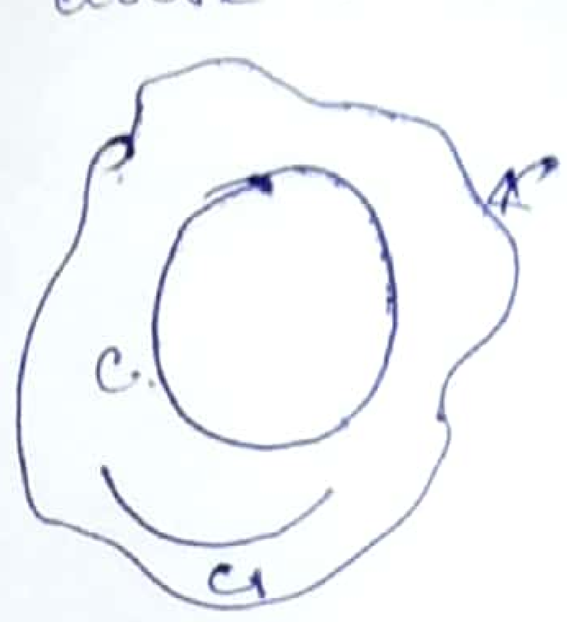
$c(x) = \int 3x^2\,dx = x^3 + c.$

Hence $\boxed{v(x,y) = x^3 - 3xy^2}$ (ignoring the constant term).

6. $f(z)$ is an analytic function in a simply connected domain A.

C is a closed curve in side the domain A.

C_1 is any arbitrary curve in side the domain A



domain A.

Hence $\oint_C f(z)dz = 0$ as it is a closed curve.

$\oint_{C_1} f(z)dz$ is path independent -- for any anti-derivative

$f(z)$ has an anti-derivative anywhere in the domain A

Hence statement 1 & 3 are correct.

7. $\int_1^3 (z-2)^3 dz$ — Let us consider $z = 2+t + 0i \quad -1 < t < 1$

$z = 1$ when $t = -1$, $z = 3$ when $t = +1$

$dz = dt$

$= \int_{-1}^{+1} t^3 dt = \left(\frac{t^4}{4}\right)_{-1}^{+1} = 0$

$$\boxed{\int_1^3 (z-2)^3 dz = 0}$$

8. $\oint_C \cos\left(\frac{z}{2}\right) dz$ — C is the circle $|z| = 1$

$\cos(z)$ is an analytic function at all points inside the domain $|z| = 1$

$f'(z) = -\sin z$ is continous everywhere inside C.

Thus by, Cauchy's theorem.

$\oint_C \cos(z)dz = 0$ where $|z| = 1$ is the domain C.

9. $z = x+iy$

$f(z) = \frac{a}{z} + bz^2$

$\overline{f(z)} = \frac{a}{\bar{z}} + b\bar{z}^2$

$\frac{a}{\bar{z}} = \frac{a}{x-iy} = \frac{a(x+iy)}{\sqrt{x^2+y^2}} = \frac{ax}{\sqrt{x^2+y^2}} + \frac{ayi}{\sqrt{x^2+y^2}}$

$b\bar{z}^2 = b(x \mp iy)^2 = b(x^2 \mp 2ixy - y^2) = b(x^2-y^2) \mp 2ibxy$

$$\overline{f(z)} = \left(\frac{ax}{\sqrt{x^2+y^2}} + bx^2 - by^2\right) + \left(\frac{ay}{\sqrt{x^2+y^2}} \mp 2bxy\right)i$$

10. $z = x+iy$

$f(z) = az + bz^2$

$\overline{f(z)} = a\bar{z} + b\bar{z}^2$

$a\bar{z} = a(x-iy) = ax - ayi$

$b\bar{z}^2 = b(x-iy)^2 = b(x^2 - 2ixy - y^2) = b(x^2-y^2) - 2ibxy$

$$\overline{f(z)} = (ax + bx^2 - by^2) - (ay + 2bxy)i$$

Solution

Week-10 Assignment-1

1. Specific heat (C_p) = 129 J/kg K.
mass (m) = 100 g.
Change in temperature (ΔT) = 50 K.

Heat energy required (Q) = $C_p m \Delta T$
$= 129 \times (100 \times 10^{-3}) \times 50$
$= 645$ J

2. Heat energy supplied (Q) = 1676 kJ
mass of water (m) = 5 kg
Change in temperature (ΔT) = 80 K.
Specific heat (C_p) = $\frac{Q}{m \Delta T} = \frac{1676 \times 10^3}{5 \times 80} = 4.19 \times 10^3$ J/kg K
$= 4190$ J/kg K

3. Heat energy supplied (Q) = 1500 J.
mass of copper ball (m) = 45 ~~kg~~ g
Specific heat (C_p) = 0.39 J/g K.
Change in temperature (ΔT) = $\frac{Q}{m C_p} = \frac{1500}{0.39 \times 45 \times \cancel{1000}}$ K
$= 85.47$ K.

4. Thermal conductivity (K) = 155 W/mK.
Density (ρ) = 2700 kg/m^3
Specific heat (C_p) = 900 J/kg K.
Thermal diffusivity (α) = $\frac{K}{\rho C_p} = \frac{155}{2700 \times 900} = 6.37 \times 10^{-5}$ m^2/s

5. In conduction process heat is transferred by means of physical contact.
In convection process heat is transferred by mass motion of any fluid.

Hence only statement 1 is correct.

6. Duhamel-Neumann constitutive relationship is expressed as

$$\sigma_{ij} = 2\mu\, \varepsilon_{ij} + \lambda\, \varepsilon_{kk}\, \delta_{ij} - (3\lambda + 2\mu)\, \alpha.(T - T_0)\, \delta_{ij}$$

where λ, μ are Lamé's constant
T, T_0 are current and ambient temperature.

7. The compatibility equation of plane strain formulation of any uncoupled thermoelastic problem is;

$$\nabla^2 (\sigma_{xx} + \sigma_{yy}) + \frac{E\alpha}{1-\nu} \nabla^2 T = 0$$

where, α is the thermal ~~diffusivity~~ coefficient.

8. Initial temperature of the mild steel bar = 300 K.
Final temperature of the mild steel bar = 375 K.
Youngs modulus (E) = 200 GPa.
Thermal coefficient (α) = 11.2×10^{-6}

The thermal stress induced $= E\alpha(\Delta T)$
$= 200 \times 10^9 \times 11.2 \times 10^{-6} \times 75$
$= 168 \times 10^6$ Pa
$= 168$ MPa

9. Change in temperature (ΔT) $= (800 - 400)$ K $= 400$ K.
Thermal conductivity $= 1.5$ W/mK.
wall thickness $= 0.20$ m.
Heat flux (q) $= \frac{1.5 \times 400}{0.20} = 3000$ W/m².
Total heat lost $= 3000 \times 0.6$ W $= 1800$ W

10. As both ends of the bar is free, there will be no thermal strain induced in the bar.
Hence thermal stress induced in the bar = 0 MPa.

Assignment Solution

Week-11 Assignment-1

1. Optically anisotropic materials differ from optically isotropic materials by being able to polarise light.

2. Photo elasticity is a full-field technique in experimental stress analysis.

3. Some transparent non-crystalline materials which are optically isotropic material in stress free state behave like an optically anisotropic material when subjected to load. This characteristic is known as temporary double refraction.

   The characteristic persist only when the material is subjected to load.

   Both of the statements are correct.

4. Wave number $(\xi) = \frac{2\pi}{T}$ T = Time period.

   frequency $(f) = \frac{1}{T}$.

   $$\boxed{\xi = 2\pi f}$$

5. $$\boxed{I_r = R\, I_i}$$

   I_r = Intensity of reflected light

   I_i = Intensity of incident light

   R = Reflection coefficient.

6. $E_1 = a_1 \cos(\omega_1 t - \phi_1)$

   $E_2 = a_2 \cos(\omega_2 t - \phi_2)$

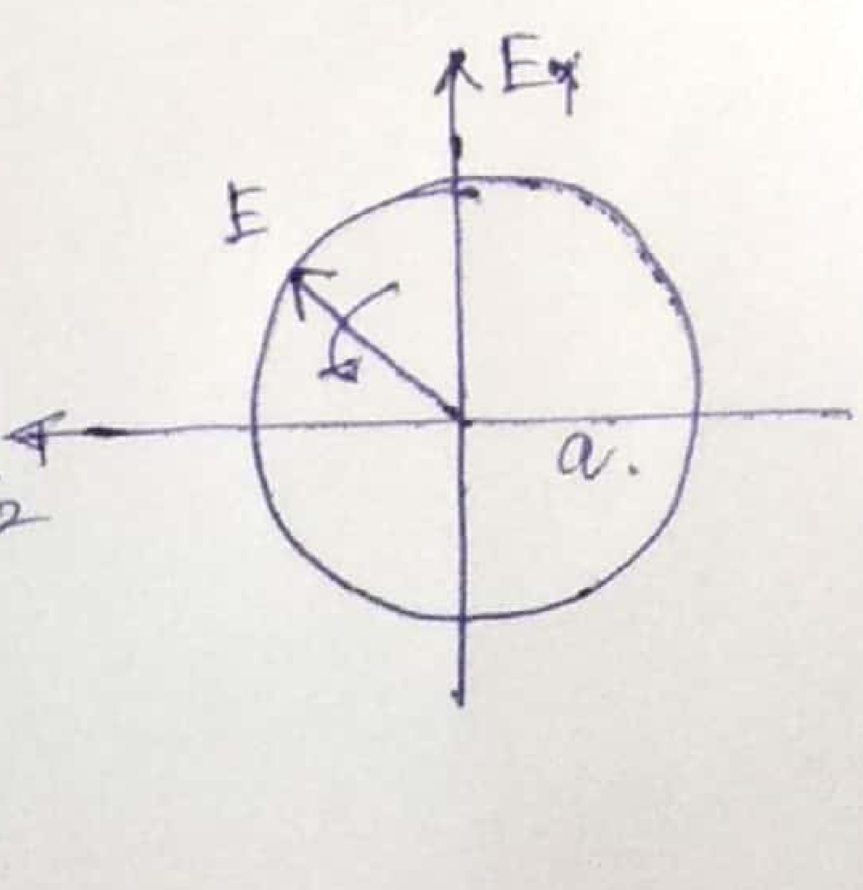
   If we eliminate the time dependency from both the expressions we end up to

   $$\frac{E_1^2}{a_1^2} - 2\frac{E_1 E_2}{a_1 a_2}\cos\frac{2\pi\delta}{\lambda} + \frac{E_2^2}{a_2^2} = \sin^2\frac{2\pi\delta}{2}$$

   where $\phi_2 - \phi_1 = \frac{2\pi\delta}{\lambda}$; $\delta = \delta_2 - \delta_1$

   Now if $a_1 = a_2 = a$ and ; $\delta = (2n+1)\frac{\pi}{4}$

   $$\boxed{E_1^2 + E_2^2 = a^2} \leftarrow \text{Equation of a circle.}$$



7. Two wavefronts are acting in the same plane;

$E_1 = a_1 \cos(\omega_1 t - \phi_1)$

$E_2 = a_2 \cos(\omega_2 t - \phi_2)$

The magnitude of the superimposed wavefront

$$\boxed{E = E_1 + E_2}$$

and it acts in the same plane

8. $E_1 = a_1 \cos(\omega_1 t - \phi_1)$

$E_2 = a_2 \cos(\omega_2 t - \phi_2)$

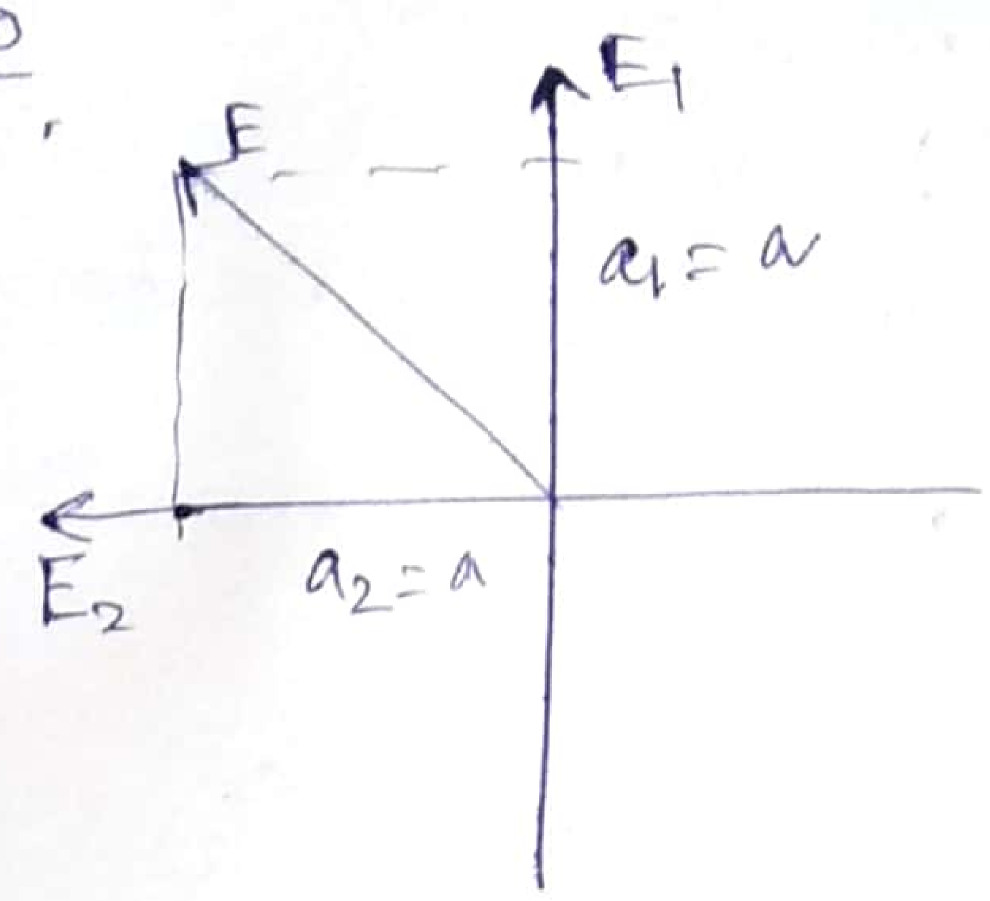
wave fronts are acting in two mutually perpendicular direction.

$$\frac{E_1^2}{a_1^2} - \frac{2E_1E_2}{a_1a_2}\cos\frac{2\pi\delta}{2} + \frac{E_2^2}{a_2^2} = \sin^2\frac{2\pi\delta}{2},$$

if $a_1 = a_2 = a$; $\delta = \frac{n\lambda}{2}$.

$$\boxed{\frac{E_1}{a_1} = \frac{E_2}{a_2}} \quad \text{or,} \quad \boxed{E_1 = E_2}$$

↑ Equation of a straight line.



9. $E_1 = a_1 \cos(\omega_1 t - \phi_1)$

$E_2 = a_2 \cos(\omega_2 t - \phi_2)$

The two wavefronts are acting in two mutually perpendicular direction.

The superimposed wave front is $\boxed{E = \sqrt{E_1^2 + E_2^2}}$

and it acts in the plane orthogal to the first two planes.

10. A polariscope tests the birefringence of a material.

The answer is <u>none of the above</u>.

Solution

Week-12 Assignment-1

1. $f(ax+by)$ is a linear function, if

$f(ax+by) = af(x) + bf(y)$ where, a and b are constants.

2. $\frac{d^4y}{dx^4} = \frac{d^2y}{dx^2} + \frac{dy}{dx}$ is a linear ordinary differential equation.

because the powers of the derivatives are 1.

3. If stress-strain constitutive relationship is a nonlinear function, the corresponding nonlinearity is termed as material nonlinearity.

4. If strain displacement relationship is nonlinear, the corresponding nonlinearity is termed as geometrical nonlinearity.

5. In case of non-linear elasticity;

1. after load reversal no residual strain retains in the material
2. The loading and unloading path is same.

only statement 2 is correct.

6. Curvature $\kappa = \dfrac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}$

7. X is undeformed configuration
x is deformed configuration

$x = \phi(X)$

The mapping ϕ is an one to one unique mapping

8. In linear elasticity the stress-strain relationship is defined only in undeformed configuration

9. In nonlinear elasticity the stress - strain relationship is defined in both deformed and undeformed configuration.

10. Non-linear elasticity encompasses both large deformation and small deformation but large rotation/ displacement problems.

****Previous Examination Questions****

1. Obtain the compatibility equation in terms of stress components for a 2-D problem of elasticity when there are no body forces. Hence obtain the general 3rd order polynomial solution for this differential equation and describe the physical stress state it depicts.
2. Evaluate the stresses and displacements for a cantilever loaded at the free end.
3. Explain stress ellipsoid and stress invariants. Evaluate the principal stress, both direct and shear, and the principal planes if the stress at a point is given as follows.

$$\begin{bmatrix} 10 & 2 & 6 \\ 2 & 8 & 4 \\ 6 & 4 & -6 \end{bmatrix}$$

4. Using general solution for an axisymmentric problem in polar coordinates obtain the stresses and displacements in a curved beam subjected to pure bending.
5. Explain the stress concentration that occurs around a hole made in an infinitely large plate. Under a uniform direct stress.
6. Explain the following
   i) Strain components in polar coordinates.
   ii) Homogeneous deformation
   iii) Rotation.

1. Explain plane stress and plane strain problems.
2. What is a strain rosette? And how is it constructed?
3. Explain Saint-Venant’s Principe.
4. Give the basic equations of equilibrium and stress-strain for axisymmetric problem neglecting body forces.
5. Explain the phenomenon “Strain Hardening”.
6. State “Maximum principal stress theory”.
7. (a) Explain the equations of compatibility.
   (b) State the stress and strain transformation laws.

   (or)
8. Establish the relationship between various constants of elasticity.
9. The state of strain at a point is given by

   $\varepsilon_X = 0.001$ $\varepsilon_Y = -0.003$ $\varepsilon_Z = 0.002$

   $\gamma_{XY} = 0.001$ $\gamma_{YZ} = 0.005$ $\gamma_{XZ} = -0.002$

   (or)
10. Determine the bending stress and shear stress at a section in a cantilever beam with a point loaded at the free end using two dimensional rectangular coordinates.
11. Using Fourier integral method, determine the solution of biharmonic equation in Cartesan coordinates.

    (or)
12. A semi-infinite elastic medium is subjected to a normal pressure of intensity “p” distributed over a circular area of radius “a” at x = 0. Determine the stress distribution by using Fourier integral.
13. Explain St.Venant’s Theory using a suitable example of torsional problem.

    (or)
14. Establish the torsional moment carrying capacity of an equilateral triangle cross sectional bar.
15. Explain any three Theories of Failure and give the governing equations. Also explain the limitations of those theories.

    (or)
16. Explain: (a) Plastic flow (b) Yield surface, and (c) Plastic potential

1.a) Explain Hooke’s law and then derive stress strain relations.

b) Define a state of (i) plane stress (ii) plane strain and explain stress & strain components. Give examples for each.

c) Write equations of equilibrium, boundary conditions & compatibility equation for 2-D problem of elasticity.

2. Obtain a 4th order polynomial solution for the differential equation in terms of stress function. Hence evaluate stresses and displacements for a cantilever beam loaded at the free end.

******<u>Previous Examination Questions</u>******

3. Derive the differential equation in terms of polar coordinates and obtain a solution for an axisymmetric problem. Obtain stress components in a circular disc with a central hole.
4. Evaluate the effect of a circular hole on stress distribution in plates subjected to uniform normal stress.
5. For a problem of bending of a curved bar by a force at the free end calculate stresses and displacements.
6. Write short notes on
 a) Stress Ellipsoid
 b) Stress invariants
 c) Principal stresses & planes for normal and shear stresses.

1. Considering as three dimensional problem of elasticity evaluate displacements in a prismatical bar under its own weight.
2. Explain the difference in behavior of a circular shaft and straight bars under torsion. Hence explain saint venants Semi inverse method. Apply the same to an elliptical cross section and obtain shear stress and displacements in the cross section.
3. How is membrane analogy applied to a problem of torsion in non-circular shafts, evaluate shear stress in a narrow rectangular section and apply the same to twist in rolled profiled steel sections.
4. If a cantilever beam is subjected to point load at the free end calculate shear stresses if the cross section is circular.
5. Explain soap film method
6. Explain briefly
   i) Torsion of hollow shaft
   ii) Strain energy of bodies
   iii) The principle of superposition
   iv) Failure theories or yield criterion in plastic behavior

1. Derive the equations of equilibrium in terms of displacements for a 3-D problem of elasticity.
2. Solve a problem of pure bending of prismatic bar as a 3-D problem of elasticity and obtain the displacements.
3. Explain membrane analogy .Apply this analogy to the problem of bar of narrow rectangular section subjected to torsion and calculate the stresses.
4. Explain the difference in behavior of a circular shaft and straight bars under torsion. Hence explain saint venants Semi inverse method. Apply the same to an elliptical cross section and obtain shear stress and displacements in the cross section**.**
5. How is membrane analogy applied to a problem of torsion in non-circular shafts, evaluate shear stress in a narrow rectangular section and apply the same to twist in rolled profiled steel sections.
6. a) Plastic deformation and molecular behaviour of material causing yielding.
   b) write the assumptions and different yield criteria and explain failure theories for elastic material.
7. Explain Saint Venants semi inverse method for evaluation of torsion in prismatic shafts. Hence calculate torsional moment and shear stresses in terms of stress function.
8. Explain membrane analogy for a obtaining behaviour of non circular shafts under torsion.
9. Calculate shear stresses and twisting moment in a narrow rectangular section. Obtain the same for a rolled profile section.
10. Write short notes on
    a. Soap Film Method
    b. Torsion of thin tubes & Hollow sections
11. Evaluate shear stresses in a rectangular section of a cantilever beam loaded at the free end.
12. Explain the different theories failure and write yield criterion for each.

1. Explain Saint Venants semi inverse method for evaluation of torsion in prismatic shafts. Hence calculate torsional moment and shear stresses in terms of stress function.
2. Explain membrane analogy for a obtaining behaviour of non circular shafts under torsion.
3. Calculate shear stresses and twisting moment in a narrow rectangular section. Obtain the same for a rolled profile section.

******Previous Examination Questions******

4. Write short notes on
   a. Soap Film Method
   b. Torsion of thin tubes & Hollow sections
5. Evaluate shear stresses in a rectangular section of a cantilever beam loaded at the free end.
6. Explain the different theories failure and write yield criterion for each.

1. Explain.
   a) i) Hooke's law ii)Compatibility Condition
   iii) Plane stress iv)Plane strain

b) Derive the Differential equation of equilibrium based on equilibrium equations.Boundary conditions, compatibility conditions for a 2 –D plane stress problem.

2. Obtain a solution for stresses in a cantilever beam with a load at the end using polynomial solution of differential equilibrium equation .Hence also obtain displacements of the beam.

3. Evaluate the stress distribution in a plate subjected to uniform tension in both directions when a small circular hole is made in the middle of the plate.

4 Derive the equations of equilibrium for a 3-D problem of elasticity.

5 Solve a problem of pure bending of prismatic bar as a 3-D problem of elasticity and obtain the displacements

6 Using saint venant semi inverse method for the problem of Torsion of straight bars derive the solution.

7 Explain membrane analogy .Apply this analogy to the problem of bar of narrow rectangular section subjected to torsion and calculate the stresses.

8. Explain briefly
   (i) Stress invariance
   (ii) Stress ellipsoid
   (iii) Principal stress& principal planes
   (iv) Homogeneous deformation

1.a) Explain Hooke's law and then derive stress strain relations.

b) Define a state of (i) plane stress (ii) plane strain and explain stress & strain components. Give examples for each.

c) Derive the differential equation for a 2-D problem of elasticity in static equilibrium.

2. Evaluate stresses and displacements for a cantilever beam loaded at the free end.

3. Write the differential equation in terms of polar coordinates for an axisymmetric problem. Obtain stress components in a circular disc with a central hole and hence evaluate.

4. Evaluate the effect of a circular hole on stress distribution in plates subjected to uniform normal stress. Hence calculate the stress concentrations in such plate.

5. For a problem of bending of a curved bar by a force at the free end calculate stresses and displacements.

6. Explain from basics
   a) Stress Ellipsoid
   b) Stress invariants and their significance
   c) Principal stresses & planes for normal and shear stresses in 3-D problem.

1.a)Define warping.

b) Derive the equations for twisting moment and shear stresses in straight bars of non-circular cross sections. Hence evaluate the same for an elliptical cross section.

2. Explain membrane analogy for torsion of prismatic shafts. Hence obtain solution to the problem of torsion. Hence obtain solution to the problem of a bar with narrow rectangular cross section.

3. Explain briefly with relevant equations
   i) Torsion of rolled profile sections
   ii) Torsion of thin tubes
   iii) Torsion of hollow sections

4. Evaluate shear stresses in a cantilever bar with a point load at the force end. Obtain stresses variation in the cross section if the bar is circular in section.

5. a) What is soap film method.
   b) Write the equation of equilibrium for a 3-D problem in elasticity in terms of displacements.

******<u>Previous Examination Questions</u>******

6. a) Derive expression for strain energy and distraction energy.
   b) Define state of plasticity
   c) Explain different theories of failure.

1. a) Obtain the strain displacement relations.
   b) Derive the D.E of equilibrium in plane stress considering body forces.
2. Explain airy's stress function, investigate the given function is stress function is not.
   $\Phi = (al^{xy} + be^{-xy} + cye^{xy} + dye^{-xy})$ x find x.
3. Investigate what problem of plane stress is satisfies by the stress function.
   $\Phi = 3f/4d\ (xy - xy^3/3d^2) + p/2\ y^2$
   Applied in the region y = o; y = d; x = o on the ride x positive.
4. Obtain the compatibility equitation is plans strain considering the body forces.
5. a) Explain stress tensor and strain tensor.
   b) The rate of stress at a point with respect xyz plane is

$$\begin{bmatrix} 10 & 4 & -6 \\ 4 & 5 & -5 \\ -6 & -5 & 2 \end{bmatrix} \text{kN/mm}^2$$

   Determine the stress tensor relation to $x^1y^1z^1$ plane by a rotation through 600 about z – axis.
6. Obtain the stress for a simply supported beam subjected to sinusoidal loading on the upper and lower edges.

1. Obtain the compatibility equation in terms of stress components for a 2-D problem of elasticity when there are no body forces. Hence obtain the general 3^{rd} order polynomial solution for this differential equation and describe the physical stress state it depicts.
2. Evaluate the stresses and displacements for a simply supported beam under uniformly distributed load
3. Using general solution for an axisymmentric problem in polar coordinates obtain the stresses and displacements in a circular disk
4. Apply the general polynomial solution to the problem of curved bar fixed at one end and bending due to a load P applied at the other end. Obtain the deflections at loaded end.
5. Evaluate the principal stress, both direct and shear, and the principal planes if the stress at a point is given as follows.

$$\begin{bmatrix} 12 & 4 & 2 \\ 4 & 6 & 0 \\ 2 & 0 & -10 \end{bmatrix}$$

6. Explain the following
   i) Strain components in polar coordinates.
   ii) Stress ellipsoid and stress invariants

1. Apply a polynomial solution to the differential equation of equilibrium to the problem of straight bar simply supported and with UDL on the entire span. Obtain the deflections at mid span .
2. From the general solution of symmetric stress distribution problem in polar coordinates derive the stresses in the case of a circular plate with a hole at center?
3. When a curved bar is bending due to force applied at one end, find out the stresses in the c/s and deformation of the bar.
4. Explain membrane analogy. Apply this analogy to the problem of bar of narrow rectangular section subjected to torsion and calculate the stress
5. Derive the saint venants solution to the problem of Torsion in straight bars and apply this solution to a bar with elliptical cross section.
6. What is meant by stress tensor. The state of stress at a point with respect to x-y-z system is

******Previous Examination Questions******

$$\begin{bmatrix} 15 & 12 & -20 \\ 12 & 20 & -10 \\ -20 & -10 & 30 \end{bmatrix} \text{ Mpa}$$

Determine the stress tensor relation to other plane by a rotation through 30^0

7. Evaluate the stress distribution and displacements in prismatic bar under its own weight treating it as a 3 – D problem

8. Explain briefly
   a) Torsion of hollow sections
   b) Soap film method

1. Apply a polynomial solution to the differential equation of equilibrium to the problem of straight bar fixed at one end and bending due to a load P applied at the other end. Obtain the deflections at loaded end.
2. From the general solution of symmetric stress distribution problem in polar coordinates derive the stresses in the case of pure bending of curved bar?
3. Evaluate the effect of a circular hole on stress distribution in infinite plate subjected to uniform tension in one direction..
4. Explain the stress distribution in rotating disk and the effect of a hole at the center of disk?
5. Explain membrane analogy. Apply this analogy to the problem of bar of narrow rectangular section subjected to torsion and calculate the stress
6. Derive the saint venants solution to the problem of Torsion in straight bars and apply this solution to a bar with elliptical cross section.
8. Evaluate the stress distribution and displacements in prismatic bar subjected to pure bending treating it as a 3 – D problem

1. Define Hooke's law and stress strain relations for a deformable body of elastic material. Obtain equilibrium equation and boundary conditions and hence arrive at compatibility condition in term of stress components for a plane stress condition.
2. Evaluate the stress components in the cross section and deformations of a simply supported beam loaded with UDL.
3. Obtain the effect of a circular hole on stress distribution in plates.
4. When a curved bar is bending due to force applied at one end, find out the stresses in the c/s and deformation of the bar.
5. Explain stress ellipsoid and stress invanants Calculate principal stresses for the following stress tensor at a point in a 3-D body.

$$\begin{matrix} 12 & 0 & 6 \\ 0 & 10 & 4 \\ 6 & 4 & 14 \end{matrix}$$

6. a) Write equations of equilibrium in term of displacements for 3-D problem of elasticity.
   b) When a prismatic bar is stretching by its own weight. obtain displacements of bar at the free end.
7. Explain membrane analogy. Apply the same to a bar of narrow rectangular section and evaluate shear stresses in cross section.
8. Explain briefly
   i) Soap film method
   ii) Torsion of rolled profiled section

1. Evaluate the displacements in pure bending of prismatic bar.
2. State and explain saint venants semi inverse method for prismatic bars under torsion. Hence arrive at shear stress and torque values in terms of stress function Ø. Applying the same to a bar of elliptic c/s obtain distribution of shear stress in the c/s and warping displacement in c/s.
3. Derive membrane analogy. Apply this to the torsion of bar of narrow rectangular cross section.
4. Evaluate the shear stress distribution in a cantilever bar of circular cross section, loaded at the fue end.
5. Explain soap film method for solving bending problem.
6. Explain

i) Torsion of thin tubes
ii) Failure theories for Elastic / Plastic behavior of materials.

******<u>Previous Examination Questions</u>******

9. Define Hooke's law and stress strain relations for a deformable body of elastic material. Obtain equilibrium equation and boundary conditions and hence arrive at compatibility condition in term of stress components for a plane stress condition.
10. Evaluate the stress components in the cross section deformations in a simply supported beam loaded with UDL.
11. Obtain the effect of a circular hole on stress distribution in plates.
12. When a curved bar is bending due to force applied at one end, find out the stresses in the c/s and deformation of the bar
    a) Write equations of equilibrium in term of displacements for 3-D problem of elasticity.
13. Explain membrane analogy. Apply the same to a bar of narrow rectangular section and evaluate shear stresses in cross section.
14. Explain briefly
    i) Soap film method
    ii) Torsion rolled profiled section
    iii) Evaluate the displacements in pure bending of prismatic bar.

7. State and explain saint venants semi inverse method for prismatic bars under torsion. Hence arrive at shear stress and torque values in terms of stress function Ø. Applying the same to a bar of elliptic c/s obtain distribution of shear stress in the c/s and warping displacement in c/s.
8. Derive membrane analogy. Apply this to the torsion of bar of narrow rectangular cross section.
9. Evaluate the shear stress distribution in a cantilever bar of circular cross section, loaded at the fue end.
10. Explain soap film method for solving bending problem.
11. Explain

iii) Torsion of thin tubes
iv) Failure theories for Elastic / Plastic behavior of materials.

7. Derive the differential equation of equilibrium for 2 – D problem of elasticity
8. Apply a polynomial solution to the differential equation of equilibrium to the problem of straight bar fixed at one end and bending due to a load P applied at the other end. Obtain the deflections at loaded end.
9. Obtain stress distribution in a rotating disk
10. Evaluate the effect of a circular hole on stress distribution in infinite plate subjected to uniform tension in one direction.
11. Derive the saint venants solution to the problem of Torsion in straight bars and apply this solution to a bar with circular cross section.
12. Evaluate the stress distribution and displacements in prismatic bar subjected to pure bending treating it as a 3 – D problem
13. Explain membrane analogy this analogy to evaluate stress distribution under Torsion of a Bar of Narrow rectangular cross section.
14. Based on saint venaints solution for Torsion evaluate the shear stress distribution in a cantilever loaded at the free end and having a circular cross section.
15. Derive the differential equation of equilibrium for 2 – D problem of elasticity

1. Apply a polynomial solution to the differential equation of equilibrium to the problem of straight bar fixed at one end and bending due to a load P applied at the other end. Obtain the deflections at loaded end.
2. Evaluate stresses in a simply supported beam cross section where a udl of q/m is acting on the beam. Also calculate maximum deflection.
3. Using a general solution to the differential equation of equilibrium in polar coordinates, calculate stresses and deflections in a circular disc with a whole at the centre.
4. Obtain stress distribution in a rotating disk
5. Evaluate the effect of a circular hole on stress distribution in infinite plate subjected to uniform tension in one direction.

****Previous Examination Questions****

1. Derive the 4^{th} order differential equation of equilibrium for a rectangular plate by explaining moment curvature relationships.
2. Obtain Navier Solution to the deflections and moments in a SS rectangular plate with a uniformly distributed lateral load.
3. Evaluate the LEVY solution for deflections to a rectangular plate with opposite edges clamped.
4. Apply a general solution to the equilibrium equation of a circular plate to a SS circular plate.
5. Obtain Navier Solution for SS Rectangular plate with pointload using strain energy formulation for deflection of plates.
6. Obtain deflection & moments in a circular plate with a hole at centrel SS on outer edge and uniformly loaded.

1. Obtain the strain – displacement relation
2. Derive the D.E of equilibriums interms of displacement components
3. a) Explain the advantages of stress tensor and strain tensor.
   b) Explain plane stress and plane strain with examples
   c) What is meant by equilibrium and compatibility conditions.
4. Considering the plane strain derive the D.E. of compatibility without body forces.
5. The state of stress at a point with respect to x,y,z system is

$$\begin{pmatrix} 10 & 5 & -15 \\ 5 & 10 & 20 \\ -15 & 20 & 25 \end{pmatrix} \text{ kN/sq.m}$$

   Determine the stress relative to x^1,y^1, z^1 coordinate systems obtained by a rotation through 45^0. about Z axis
6. What do you understand about stress function. Derive the D.E for stress function.
7. Investigate what problem of plane stress is satisfied by the stress function.
   $\Phi = 3f/4d \ (xy - xy^3/3d^2) + py^2/2$
   Applied in the region y = 0, y = d and x = 0
8. a) What are the advantages of fourier series
   b) Obtain the equation of stress function by fourier series.
9. Obtain the strain – displacement relation
10. Derive the D.E of equilibriums interms of displacement components
11. a) Explain the advantages of stress tensor and strain tensor.
    b) Explain plane stress and plane strain with examples
    c) What is meant by equilibrium and compatibility conditions.
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10	5	-15	
5	10	20	Kn/sq.m
-15	20	25	

    Determine the stress relative to x^1,y^1, z^1 coordinate systems obtained by a rotation through 45^0 about Z axis
14. What do you understand about stress function. Derive the D.E for stress function.
15. Investigate what problem of plane stress is satisfied by the stress function.
    3f(xy-xy3) + py2
    Applied in the region y = 0, y = d and x = 0
16. a) What are the advantages of fourier series
    b) Obtain the equation of stress function by fourier series.

1. Obtain the equilibrium equation in 2 D – problems in polar coordinates.
2. For a hallow cylinder under uniform pressure obtain the radial, circumferential and longitudinal stresses.
3. Obtain the stresses distribution with the effect of circular hole in a plate.
4. Explain max well bettis and castigtianos's theorems for stresses.

******<u>Previous Examination Questions</u>******

5. Derive the D.E for bending of a cantilever by terminal loads with (i) circular section and (ii) with elliptical section.
6. Draw the stress distribution for torsion of elliptical cross section.
7. For an elastic body explain the following using stress and strain components in three dimensions.

1. Principal stresses and stress ellipsoid
2. Explain STRESS Invariants and determine principal stress and max shearing stresses for the following stress state.
   $\sigma_x = 4$ N/mm^2
   $\sigma y = 2.5$ N/mm^2
   $\sigma_z = 1$ N/mm^2
3. Explain strain energy formulation.
4. Explain homogeneous deformation and rotation
5. Derive using St. Venants semi inverse method the stress function for Torsion of non circular shafts and obtain Twisting moment in term of this stress function. Hence apply this to an elliptic c/s and obtain distribution of shear stresses in a c/s.
6. Explain membrane analogy and derive its formulation for Torsion of non circular shafts. Hence obtain solution in terms of shear stresses in a bar of Narrow rectangular cross section subject6ed to Twisting moment.
7. Explain briefly
   a. Torsion of Rolled profile sections.
   b. Torsion of Hollow shafts.
   c. Torsion of Thin tubes.
8. Obtain displacements in a prismatic bar subjected to pure bending.

1. Derive the differential equation of equilibrium in term of stress for a 2 – D problem of Elasticity and write the general polynomial form of solution to the above different equations?
2. Evaluate the displacements of a cantilever beam subjected to a point load at free end?
3. From the general solution of symmetric stress distribution problem in polar coordinates derive the stresses in the case of pure bending of curved bar?
4. Explain the stress distribution in rotating disk and the effect of a hole at the center of disk?
5. How does a circular hole effect the stress distribution in a plate under uniform stress distribution .Explain and sketch the distribution ?
6. If an infinite large plate is loaded at the straight boundary with a concentrated point load .Derive the radial solution for the stress distribution in the plate .sketch the variation of stresses?

1. Derive the differential equation for equilibrium and compatibility in term of stress for a 2 – D problem of Elasticity and write the general polynomial form of solution to the above differential equations?
2. Apply a general polynomial solution of governing differential equation to the case of bending of cantilever loaded at the end, and obtain stresses, strains and displacements.
3. Write a general solution for a problem in polar coordinates when stress distribution is symmetrical about an axis. Hence obtain stresses for a circular plate with a hole at centre.
4. Obtain a solution (stress component and displacements) to the problem of rotating disk
5. How does a circular hole effect the stress distribution in a plate under uniform Stress distribution .Explain and evaluate the distribution and sketch the results
6. If an infinite large plate is loaded at the straight boundary with a concentrated point load, Derive the radial solution for the stress distribution in the plate .sketch the variation of stresses on a horizontal plane.

1. Apply a general polynomial solution of governing differential equation to the case of bending of cantilever loaded at the end, and obtain stresses, strains and displacements.
2. From the general solution of symmetric stress distribution problem in polar coordinates derive the stresses in the case of pure bending of curved bar?
3. Explain the stress distribution in rotating disk and the effect of a hole at the center of disk?

******Previous Examination Questions******

4. How does a circular hole effect the stress distribution in a plate under uniform stress distribution .Explain and sketch the stress distribution?
5. If an infinite large plate is loaded at the straight boundary with a concentrated point load .Derive the radial solution for the stress distribution in the plate .sketch the variation of stresses?
6. For the following stress tensor generate a stress ellipsoid and obtain principal stresses principal planes and hence formulate the stress invariants

$$\begin{bmatrix} 20 & 16 & 10 \\ 16 & 30 & 12 \\ 10 & 12 & 15 \end{bmatrix}$$

1. Apply a polynomial solution to the differential equation of equilibrium to the problem of straight bar fixed at one end and bending due to a load P applied at the free end. Obtain the deflections at loaded end.
2. Evaluate the effect of a circular hole on stress distribution in infinite plate subjected to uniform tension in one direction.
3. Evaluate stresses in a simply supported beam cross section where a udl of q/m is acting on the beam. Also calculate maximum deflection.
4. Obtain stress distribution in a rotating disk
5. a) Write the equation of equilibrium in terms of displacements and hence write general solution to differential equation.

   b) Determine displacements by writing strain displacement solution and hence obtain general form of displacements that include rigid body displacements.
6. Explain membrane analogy. Apply this analogy to the problem of bar of narrow rectangular section subjected to torsion and calculate the stress.
7. Explain briefly
   (i) Stress invariance
   (ii) Stress ellipsoid
   (iii) Principal stress& principal planes
   (iv) Homogeneous deformation
8. Derive the saint venants solution to the problem of Torsion in straight bars and apply this solution to a bar with circular cross section.

1. Obtain the Governing D.E for two dimensional problem in polar coordinates using compatibility.
2. a) Obtain the expressions for strain components in polar coordinates.
   b) Obtain the stress components for a thin rotating hallow disk.
3. Using general theorems obtain the expression for condition of compatiability
4. Explain Maxwell's bettis and castiglianos theorems.
5. obtain the displacements in bending of prirmatic bar subjucted to pure bending
6. Explain saint venant torsion for elliptical cross section and torsion of thin walled tubes.

1) a) obtain strain displacement relations
   b) Derive the Differential equation of equilibrium for plane stress neglecting body forces

2)a) What is meant by compatibility and obtain the condition for compatibility
   b) Considering plane strain problem obtain the expression for compatibilityinterms of stresses.

4) a) Explain airy's stress function by considering body force
   b) Explain plane stress and plane strain

5) Given the following stress function
   $Ø = -Fx/y^2\ (3d-2y)$ determine the stress components and sketch them

6) A cantilever beam of uniform cross section is subjected to a point load p at its end.
   Determine the constants C_1, C_2, C_3 if the stresses are $\sigma x = C_1 xy$; $\sigma y = 0$; $Txy = C_2 + C_3 y^2$. Also determine the strain components and find whether these are compatible or not
   Boundary Conditions are T xy at y = ± C = 0
   $\int T\, xy\, dy = -p$
   $\int \sigma x\, y\, dy = -px$

******Previous Examination Questions******

1. Derive the governing differential equation for circular plates ?
2. Obtain expression for deflection for a circular plate with a central-hole bent by moments M1 & M2 uniformly distributed along inner and outer boundaries ?
3. Derive the governing differential equation for bending of isotropic plates ?
4. Derive the governing differential equation for plates subjected to lateral loading and in-plane forces ?
5. Using finite difference techniques, find the maximum deflection and bending moment for a square plate (a x a) loaded with udl of intensity 'P' if the plate is fixed at the edges ? (consider α =a/2 and γ =0.3).
6. Find the maximum deflection for a square plate fixed at edges and loaded with udl of intensity 'P_o', using Galerkin's method? Take poisson's ratio as 0.3.

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6. Find the maximum deflection for a square plate fixed at edges and loaded with udl of intensity 'P_o', using Galerkin's method? Take poisson's ratio as 0.3.

****Previous Examination Questions****

(a) Explain about plane stress and plane strain problems. Give two examples also.

(b) Derive the compatibility equation in terms of stress for a plane stress problem. Is this equation valid for plane strain also ?

(c) The general displacement field in a body in certain coordinates is given as:-

$u = 0.015 \ x^2y + 0.03$

$v = 0.005y^2 + 0.03 \ xz$

$w = 0.003 \ z^2 + 0.001yz + 0.005$

Find all the strains for the point (1,0,2)

(a) Derive an expression for strain energy per unit volume for a two dimensional linearly elastic body for plane stress or plane strain in terms of Airy's stress functions.

(b) How do you determine the stress distribution due to cracks? Explain with a suitable example.

2 Attempt any **two** parts of the following : **10×2=20**

(a) Derive the expression for circumferential stress in a curved beam with large initial armature and subjected to pure binding. State clearly the assumptions and its limitations.

(b) A circular plate with a circular hole is simply supported around its edge and subjected to linearly varying distributed load. Derive the expressions for maximum stress.

(c) A narrow, simply supported beam of rectangular cross-section is subjected to a uniformly distributed load. Determine the stress distribution in the beam.

3 Attempt any **one** part of the following : **20×1=20**

(a) Determine the distribution of stress is a circular cylindrical shell having the ends supported by the diagraphs. The shell has been filled with oil of density **P** such that

$\mathbf{P(Q) = 10^{-pa \cos Q}}$

Where a = radius

(b) Derive the expressions for the stress resultants and displacements for the case of a cylindrical shell with a uniform pressure.

4 Attempt any **one** part of the following: **20×1=20**

(a) Derive an expression for strain energy per unit volume for a two-dimentional linearly elastic body for plane stress or plane strain in terms of Airy's stress function.

(b) How do you determine the stress distribution due to cracks? Explain with a suitable example.

******Previous Examination Questions******

1. The stress components at a point are $\sigma_x = 100$, $\sigma_y = 50$, $\sigma_z = 40$, $\tau_{xy} = 20$, $\tau_{yz} = -40$, $\tau_{zx} = -60$ MPa. Determine the resultant stress on a plane whose direction cosines are $(1/3, -2/3, 2/3)$.

2. The displacement components are given by the relations $u = x - 2y$, $v = 2x + 2y$, $w = 5z$. Show that the displacement vector is physically possible for a continuously deformed body.

3. What do you mean by inverse method in elasticity?

4. Determine the radial and shear stresses for the Airy's stress function, $\phi = \dfrac{\cos^3\theta}{r}$.

5. Show that $\nabla^2\psi = 0$ where ψ is, St. Venant's warping function.

6. A hollow tube 50 mm mean diameter and 2 mm wall thickness with a 2 mm wide saw cut along it's length is subjected to a twisting moment. If the maximum shear stress induced is 5 N/mm², find the value of the twisting moment.

7. State Engessor's theorems.

8. Write the expression of finding displacement at any section of a loaded beam using the principle of virtual force.

9. State any four advantages of true stress–strain diagram.

10. Explain soap film analogy for plastic torsion.

******Previous Examination Questions******

11. (i) Derive the Navier's equilibrium equation in Cartesian coordinates in terms of displacements. (12)

(ii) The stress components at a point are given by $\sigma_x = 200$, $\sigma_y = -240$, $\sigma_z = 160$, $\tau_{xy} = 160$, $\tau_{yz} = 100$, $\tau_{zx} = -120$ N/mm². Determine the normal strain components at this point. Assume the modulus of elasticity and Poisson's ratio of the material as 210 kN/mm² and 0.3 respectively. (4)

12. (a) Show that $\phi = \frac{q}{8c^3}\left[x^2(y^3 - 3c^2y - 2c^3) - \frac{1}{5}y^3(y^2 - 2c^2)\right]$ is a stress function and find what problems it solves when applied to the region included in $y = \pm C$, $x = 0$ on the side x positive.

Or

(b) Derive the expression for stress components in a thin plate of infinite dimension with a central circular hole under uniform uniaxial tension.

3. (a) Derive the expression for the angle of twist, shear stress at any point and hence maximum shear stress in a bar of elliptical section due to a twisting moment.

Or

(b) A thin walled box section of dimensions $2a \times a \times t$ is to be computed with a solid section of diameter d (fig. Q. 13 (b)). Find the thickness so that two sections have

(i) the same maximum shear stress for the same torque

(ii) the same stiffness.

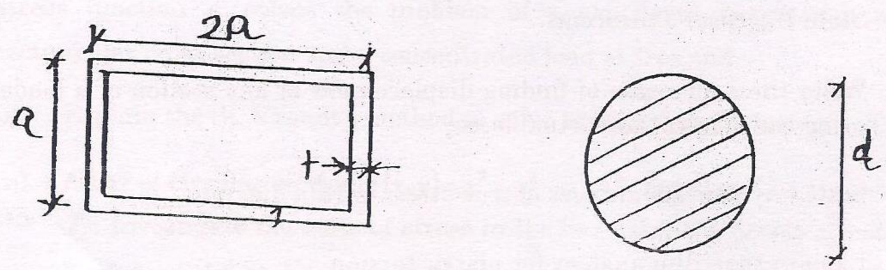


Fig. Q. 13 (b)

******<u>Previous Examination Questions</u>******

14. (a) Determine the expression for the total strain energy in terms of components of stress and strain.

Reduce the above expression for the case of (i) plane stress (ii) simple tension (iii) symmetrical bending and (iv) torsion.

Or

(b) Using Rayleigh Ritz method, find the critical load of a long column fixed at one end and free at the other end.

15. (a) (i) What do you understand by yield criteria? (4)

(ii) A thin walled tube of mean radius 100 mm and wall thickness 4 mm is subjected to a torque of 10 N-m. If the yield strength of the tube materials is 120 N/mm², determine the value of the axial load applied to the tube so that the tube starts yielding according to the Von Mises criteria. (12)

Or

(b) (i) What is meant by residual stress with respect to torsion? (4)

(ii) A solid circular shaft of 100 mm radius is subjected to a twisting moment so that the outer 50 mm deep shell yields plastically. If the yield stress in shear for the shaft material is 175 N/mm², determine the twisting couple applied and the associated angle of twist. Assume the shear modulus of the shaft material as 84 kN/mm². (12)

****Previous Examination Questions****

1. Derive Equilibrium Equations for a 3 Dimensional State of Stress?

2. The state of stress at a point is given by

   $\sigma_{xx} = 10, \quad \tau_{XY} = 8$
   $\sigma_{YY} = -6, \quad \tau_{YZ} = 0$
   $\sigma_{ZZ} = 4, \quad \tau_{ZX} = 0$

   Consider another set of Co-ordinate axis X^1, Y^1, Z^1 in which Z^1 coincides with Z-axis and X^1 is rotated by 30^0 anticlock wise from the X axis. Determine the stress components in the new system?

3. Derive Equilibrium & Compatibility equations for a body in polar co-ordinate system?

4. What is plane strain & plane stress problems? Explain with an example and derive appropriate equations for the above problems?

5. By assuming appropriate stress function "Φ", derive deflection equation for a simply supported beam carrying a u.d.l of **q** kN/m.

6. Calculate the Torque carrying capacity for an elliptical cross section by stress function approach?

7. What is membrane analogy? By Membrane analogy calculate the Torsion in Circular body?

8. Explain in Detail the Following yield criteria with neat Sketches?
   a) Maximum Shear Criteria
   b) Distortion Energy Criteria

******Previous Examination Questions******

1. Define the terms:

   (a) Homogeneous

   (b) Isotropy.

2. Write down the partial differential equation of equilibrium in polar coordinate system.

3. Mention a practical example for plane stress and plane strain problem.

4. Write the bihormonic equation in Cartesian system used to solve a torsional problem in semi-inverse approach.

5. Give the concept of membrane analogy.

6. Express the maximum shear stress and angle of twist per unit length of a thin rectangular section of size b × d.

7. Give the principle of Finite Difference method.

8. List the various energy theorems.

9. What is strain hardening?

10. Define yield criteria.

******<u>Previous Examination Questions</u>******

11. (a) The state of stress at a point in a strain material are given by the following array, $\begin{bmatrix} 9 & 15 & 24 \\ 15 & 1 & 0 \\ 24 & 0 & 2 \end{bmatrix} N/mm^2$. Determine the principle stresses and the associated direction cosines.

Or

(b) The state of stress at a point for a given reference axis xyz is given by the following array of terms. The stresses are in MPa.

$$\begin{bmatrix} 60 & 30 & -20 \\ 30 & 30 & 25 \\ -20 & 25 & 20 \end{bmatrix}.$$

(i) Determine the stress invariants

(ii) If a set of new axes $x'y'z'$ is formed by rotating about the z axis in anticlockwise direction by 45°, determine the stress components in the new coordinate system.

12. (a) Show that in the absence of body forces the displacements in problems of plane stress must satisfy:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\nu}{1-\nu}\right)\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0.$$

Or

(b) A stress function is given by $\phi = \frac{-2P}{d^3 b} xy^3 + \frac{3Pxy}{2bd} + K_1 x + K_2$. Show that stress function ϕ solves the problem of a cantilever beam with a rectangular cross section and a concentrated load at free end.

13. (a) (i) Explain the St. Venant's method to solve torsional problems. (8)

(ii) A bar of circular section $f(x,y) = x^2 + y^2 - a^2 = 0$ is twisted by torque T_z. Investigate the state of stress in the bar using a suitable stress function using St. Venant's method. (8)

Or

******Previous Examination Questions******

(b) A hollow multi-cells aluminium tube of cross section as shown in Fig. Q.13(b) resist a torque of 5kN-m. The wall thickness are $t_1 = t_2 = t_4 = t_5 = 0.5\ mm$, $t_3 = 0.75\ mm$. Determine the maximum shear stress and angle of twist per unit length. Take $G = 25\ GPa$. All dimensions in the figure are in 'm'.

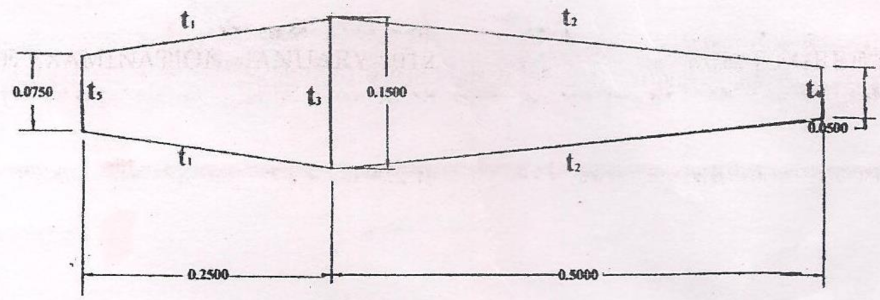


Fig. Q.13(b)

14\. (a) Explain the strain energy of a 3D stress system by applying to an elastic body subjected to stresses σ_1, σ_2 and σ_3 [principal stresses].

Or

(b) Explain in detail about the principle of virtual work. Also discuss about the applications.

15\. (a) A steel bolt is subjected to a bending moment of 240 kN-m and torque 140 kN-m. If the yield-stress in tension for the bolt material is 250 MPa, find the diameter of the bolt, according to (i) Tresca's (ii) Von Mises.

Or

(b) A member is subjected to design loads. The calculated stresses are $\sigma_x = 80\ MPa, \sigma_y = 240\ MPa, \tau_{xy} = -80\ MPa.$ The yield stress of material is $\sigma_y = 500\ Mpa.$ Determine the factor of safety as per (i) Tresca criteria and (ii) Von Mises Criteria.

******Previous Examination Questions******

1. Explain strain tensor.
2. Octahedral stresses.
3. Write short notes on Prandtl's membrane analogy.
4. Explain briefly about St.Venant's Approach for torsion.
5. Write down polynomial of the second degree.
6. Define stress concentration factor.
7. Define Winkler's constant.
8. Compare Kelvin's and Boussinesq's solutions.
9. State Von-Mises criterion
10. Write the final equation for plastic stress-strain relationship.

PART B — (5 × 16 = 80 marks)

11. (a) The state- of-stress at a point is given by the following array of terms

$$\begin{bmatrix} 9 & 6 & 3 \\ 6 & 5 & 2 \\ 3 & 2 & 4 \end{bmatrix} \text{MPa.}$$

Determine the principal stresses and principal directions.

******Previous Examination Questions******

(b) The components of strain at a point is given by

$$\varepsilon_x = 0.15\,,\ \varepsilon_y = 0.25\,,\ \varepsilon_z = 0.40\,,\ \gamma_{xy} = 0.10\,,\ y_{yz} = 0.15\,,\ \gamma_{xz} = 0.20\,.$$

(i) If the coordinate axis are rotated about z axis through 60 degree in the anticlockwise direction determine the new stress components.

(ii) Also find principal stress and its orientation.

12\. (a) (i) Discuss the effect of radial and tangential stress for a circular hole on a plate. (8)

(ii) Find the expression for normal and shear for a circular disc subjected to compression along the diameter. (8)

Or

(b) Show that the following stress function satisfies the boundary condition in a beam of rectangular cross–section of width 2h and depth d under a total shear force W. $\phi = \left[\frac{W}{2nd^3}xy^2(3d - 2y)\right]$.

13\. (a) A thin walled steel section shown in figure 1 is subjected to a twisting moment T. Calculate the shear stresses in the walls and the angle of twist per unit length of the box.

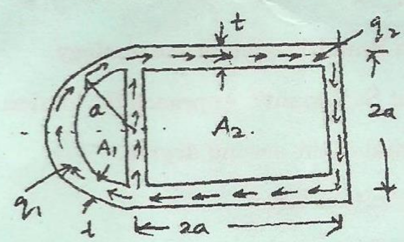


Figure – 1

Or

(b) Discuss the effect of shear and torsion on (i) elliptical cross section and (ii) triangular cross section of bar. (8+8)

14\. (a) Find out bending moment and shear force for Semi-Infinite beams with concentrated loads.

Or

(b) Find out bending moment and shear force for Infinite beams with concentrated loads.

******Previous Examination Questions******

15. (a) (i) A steel bolt is subjected to a bending moment of 300 Nm and a torque of 150 Nm. If the yield stress in tension for the bolt material is 250 MPa, determine the diameter according to (i) Tresca criteria and (ii) Von-Mises criteria. (8)

(ii) A cantilever beam 10cm wide, 12cm deep is 4m long and is subjected to an end load of 500 kg. if the $\sigma\varepsilon$ curve for the material is given by $\sigma = 7000(\varepsilon)^{0.2}$ (in kg cm unit) determine the maximum stress method and the radius of curvature. (8)

Or

(b) The state of stress at a point is given by $\sigma_x = 70$ MPa, $\sigma_y = 120$ MPa and $\tau_{xy} = 35$ MPa, if the yield strength for the material is 125 MPa, check weather yielding will occur according to Tresca's and Von Mises condition.

****Previous Examination Questions****

1 a. Derive the equations of equilibrium for a 3-D stress state. **(10 Marks)**

b. A point P in a body is given by

$$Z = \begin{bmatrix} 100 & 100 & 100 \\ 100 & -50 & 100 \\ 100 & 100 & -50 \end{bmatrix} mN/mm^2$$

Determine the total stress, normal stress and shear stress on a plane which is equally inclined to all the three axes. **(10 Marks)**

2 a. What is meant by stress invariants? With a sketch show that stress invariants are the same. **(10 Marks)**

b. The state of stress at a point is characterized by

$$Z = \begin{bmatrix} 12 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 10 \end{bmatrix} MPa.$$

Determine the principle stresses and directions for any principal stress. **(10 Marks)**

3 a. Derive the compatibility relation of strain in a 3-D elastic body. What is its significance? **(10 Marks)**

b. The state of stress at a point is given by

$\sigma_x = 200$ MPa ; $\sigma_y = -100$ MPa and $\sigma_z = 50$ MPa

$\tau_{xy} = 40$ MPa ; $\tau_{yz} = 50$ MPa and $\tau_{zx} = 60$ MPa.

If $E = 2 \times 10^{+5}$ N/mm² and $G = 0.8 \times 10^5$ N/mm². Find out the corresponding strain components from Hook's law. Take $\gamma = 0.2$. **(10 Marks)**

4 a. Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$ for a 2-D elastic body. **(10 Marks)**

b. What is stress function (ϕ)? Show that $\nabla \overset{\uparrow}{\phi} = 0$. **(10 Marks)**

PART – B

5 a. Derive the stress components for a plate with circular hole subjected to an uniazxial load. **(10 Marks)**

b. Derive the equilibrium equation in cylindrical coordinates for 2-D elastic body. **(10 Marks)**

6 a. Starting from the fundamentals derive the expression for hoop and radial stresses for a rotating hollow disc. **(10 Marks)**

b. Show that $M_t = GJ\theta$ in torsion of shafts with usual notations. Where G – modulus of rightly J – polar moment of inertia and θ - angular twist for unit length. **(10 Marks)**

7 a. Write the thermo elastic stress-strain relationships for 3-D elastic body. **(10 Marks)**

b. Derive the thermal stresses in a thin circular disc. **(10 Marks)**

8 Write a short notes on : a. Saint –Venants principle
b. Plane stress and plane strain
c. Principle of super –position
d. Membrane analogy. **(20 Marks)**

* * * * *

******<u>Previous Examination Questions</u>******

1. Define body forces with examples.
2. State any two examples each for plane stress and strain problems.
3. What is Airy's stress equation?
4. Write the polynomial equation for first and second degree functions if $\varphi = a_1x + b_1y$.
5. State the membrane analogy for torsion.
6. Define warping torsion.
7. Define virtual work.
8. What is strain energy?
9. What is plastic potential?
10. State the assumptions in yield criteria.

******<u>Previous Examination Questions</u>******

11. (a) (i) The strain components at a point are given by

$$\xi_x = 10xy + 12z \qquad r_{xy} = 4xy^2$$

$$\xi_y = 6xy^2 + 2yz \qquad r_{xy} = 2yz^2$$

$$\xi_z = 2x^2z + 2y \qquad r_{zx} = 2xz^2$$

Verify whether the compatibility equations are satisfied or not. (6)

(ii) The strain components at a point are given by

$$\begin{bmatrix} 10 & 15 & 20 \\ 15 & 25 & 15 \\ 15 & 15 & 30 \end{bmatrix} \text{MPa}$$

If the system is rotated by 45° about the z-axis in the anticlockwise direction, find the new stress tensor. (10)

Or

(b) (i) Explain generalized Hooke's law. (6)

(ii) Derive the equations of equilibrium and compatibility conditions in Cartesian co-ordinates for a two-dimensional stress field. (10)

12. (a) The stress tensor at a point is given by

$$\begin{bmatrix} 10 & 6 & -12 \\ 6 & 16 & 9 \\ -12 & 9 & 21 \end{bmatrix} \text{MPa}$$

Determine the principal stresses and principal planes. (16)

Or

(b) Apply the stress function $\Phi = -\left(\frac{F}{2hd^3}\right)xy^2(3d-2y)$ on a beam of rectangular section of breadth '$2h$' and depth 'd'. Determine what kind of problem is solved by this stress function. Is the solution perfect or imperfect? Comment on the results. (16)

******Previous Examination Questions******

13. (a) A closed thin walled tube has a perimeter 'L' and a uniform wall thickness 'h'. An open tube is made by making fine silt in it. Show that when the maximum shear stress is the same in both closed and open tubes,

$$\left(\frac{T_{open}}{T_{closed}}\right) = \frac{LH}{6A} \text{ and}$$

$$\left(\frac{\theta_{open}}{\theta_{closed}}\right) = \frac{2A}{LH}$$

where A is the silt. (16)

Or

(b) (i) Obtain the St. Venant's torsion equation and state how will you obtain the shear stresses and angle of twist. (7)

(ii) A thin walled box section having dimensions 200 mm × 100 mm × 't' mm is to be compared with a solid circular section of diameter 100 mm. Determine the thickness 't' so that the two sections have

(1) Same maximum shear stress for the same torque

(2) The same stiffness. (9)

14. (a) A square bar of cross section 60 mm × 60 mm is subjected to a twisting moment of 180 Nm at the ends. G = 80 GPa. Find the maximum shear stress and the angle of twist per unit length. Adopt strain energy method and proceed from fundamentals. (16)

Or

(b) (i) Briefly discuss Rayleigh-Ritz method. (5)

(ii) Assuming a suitable equation for the deflection curve, determine the deflection of a cantilever beam of span 'l', carrying a concentrated load 'P' at the free end. (11)

******Previous Examination Questions******

14. (a) Compute the first three natural frequencies and the corresponding mode shapes of the transverse vibrations of a uniform beam if the ends are simply supported. Proceed from fundamentals and derive any equation that you may adopt.

Or

(b) (i) For a cantilever beam with mass and stiffness matrices as given below, determine the fundamental frequency by Rayleigh's method. (6)

$$[m]=\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0.5m \end{bmatrix}; [K]=\begin{bmatrix} 2K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{bmatrix}$$

(ii) Determine the first two modes of the above problem by Rayleigh-Ritz method by assuming,

$$[\bar{\varphi}]=\begin{bmatrix} 1.00 & 1.00 \\ 0.670 & -0.68 \\ 0.200 & -1.33 \end{bmatrix}.$$ (10)

15. (a) Describe briefly how will you idealise and formulate a structure subjected to blast loading.

Or

(b) Write short notes on the following :

(i) Deterministic analysis of Earthquake

(ii) Gust phenomenon.

******Previous Examination Questions******

1. (i) Let x_1, x_2, x_3 be rectangular Cartesian co-ordinates and θ_1, θ_2, θ_3 be spherical polar co-ordinates having the following relationship:

$$x_1 = \theta_1 \, Sin\theta_2 \, Cos\theta_3; \; x_2 = \theta_1 \, Sin\theta_2 \, Sin\theta_3; \; x_3 = \theta_1 \, Cos\theta_2$$

Get the components of Euclidian Metric tensor and the length of the line element. (12)

(ii) What do you understand by Cauchy's Stress Ellipsoid? Explain. (8)

2. (i) Derive the relation between the Lame's Coefficient and the elastic constants. (10)

(ii) State the conditions under which the following is the possible system of strains:

$$\epsilon_{xx} = a + b\,(x^2+y^2) + x^4 + y^4$$

$$\epsilon_{xx} = \alpha + \beta\,(x^2+y^2) + x^4 + y^4$$

$$\gamma_{xy} = A + Bxy\,(x^2+y^2 - C^2)$$

$$\gamma_{yz} = 0; \;\; \gamma_{xz} = 0; \;\; \epsilon_{zz} = 0$$ (10)

3. As a result of measurements made on the surface of a machine component with strain gages oriented in various ways, it was established that the principal strains on the free surface are $\epsilon_a = +400 \times 10^{-6}$; $\epsilon_b = -50 \times 10^{-6}$.

   (i) Calculate the value of maximum in plane shearing strain.

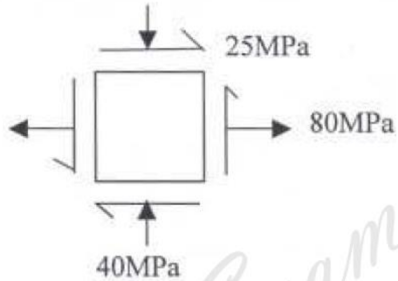
   (ii) Find absolute maximum shearing strain for the system (Given that $\sigma_c = 0$ for the free surface and Poisson ratio, $\nu = 0.3$). (20)

4. (i) Explain the development of Tressca Yield criteria. (10)

(ii) Write a short note on Plastic stress – strain relations. (10)

******Previous Examination Questions******

5. A state of plane stress shown in figure occurs at a critical point of a steel machine component.



(i) Determine whether the machine will fail or not if the tensile yield strength is σ_y = 250MPa for the grade of steel used by using maximum shearing stress criteria.

(ii) Determine the factor of safety with respect to yield using both the maximum shearing stress criteria and maximum distortion energy criteria. (20)

6. (i) What is a Viscoelastic material. Explain the different ways to model its behaviour. (10)

(ii) Explain the true Stress – strain curve for a ductile material. Also, illustrate the influence of Bauschinger Effect, strain rate and temperature on the curve. (10)

******Previous Examination Questions******

1.a) Derive the boundary conditions in Cartesian coordinates of a three dimensional system.

b) Determine the principal stresses, maximum shear stress, octahedral normal and shear stress at a point

$\sigma_x = 4MPa, \sigma_y = 8MPa, \sigma_z = -12MPa$

$\tau_{xy} = \tau_{yz} = 0, \tau_{xz} = 2MPa$

2.a) Determine the principal strain and principal plane for the given state of strain $\varepsilon_x = 0.1, \varepsilon_y = -0.05, \varepsilon_z = -0.05, \gamma_{xy} = 0.3.\ \gamma_{yz} = 0.1$ and $\gamma_{xz} = -0.08$

b) Write down the strain transformation formula. The state of strain is given by $\varepsilon_x = -200(10)^{-06}$ $\varepsilon_y = -0.05; \varepsilon_z = 0.05,\ \gamma_{xy} = 0.3.\ \gamma_{yz} = 0.1$ and $\gamma_{xz} = -0.08$

Determine the strain in another set of axis if the X axis is rotated 30 degrees in the clockwise direction.

3.a) What are the compatibility conditions. Derive the compatibility conditions in terms of strains.

b) Prove that $(\lambda + 2G)\nabla^2 e = 0$.

4. Derive the elastic curve expression of a cantilever subjected to a point load at the free end.

5.a) Derive the equilibrium equations in polar coordinates.

b) A thick cylinder is subjected to internal pressure. Prove that the circumferential stress is numerically greater than the internal pressure in the inner surface of the cylinder.

6.a) Derive the equilibrium equation and boundary he of a bar subjected to pure torsion.

b) Explain membrane analogy applied to narrow rectangular sections and derive the torsional constant and maximum shear stress for a narrow rectangle.

7.a) Discuss the yield criteria and flow rules for perfec tly plastic and strain hardening materials.

b) Discuss the elasto plastic analysis for a beam subjected to torsion.

8. Write short notes on the following:

a) Plane stress problem and plane strain problem.
b) Reciprocal theorem.
c) Principle of superposition.
d) Saint venants principle.

****<u>Previous Examination Questions</u>****

1. (*a*) Derive equations of equilibrium for 3-D cartesian system of coordinates. **8**

   (*b*) Derive strain-displacement relationships for 3-D cartesian system of coordinates. **12**

2. Derive the expressions for finding out radial stress, tangential stress and shear stress on a large plate with a small hole when subjected to direct tensile stress, *s* (uniaxial). **20**

3. Stress tensor at a point is given by:

$$\tau_{ij} = \begin{pmatrix} 10 & 15 & 20 \\ 15 & 25 & 15 \\ 20 & 15 & 30 \end{pmatrix}.$$

Find out:

(*i*) Principal stresses and their directions. 10

(*ii*) Maximum and minimum shear stresses alongwith their planes. 10

4. Find out stresses in a cantilever beam by Airy's stress function approach when it is subjected to a point load at the free end. The width of the beam is h and depth of the beam is d. 20

5. A rectangular beam 8 cm wide and 10 cm deep is 2 m long and is simply supported at the ends. The yield strength of the material is 250 MPa. Determine the value of the concentrated load applied at the midspan of the beam if (*a*) the outermost fibres of the beam just start yielding, (*b*) the outer shell upto 3 cm depth yielded, and (*c*) whole of the beam yielded. Assume the material is linearly elastic and perfectly plastic. 20

6. A solid circular shaft of 10 cm radius is subjected to a twisting couple so that the outer 5 cm deep shell yields plastically. If the yield strength in shear for the shaft material is 175 MPa, determine the twisting couple applied and the associated angle of twist. $G = 0 \cdot 84 \times 10^5$ N/mm^2. 20

7. A thick cylinder of internal radius 15 cm and external radius 25 cm is subjected to an internal pressure p

MPa. If the yield strength of the cylinder material is 240 N/mm^2, determine (*a*) pressure at which the cylinder will start yielding just at inner radius, (*b*) the stresses when the cylinder has a plastic front of radius 20 cm, and (*c*) stresses when whole of the cylinder has yielded.

Assume Tresca yield criterion and plane strain condition. **20**

3. A thin circular disc of uniform thickness is of 50 cm outer diameter and 20 cm inner diameter. Determine (*a*) speed of rotation so that the disc just starts yielding plastically at the inner radius, (*b*) stresses in the disc when disc has yielded upto 15 cm radius and (*c*) the speed for full yielding. Given: ρ=7850 kg/m^3, σ_y=250 N/mm^2 and ν=0·30. **20**

****Previous Examination Questions****

1(a) Define surface force and body force.
(b)Define plane stress in (3-D) system.
(c) Define plane strain in (3-D) system.
(d)Define stress in (2-D) and (3-D) system (4 x 5)

SECTION A

2(a)Prove that shear stress $\tau_{xy}=\tau_{yx}$,$\tau_{xz}=\tau_{zx}$ and $\tau_{yz}=\tau_{zy}$.

(b)Derive a relationship between Bulk modulus (K) and modulus of elasticity (E).

(c)Define stress function (∅).

(d)Derive the differential equation of equilibrium of (3-D).
(4 x 5)

3(a)Prove lame stress ellipsoid in three dimensional system.

(b)Show that plane strain case is reduce to plain stress case.

(c)Prove Hooks law in three dimension system.

(d)Derive a relationship between shear modulus (G) and modulus of elasticity (E). (4 x 5)

4. An elastic layer sandwitch between two perfectly rigid plate to which it is bounded. The layer is compressed between the plates in such a way at the attachments to the plates prevent lateral strain completely.find the apparent modulus of elasticity and apparent poisson'sratio.also prove that the apparent modulus of elasticity is many times of the actual modulus of elasticity.ifpoisson's ratio is slightly less than 0.5. (20)

SECTION B

5. Obtain the compatibility equation for plane strain case. (20)
6. The state of stress at a point for a given reference is given bellow as τ_{ij} .if a new set of axes is formed by rotating xyz through 45° about z-axis .Find the new stress tensor τ_{nx}.

$$\tau_{ij}=\begin{matrix} 300 & 100 & 0 \\ 100 & 200 & 0 \\ 0 & 0 & 105 \end{matrix}$$

(20)

7. For the given function (∅)

$\emptyset = -\frac{F}{d^3}xy^2(3d-2y)$, determine the stress component. (20)

M.TECH. (STE)
FLIPPED CLASSROOM ACTIVITY

ADVANCED SOLID MECHANICS

Dr V Srinivasa Reddy

TOPIC:	ASM Fundamentals
COURSE:	Advanced Solid Mechanics
DOMAIN:	Structural Engineering
TARGET AUDIENCE:	1st Year M. Tech. Structural Engineering Students
AFFILIATION:	Gokaraju Rangaraju Institute of Engineering and Technology

Out-of-class Activity Design -1

Learning Objective(s) of Out-of-Class Activity:

At the end of watching the videos student should be able to,

1. Explain the significance of ASM (Understand)
2. Classify various types of Stresses and Strains (Understand)
3. List the outcomes of Analysis of Stresses and Strains (Recall)

Key Concept(s) to be covered:

Types of stresses
Analysis of Stresses
MOHR'S STRESS CIRCLE

Out-of-class Activity Design - 2

Uploaded Video URL https://www.youtube.com/watch?v=cMdVzMRWZTk

License of Video Creative Commons Attribution license

Duration of Screencast 12:35 min

Out-of-class Activity Design - 3

Aligning Assessment with Learning Objective

Learning Objective	Assessment Strategy	Expected duration (in min)	Additional Instructions (if any)
Concepts of Stress and Strain	Q.1 Explain the significance of Stress and Strain Q.2 Classify the Stresses and Strains	5 min	Watch Video and then answer Q1, Q2. Submit the solution at teachers desk before coming to class.

Additional Slides for Out-of-Class Design

Aligning Assessment with Learning Objective

Learning Objective	Assessment Strategy	Expected duration (in min)	Additional Instructions (if any)
Examples of stress and strain?	Q.3. **What are the practical examples of stress and strain?** Q4. What are the Examples of stress and strain?	5 min	Watch Video and then answer Q3, Q4

Expected activity duration 10 min

In-class Activity Design -1

Learning Objective(s) of In-Class Activity:

At the end of the class, students will be able to

1. Explain what is stress and strain
2. Know the practical uses of stress and strain
3. List the examples of stress and strain

Key Concept(s) to be covered:

1. Stress and Strain
2. Practical purpose
3. Examples

In-class Activity Design -2

Active Learning activities planed to do

Real world problem solving using

1. Think-Pair-Share

Concept clarification using

1. Peer Instruction

In-class Activity Design -2

Peer Instruction Strategy – What Teacher Does Duration : 10 min

After watching the out of class video, students have got the basic knowledge on stress strain analysis. Now pose the two PI questions at the start of the class and provide summary of basic identities :

Q1: What do you understand by stress and Strain

1) Stress is internal resistance offered by the body
2) Strain is the measure of deformation
3) Strain is independent and Stress is dependent

Q2: What are various types of stresses and strains

1) Direct stresses
2) Shear and torsional stresses
3) Body and surface stresses

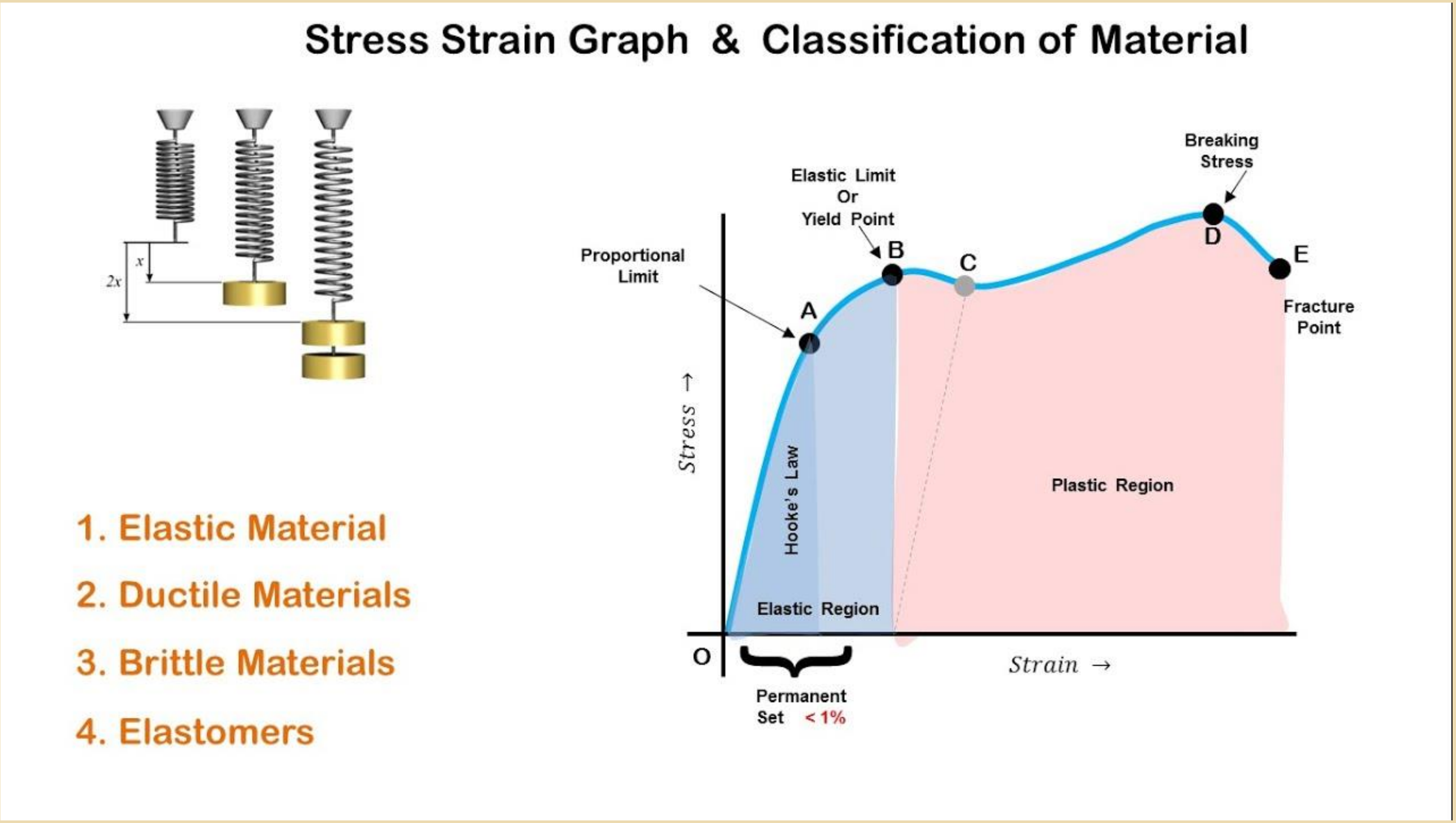
In-class Activity Design -2

Peer Instruction Strategy – What Student Does

For each question they will first think individually
Then they will discuss with peers and come to consensus
Listen to instructors explanation

In-class Activity Design -2

TPS Strategy – What Instructor does



In-class Activity Design -2

TPS Strategy – What Instructor does

Think (3 minutes)
Instruction: Observe the stress-strain curve shown in the figure. Identify the points on it

Think individually and suggest the various names and also list the stages of stress –strain phases.

In-class Activity Design -2

TPS Strategy – What Instructor does

Pair (~5 minutes)

Instruction: Now pair up and compare your answers. Agree on one final answer.

While students are pairing and discussing, instructor goes to 2~3 sections to see what they are doing.

Now try to identify the stages of the curve.

In-class Activity Design -2

TPS Strategy – What Instructor does

Share (~8 minutes)
Instructor asks a group to share their answer with class and see whether there are different answers. After sharing is done, instructor gives clarification.

In the next iteration of TPS, in the Think Phase we ask students to write the examples of stresses and strains
In the pair phase we ask students to compare the answers
In the share phase again the different answers are sought.

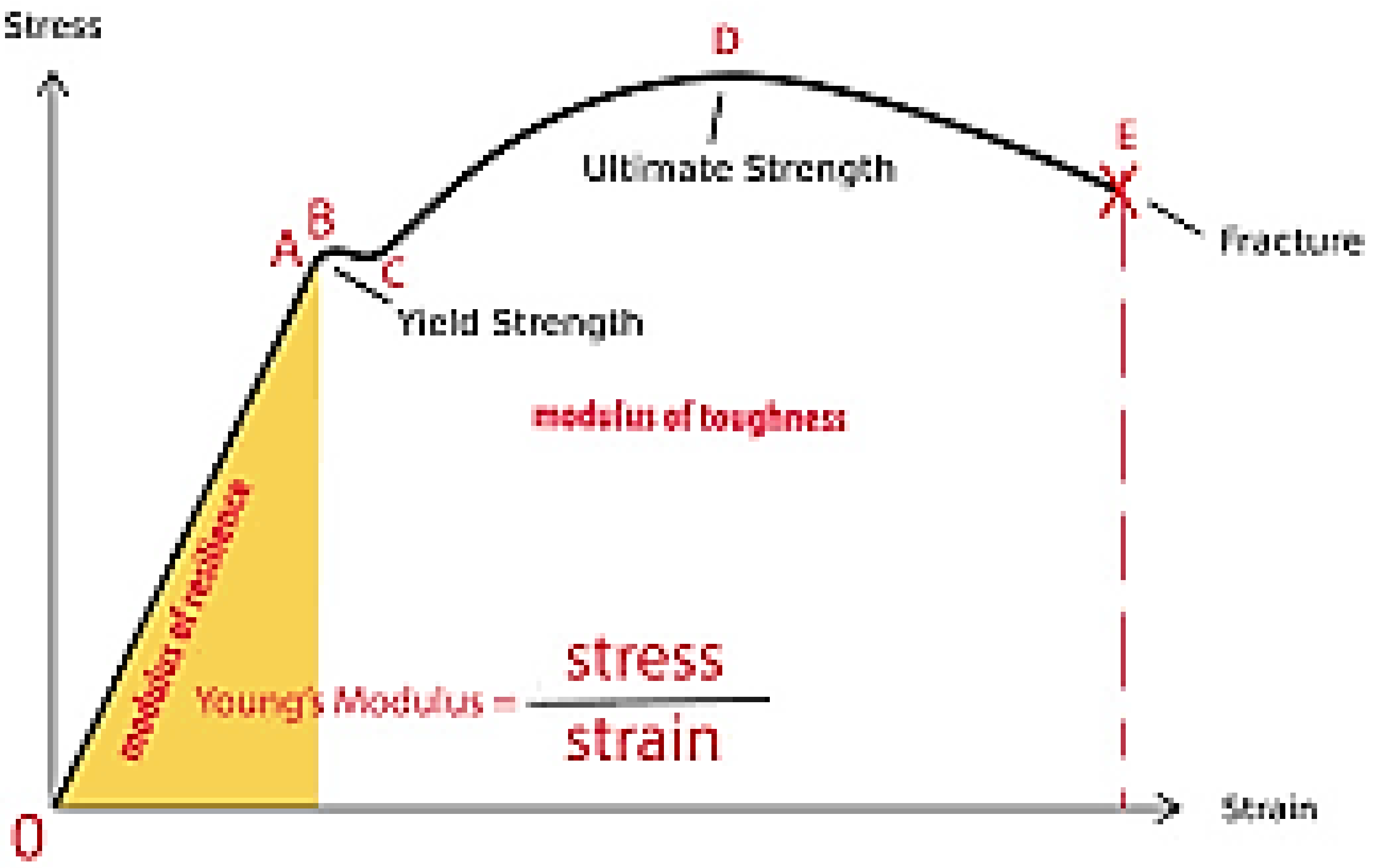
In-class Activity Design -2

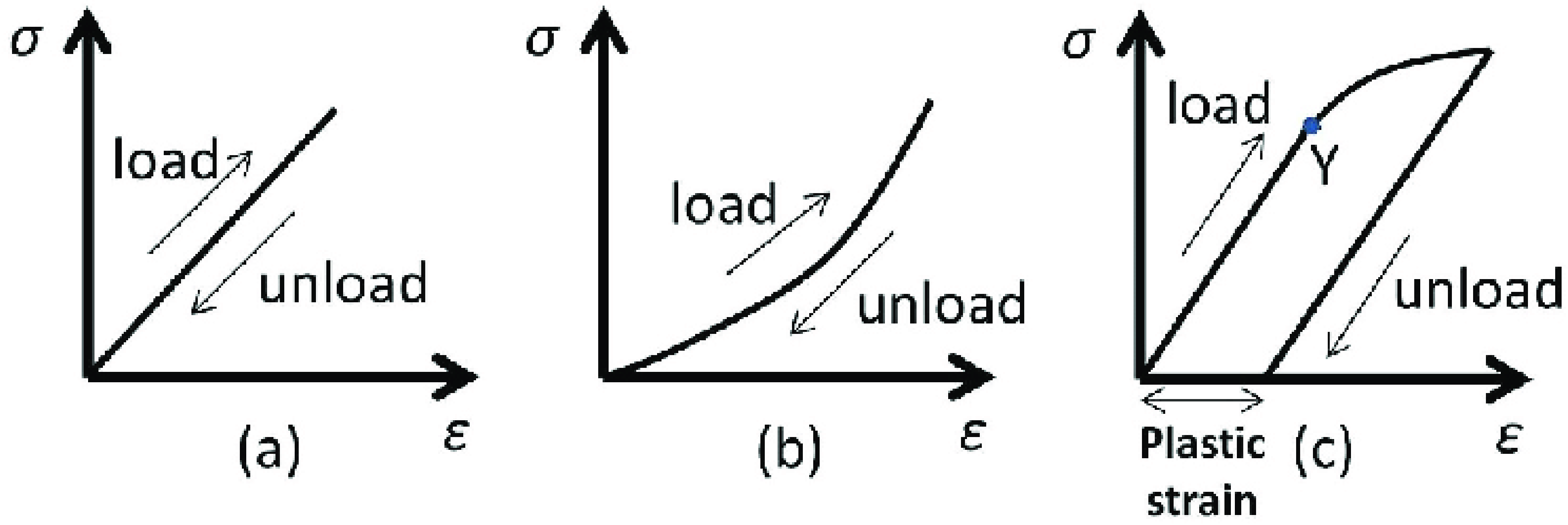
Justification for why the above is an active learning strategy

In both the above strategies, students are required to go beyond mere listening and execution of prescribed steps. They are required to think deeply about the content they were familiarized in out-of-class and do higher order thinking.
There is also feedback provided (either through peer discussion or instructor summary)

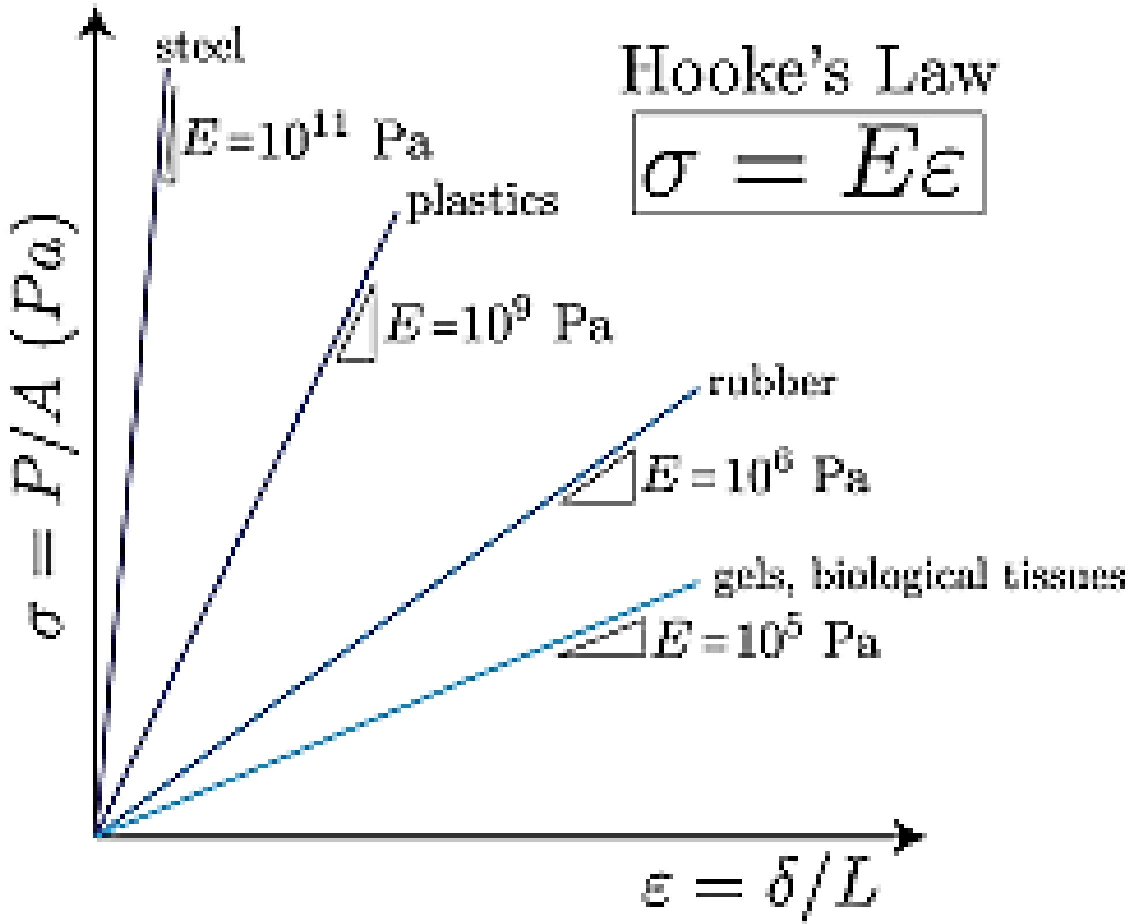
Theory of Elasticity

Dr V Srinivasa Reddy

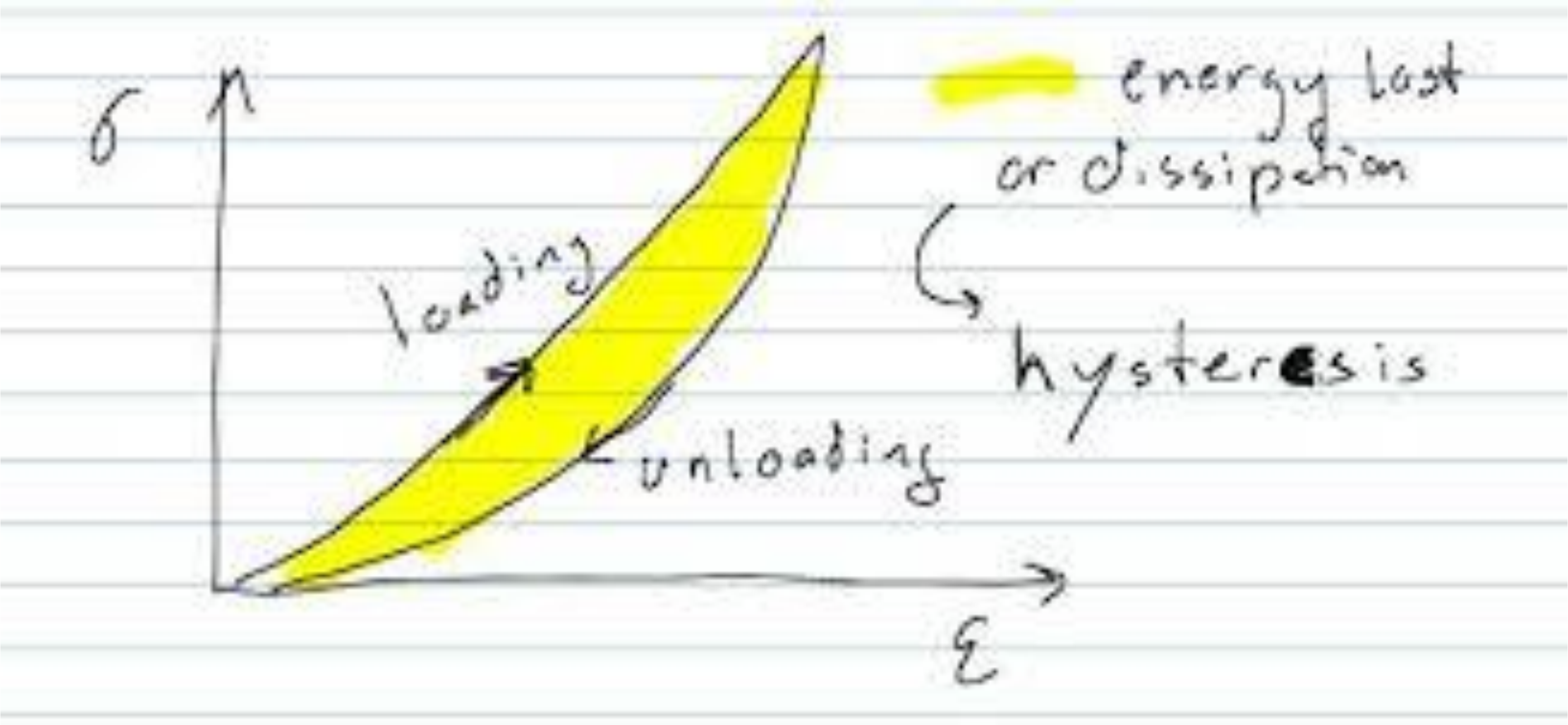

Stress
D
Ultimate Strength
E
Fracture
A
B
C
Yield Strength
modulus of toughness
modulus of resilience
Young's Modulus = stress / strain
0
Strain

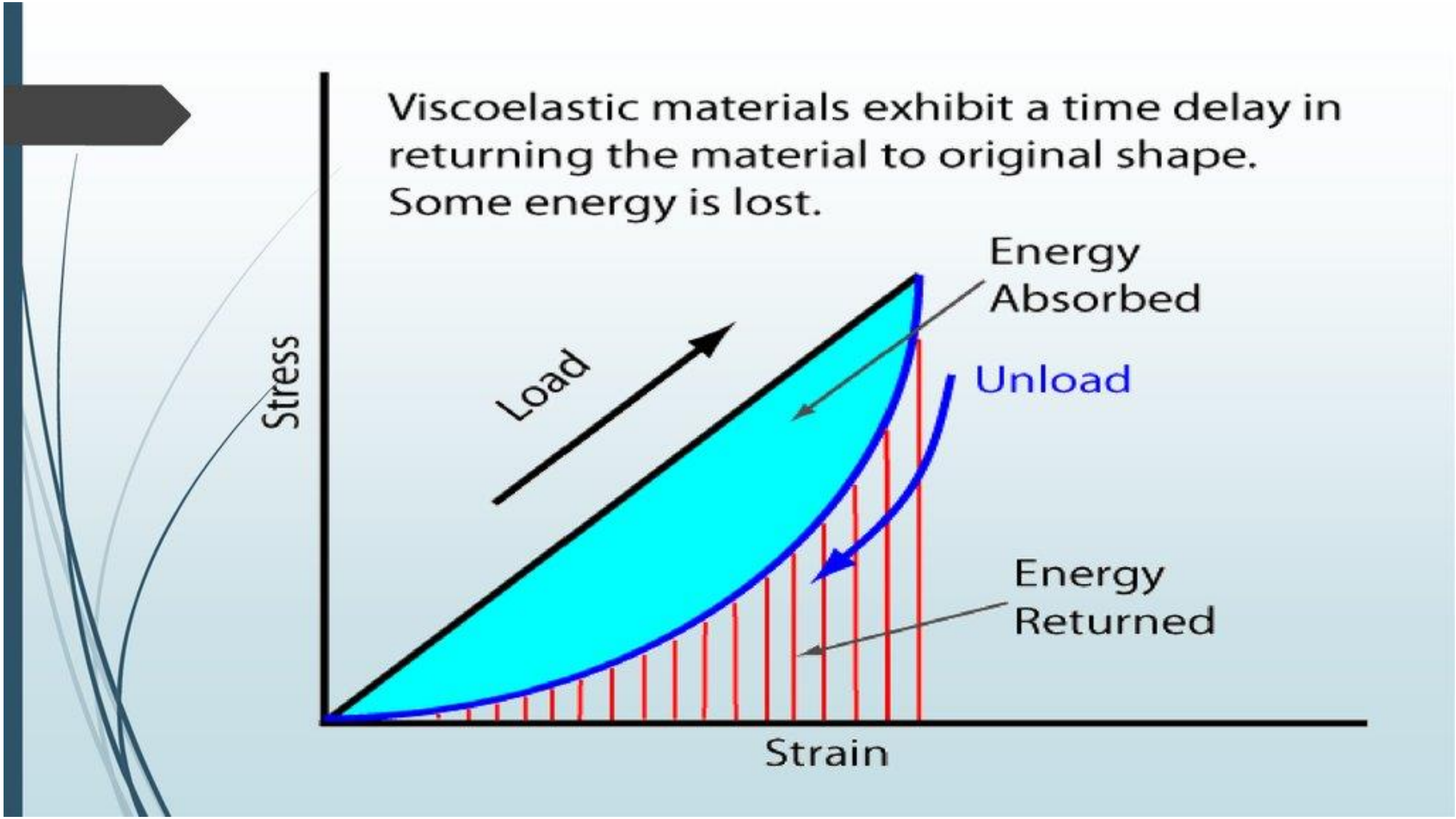


σ
load
unload
ε
(a)
σ
load
unload
ε
(b)
σ
load
Y
unload
ε
Plastic strain
(c)



Hooke's Law
$\sigma = E\varepsilon$
steel
$E = 10^{11}$ Pa
plastics
$E = 10^{9}$ Pa
rubber
$E = 10^{6}$ Pa
gels, biological tissues
$E = 10^{5}$ Pa
$\sigma = P/A\ (Pa)$
$\varepsilon = \delta/L$


σ
ε
loading
unloading
energy lost
or dissipation
hysteresis


Viscoelastic materials exhibit a time delay in returning the material to original shape. Some energy is lost.
Energy Absorbed
Load
Unload
Stress
Energy Returned
Strain

Α α	alpha		Ν ν	nu
Β β	beta		Ξ ξ	ksi
Γ γ	gamma		Ο ο	omicron
Δ δ	delta		Π π	pi
Ε ε	epsilon		Ρ ρ	rho
Ζ ζ	zeta		Σ σς	sigma
Η η	eta		Τ τ	tau
Θ θ	theta		Υ υ	upsilon
Ι ι	iota		Φ φ	phi
Κ κ	kappa		Χ χ	chi
Λ λ	lambda		Ψ ψ	psi
Μ μ	mu		Ω ω	omega

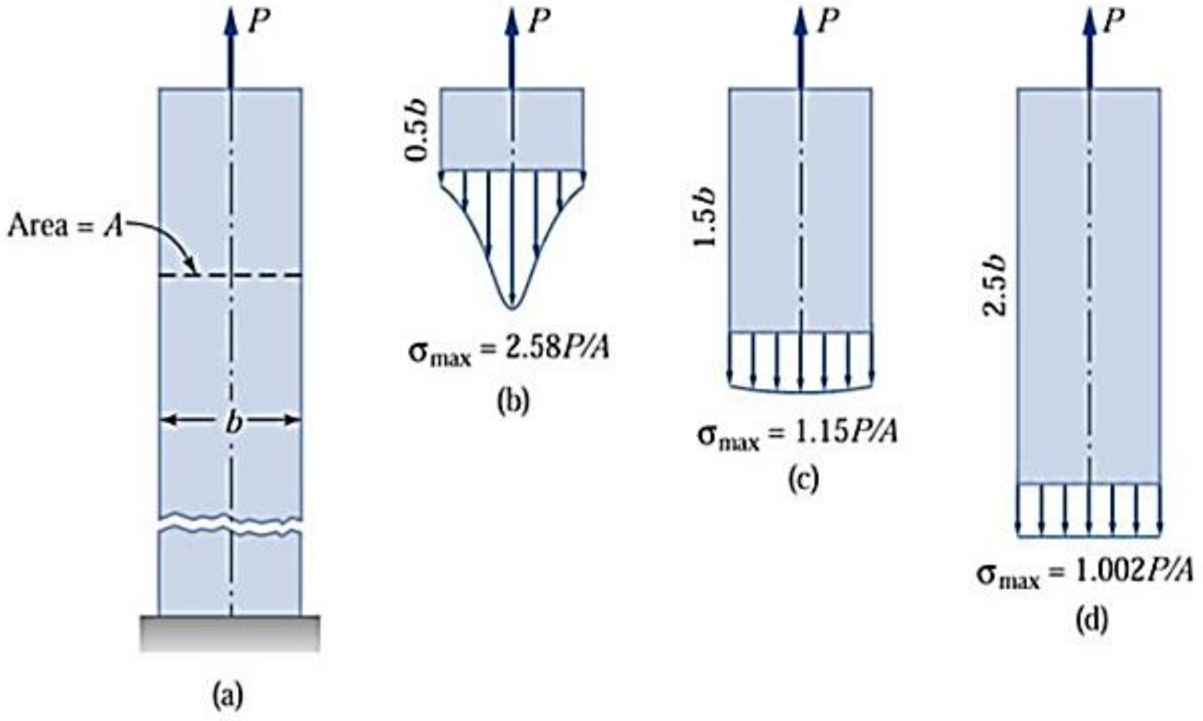
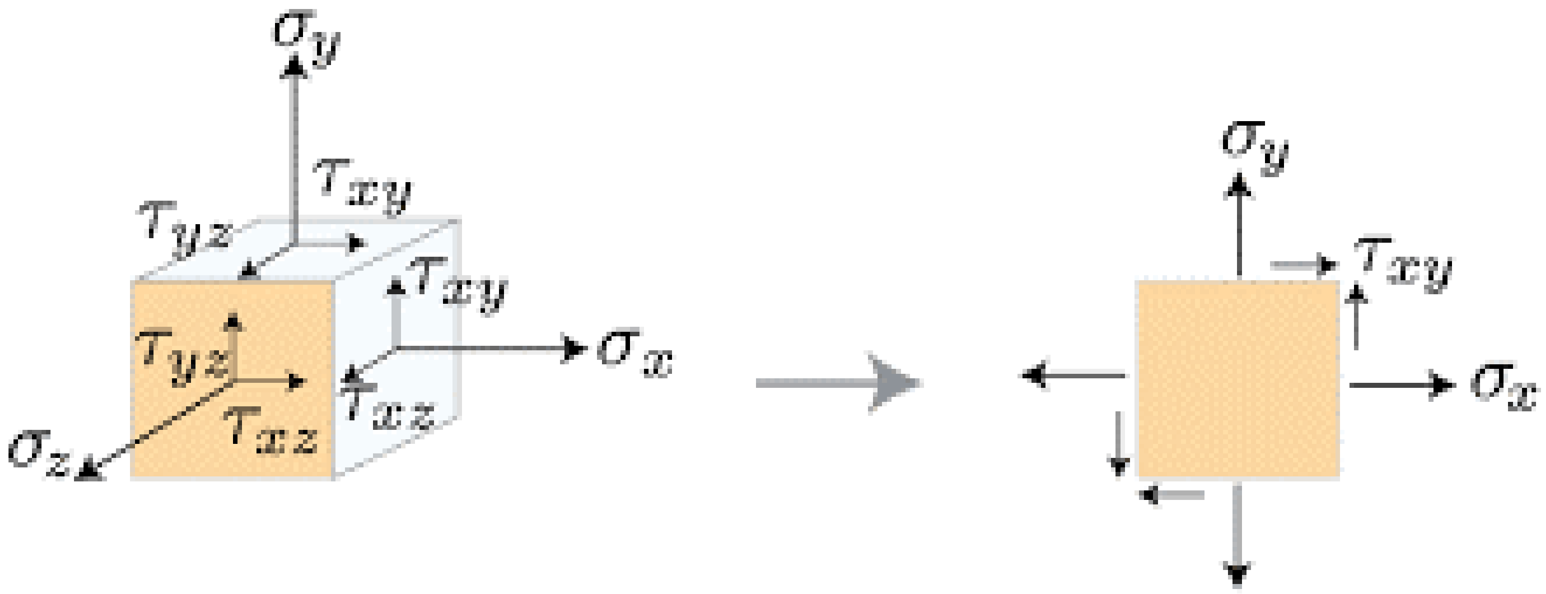


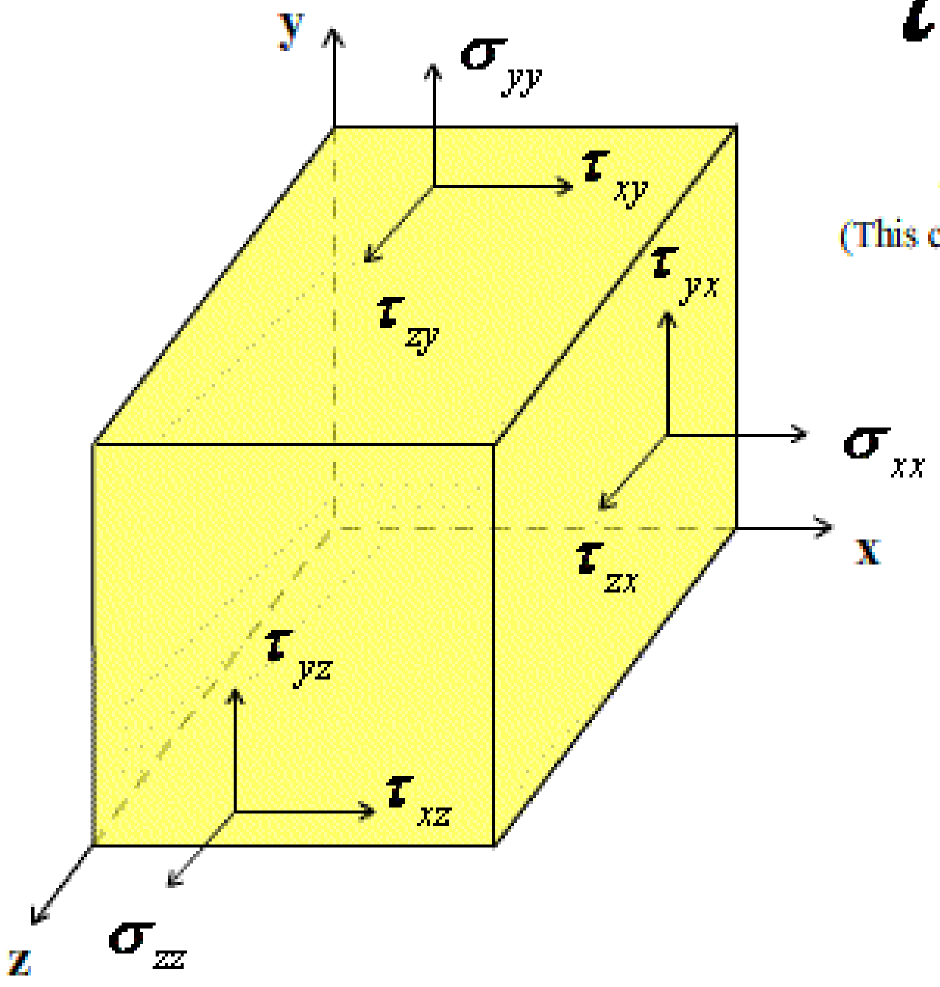
FIG. 1.7 Normal stress distribution in a strip caused by a concentrated load

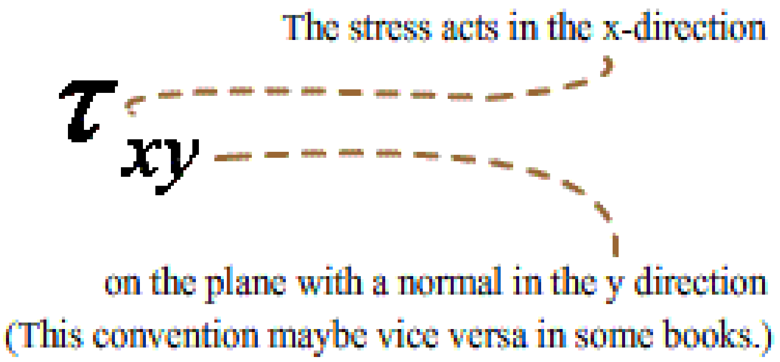
ILLUSTRATING ST. VENANT'S PRINCIPLE



3D Stress State   Plane Stress

The 9 components of a stress tensor:





$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Tensor Equation: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Matrix Equation: $\sigma_p = C_{pq}\varepsilon_q$

BASIC ASSUMPTIONS IN THEORY OF ELASTICITY

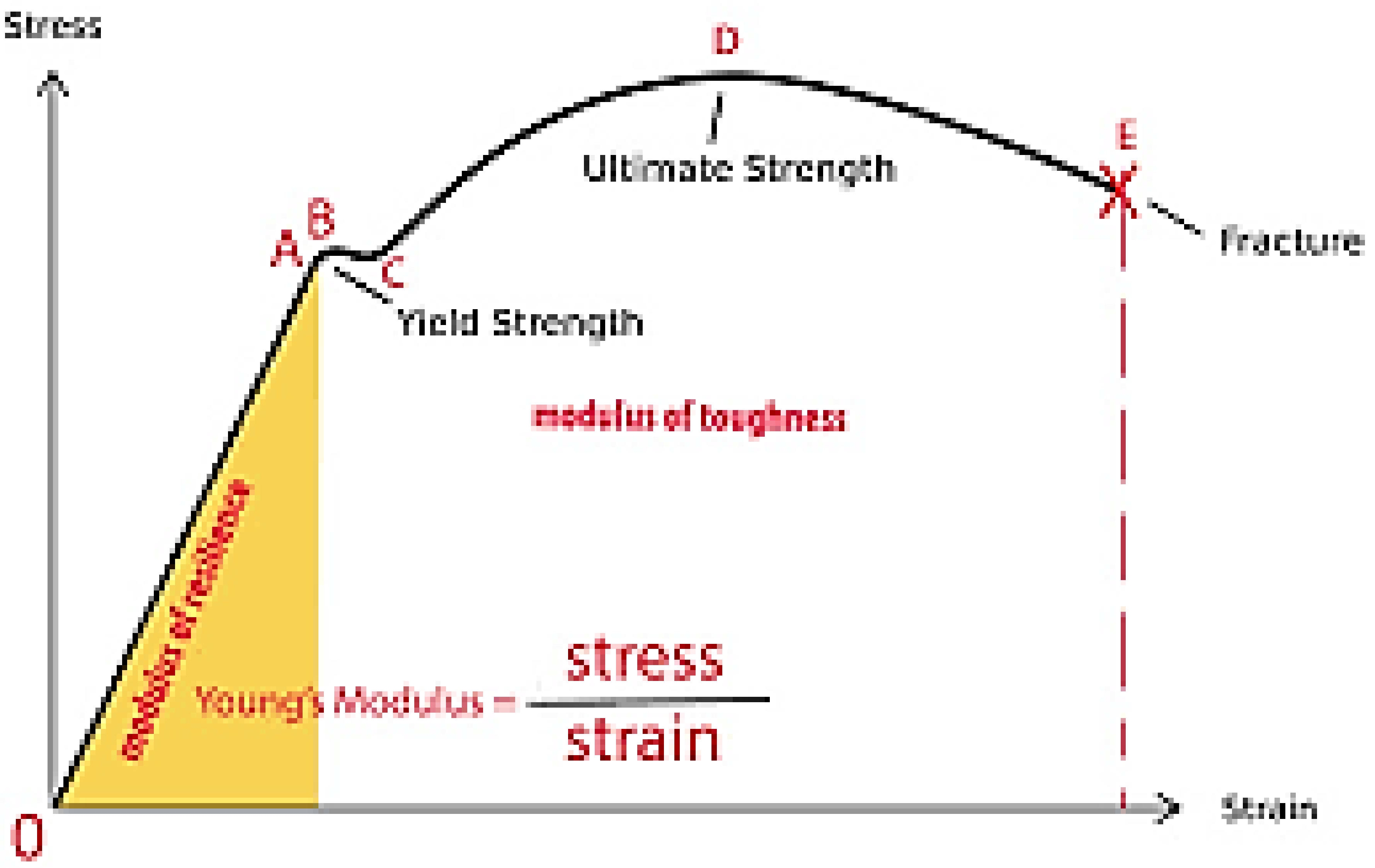
- The body is continuous
- The body is perfectly elastic
- The body is homogeneous
- The body is isotropic

  example: polycrystalline ceramics and steel

  wood and fiber reinforced composite
- The displacements and strains are small

Theory of Elasticity

Dr V Srinivasa Reddy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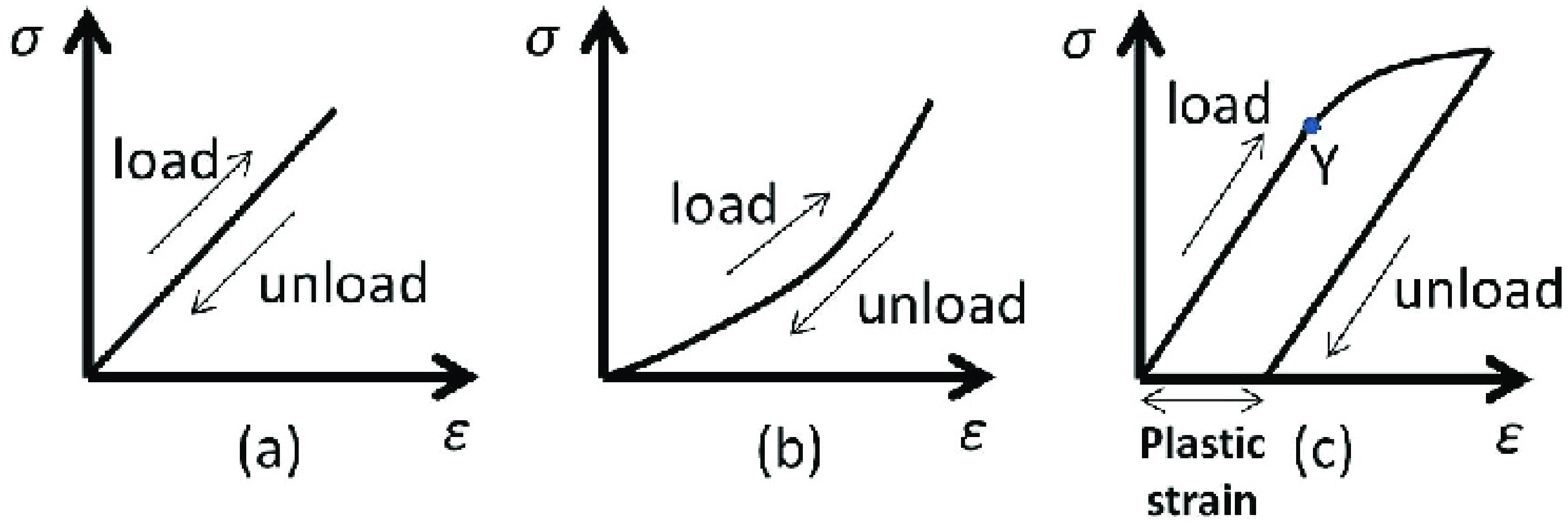
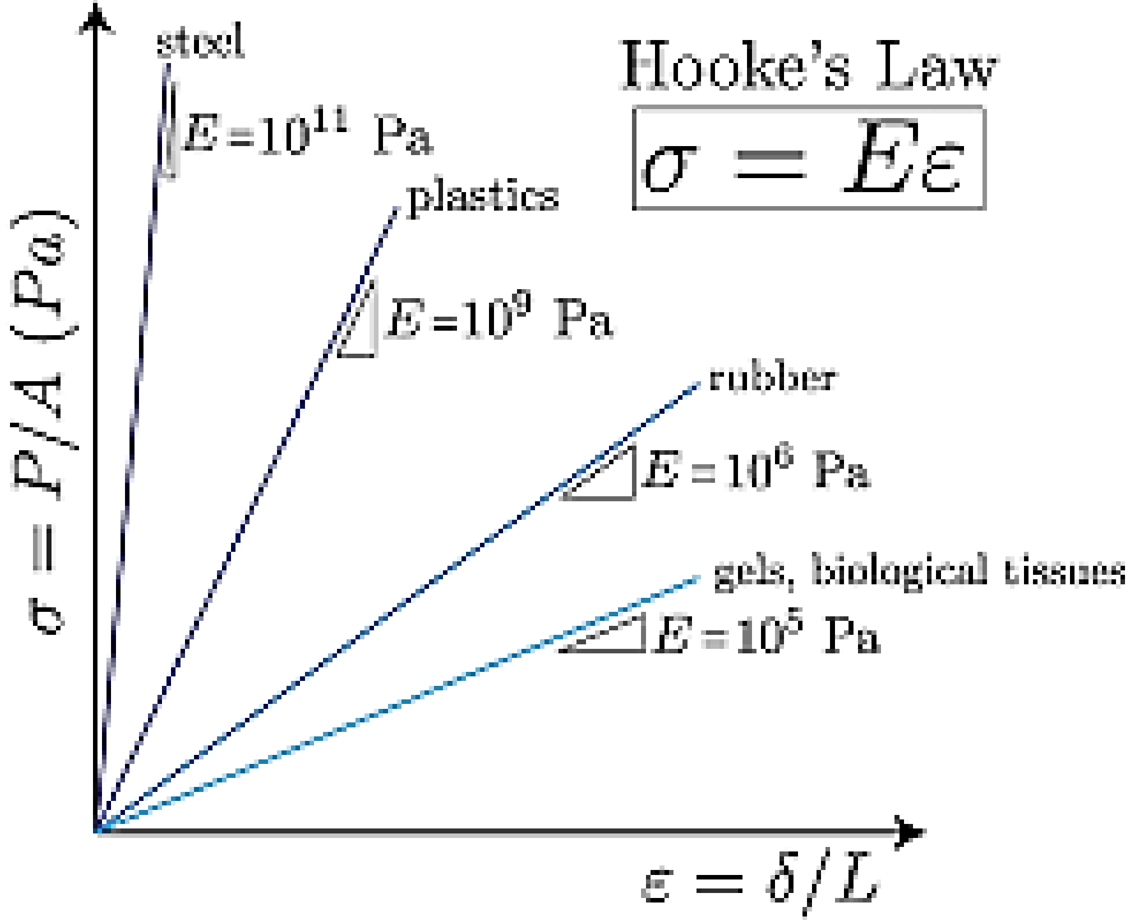

σ
load
unload
ε
(a)
σ
load
unload
ε
(b)
σ
load
Y
unload
ε
Plastic
strain
(c)



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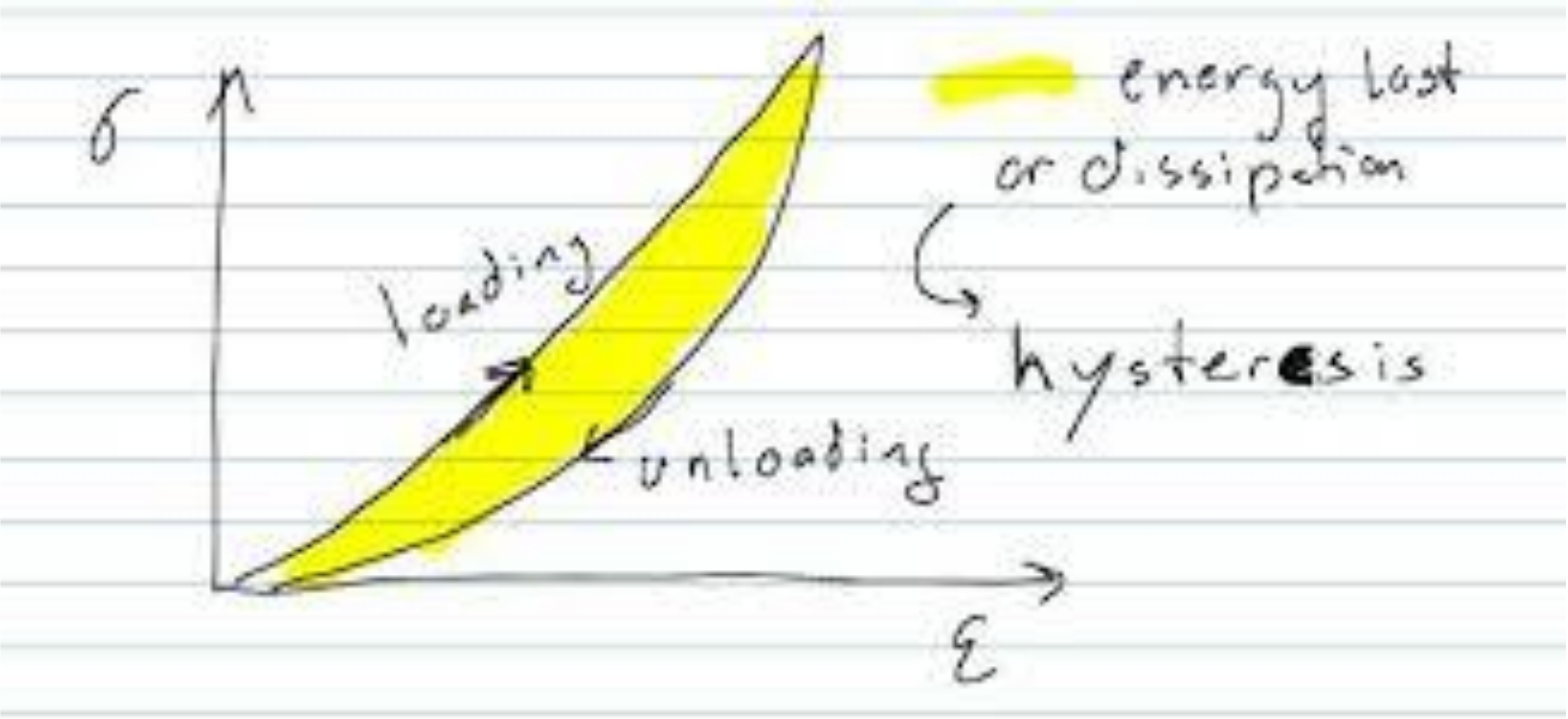
σ
ε
loading
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energy lost
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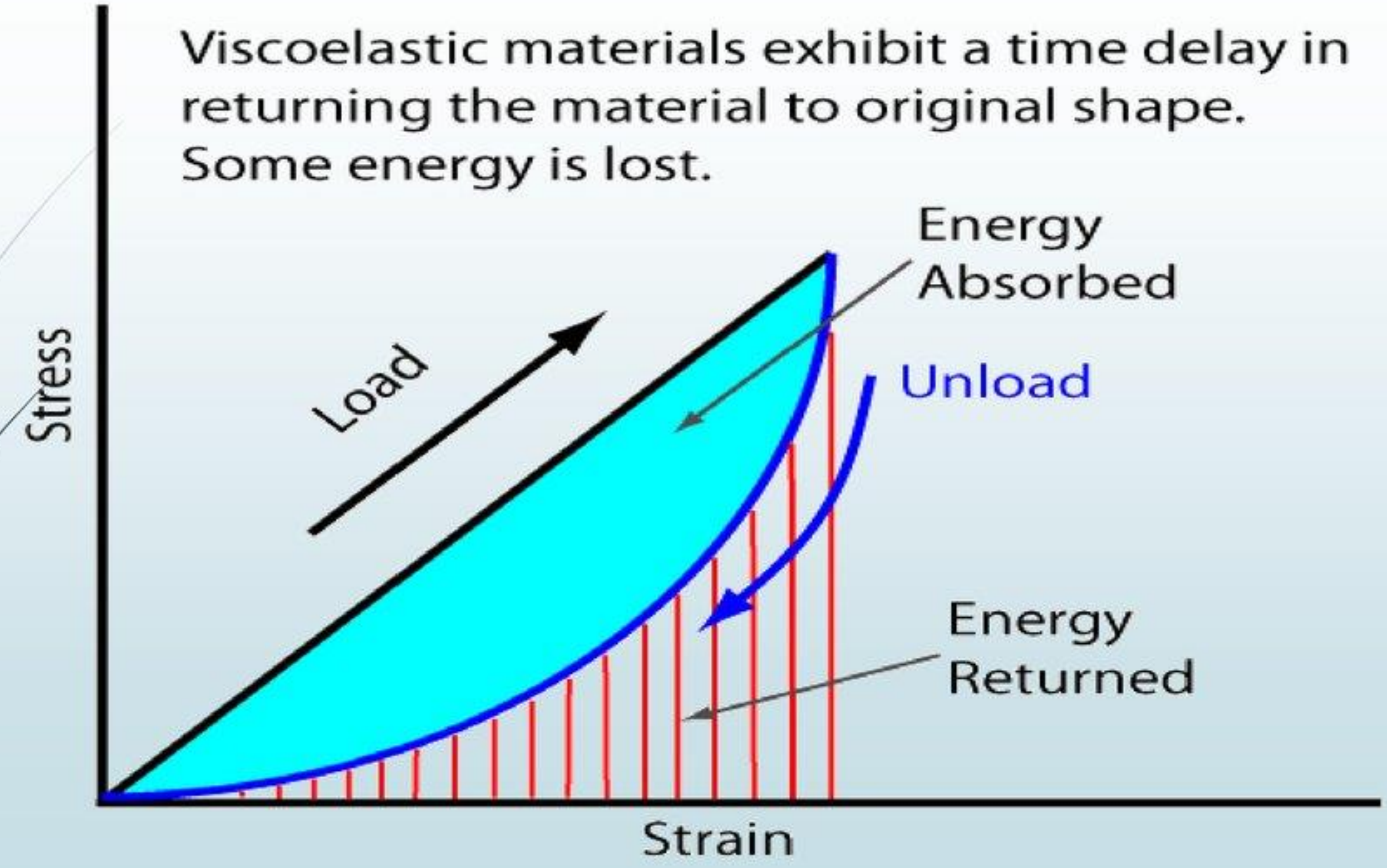
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Γ	γ	gamma	Ο	ο	omicron
Δ	δ	delta	Π	π	pi
Ε	ε	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σς	sigma
Η	η	eta	Τ	τ	tau
Θ	θ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	φ	phi
Κ	κ	kappa	Χ	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
Μ	μ	mu	Ω	ω	omega

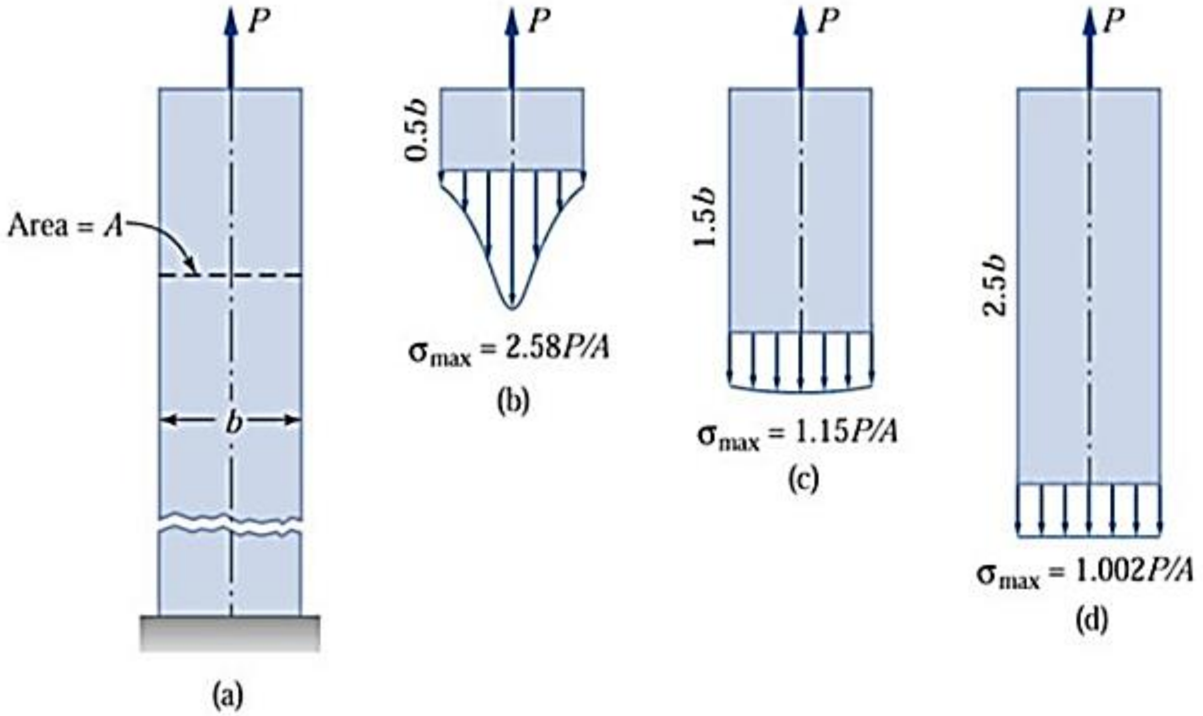
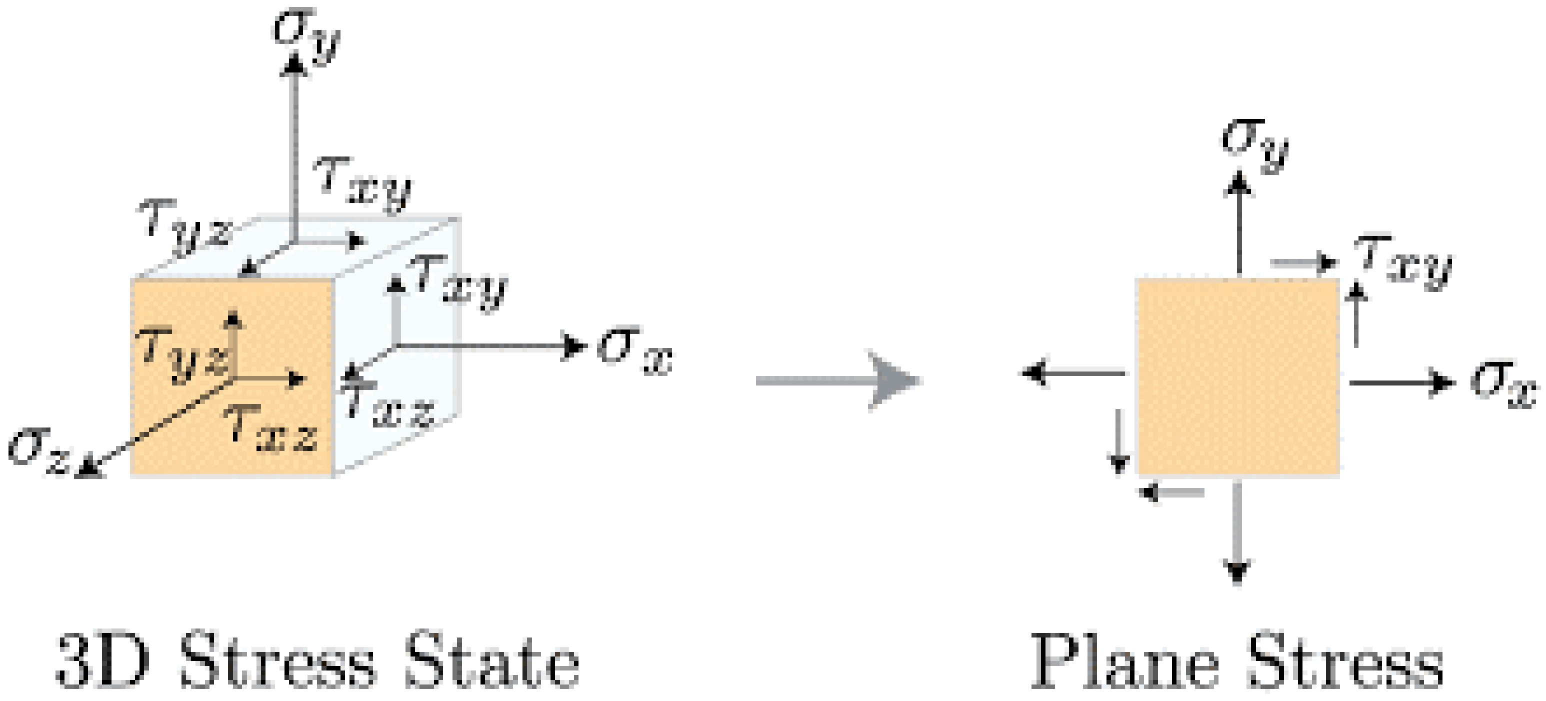
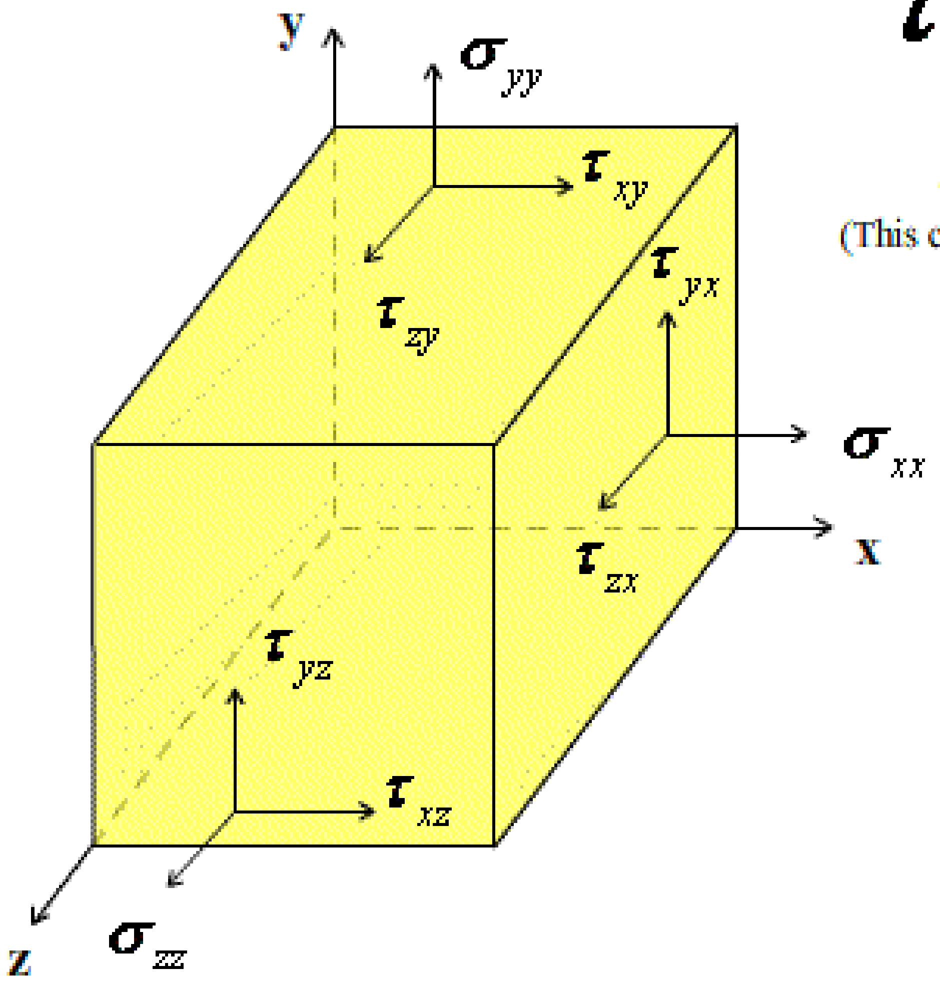


FIG. 1.7 Normal stress distribution in a strip caused by a concentrated load

ILLUSTRATING ST. VENANT'S PRINCIPLE


σ_y
τ_{xy}
τ_{yz}
τ_{xy}
τ_{yz}
σ_x
τ_{xz}
τ_{xz}
σ_z
3D Stress State
σ_y
τ_{xy}
σ_x
Plane Stress

The 9 components of a stress tensor:



The stress acts in the x-direction

$$\tau_{xy}$$

on the plane with a normal in the y direction

(This convention maybe vice versa in some books.)

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Tensor Equation: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

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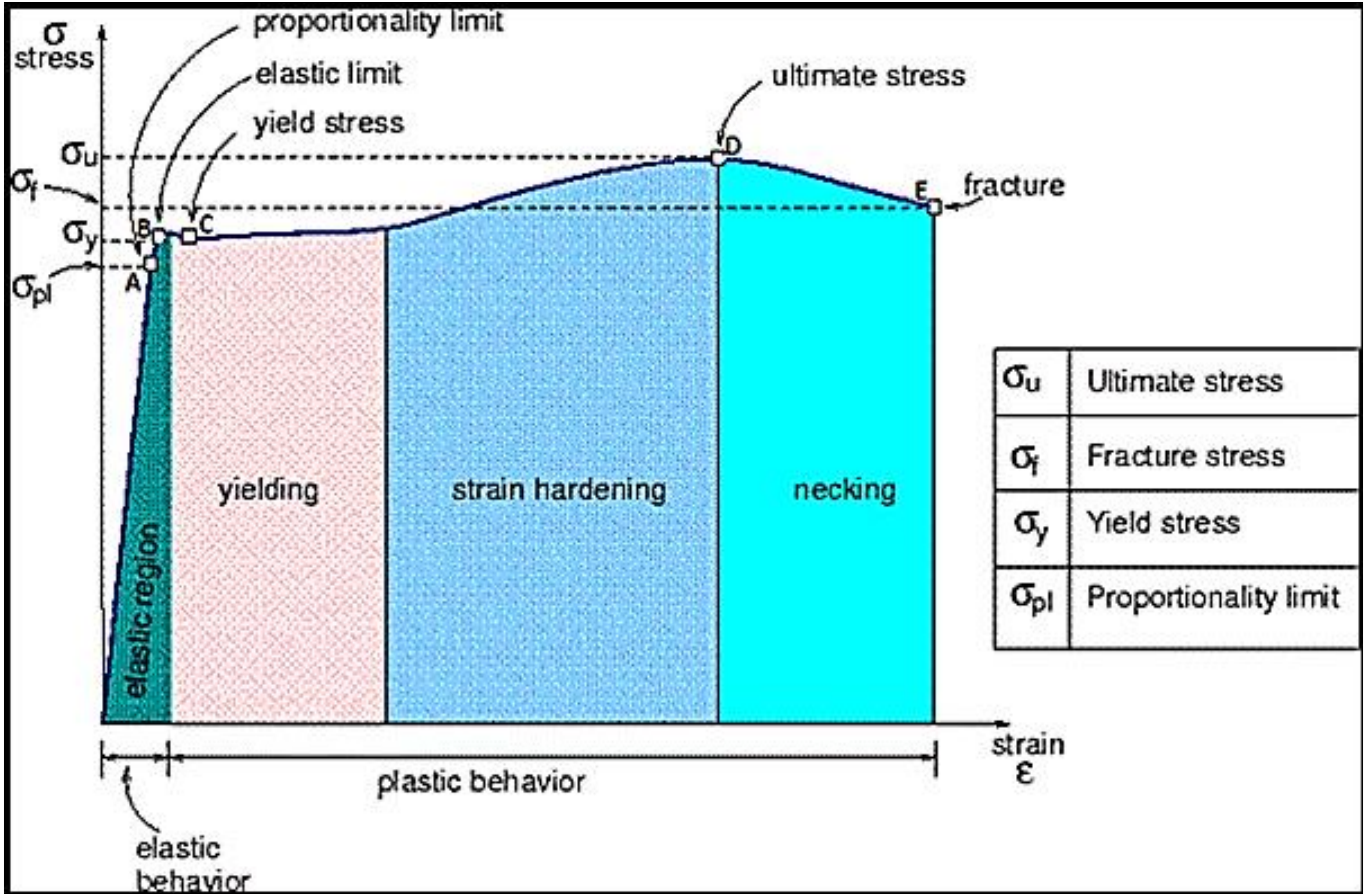
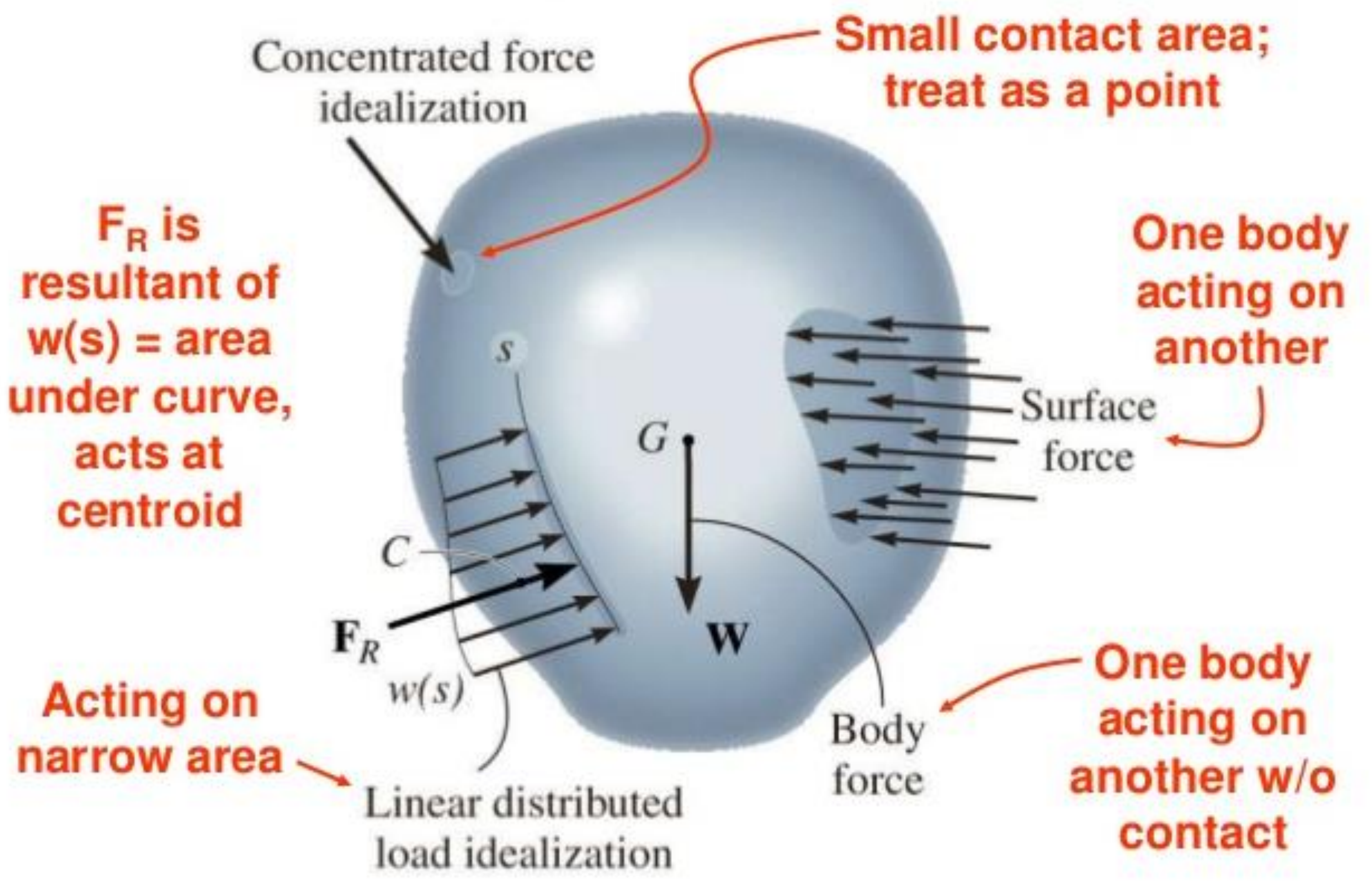
BASIC ASSUMPTIONS IN THEORY OF ELASTICITY

- The body is continuous
- The body is perfectly elastic
- The body is homogeneous
- The body is isotropic

  example: polycrystalline ceramics and steel
  wood and fiber reinforced composite
- The displacements and strains are small


σ
stress
proportionality limit
elastic limit
yield stress
ultimate stress
fracture
σu
σf
σy
σpl
A
B
C
D
E
elastic region
yielding
strain hardening
necking
strain
ε
plastic behavior
elastic
behavior
σu Ultimate stress
σf Fracture stress
σy Yield stress
σpl Proportionality limit

Statics Review: External Loads



Static Equilibrium

- Vectors: $\Sigma \mathbf{F} = 0$ $\Sigma \mathbf{M} = 0$

- Coplanar (2D) force systems:

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

$$\Sigma M_o = 0$$

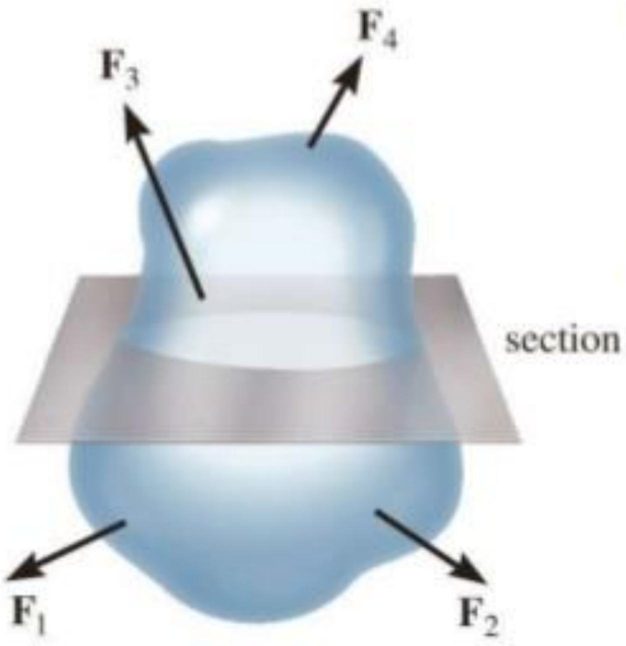
← **Perpendicular to the plane containing the forces**

- Draw a FBD to account for ALL loads acting on the body.

STATICS: You need to be able to…

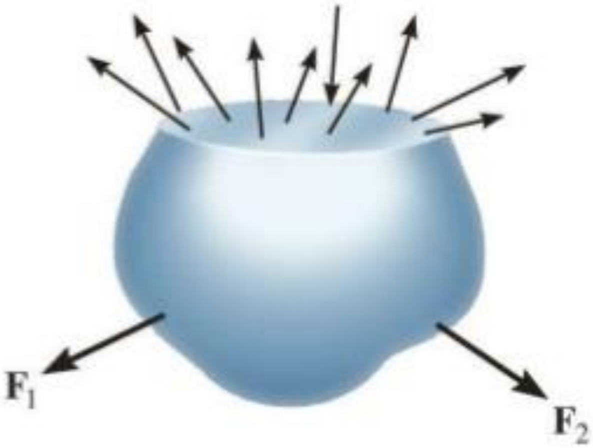
- Draw free-body diagrams,
- Know support types and their corresponding reactions,
- Write and solve equilibrium equations so that unknown forces can be solved for,
- Solve for appropriate internal loads by taking cuts of inspection,
- Determine the centroid of an area,
- Determine the moment of inertia about an axis through the centroid of an area.

Internal Reactions



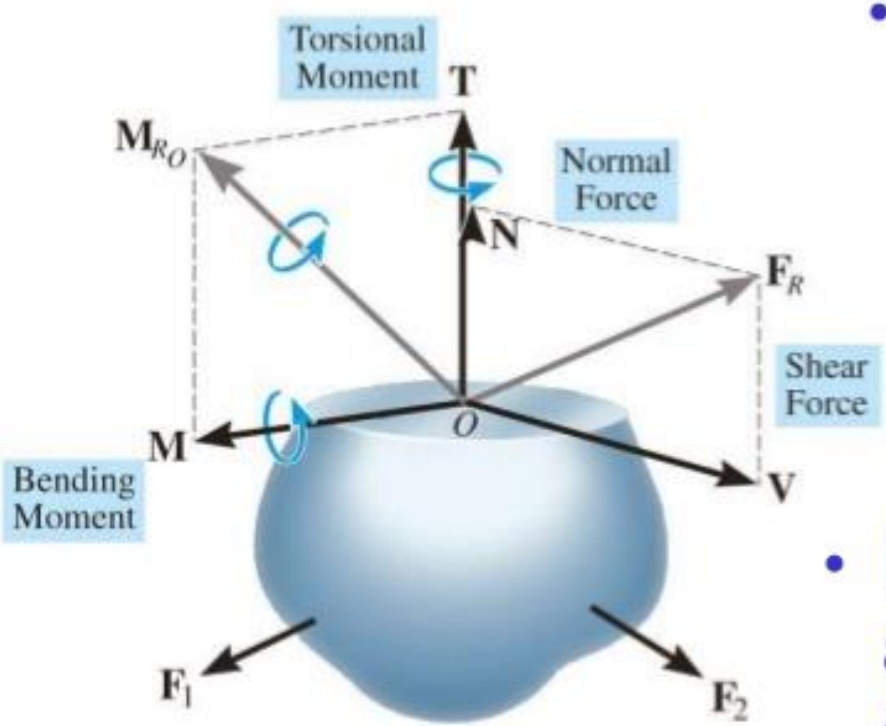
- Internal reactions are necessary to hold body together under loading.
- Method of sections - make a cut through body to find internal reactions at the point of the cut.

FBD After Cut



- Separate the two parts and draw a FBD of either side
- Use equations of equilibrium to relate the external loading to the internal reactions.

Components of Resultant



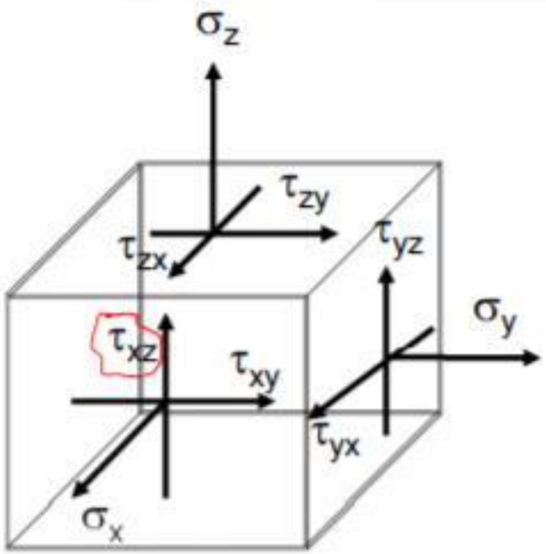
- Components are found perpendicular & parallel to the section plane.
- Internal reactions are used to determine stresses.

3D Differential Equations of Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$$



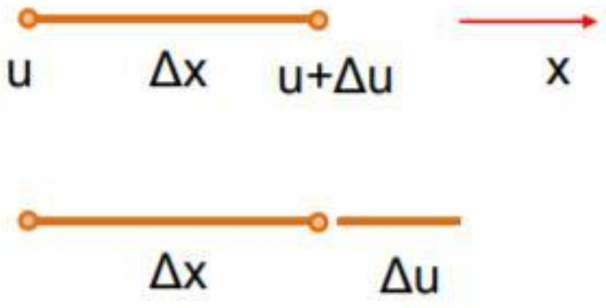
Governing equations for 3D elasticity

3 equations, 6 unknown functions

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{yx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{yx} \end{bmatrix}$$

Strain-Displacement Relations: Normal Strain

Cornell Engineering



$$u + \Delta u = u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} + H.O.T.$$

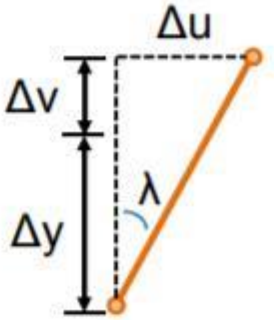
Similarly: $\varepsilon_y = \dfrac{\partial v}{\partial y}$ $\quad \varepsilon_z = \dfrac{\partial w}{\partial z}$

$$\varepsilon_x = \frac{\Delta u}{\Delta x}$$

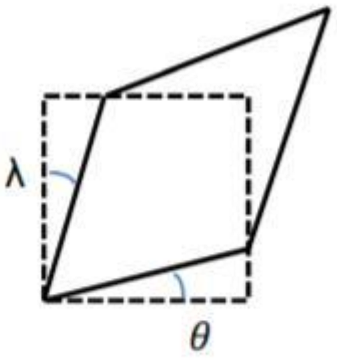
$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \Delta x \to 0$$

Cornell Engineering

Strain-Displacement Relations: Shear Strain



$$\lambda \approx \frac{\partial u}{\partial y}$$



$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

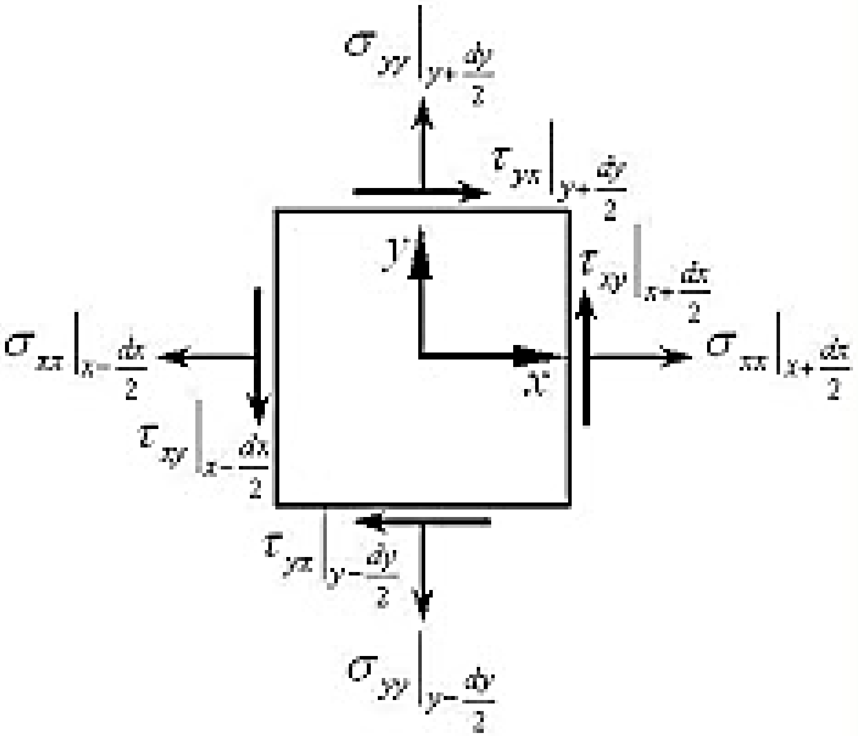
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

EQUILIBRIUM EQUATIONS

- Equilibrium Relation (2D)

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} = 0 \end{cases}$$



- Equilibrium Relation (3D)

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} = 0 \end{cases}$$

There are two basic conditions of equilibrium.

- ✓ Translational equilibrium.
- ✓ Rotational equilibrium.

- The term "translational equilibrium" describes an object that experiences no linear acceleration. (First condition of equilibrium)
- An object experiencing no rotational acceleration (a component of torque) is said to be in rotational equilibrium. (Second condition of equilibrium)
- Typically, an object at rest in a stable situation experiences both linear and rotational equilibrium.

There are two kinds of mechanical equilibrium:

- ✓ static equilibrium and
- ✓ dynamic equilibrium.

- Any object which is in static equilibrium has zero net force acting on it and is at rest.
- Any object which is in dynamic equilibrium has zero net force acting on it and is moving at a constant velocity.

Newton’s First Law of Physics:

- *A body at rest will stay at rest and a body in motion will stay in motion unless acted upon by an unbalanced force.*
- Therefore, sum of all forces must be zero.

$$\sum F = 0$$

Resultant of all forces acting on a particle is zero.

Equilibrium

Equilibrium Equation from Newton's Law

- If an object is in equilibrium, then the resultant force acting on an object equals zero. This is expressed as follows:

$$\overline{F}_R = \sum \overline{F} = 0 \quad \text{(vectorequation)}$$

Some problems can be analyzed using only 2D, while others require 3D.

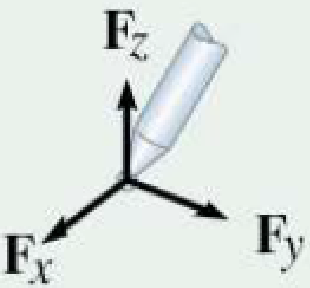
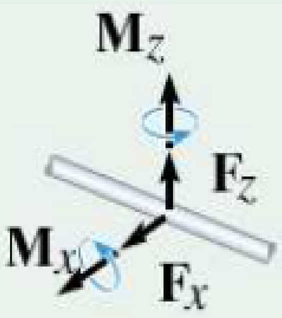
Necessary Condition for Equilibrium

- The necessary conditions for **equilibrium** are:

(i) the vector sum of all *external forces* is zero.

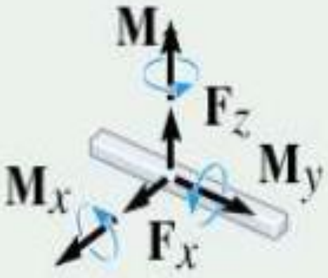
(ii) the sum of the moments of all *external forces* about any line is zero.

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a force which acts away from the member in
(2)  smooth surface support		One unknown. The reaction is a force which acts perpendicular to the surface

Types of Connection	Reaction	Number of Unknowns
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface
(4)  ball and socket		Three unknowns. The reactions are three rectangular force components.
(5)  single journal bearing		Four unknowns. The reactions are two force and two couple-moment

(6)

single journal bearing with square shaft



Five unknowns. The reactions are two force and three couple-moment components.

(7)

single thrust bearing

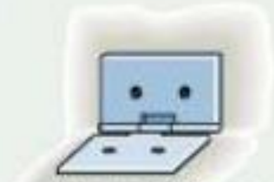
Five unknowns. The reactions are three force and two couple-moment components.

(8)

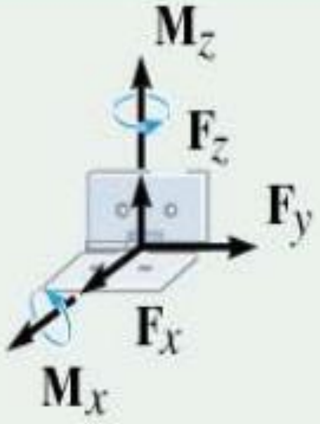
single smooth pin

Five unknowns. The reactions are three force and two couple-moment components.

(9)



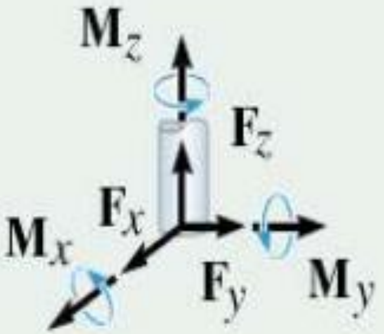
single hinge



Five unknowns. The reactions are three force and two couple-moment components.

(10)

fixed support



Six unknowns. The reactions are three force and three couple-moment components.

THE WHAT, WHY AND HOW OF A FREE BODY DIAGRAM (FBD)

Free Body Diagrams are one of the most important things for you to know how to draw and use.

What ? - It is a drawing that shows all external forces acting on the particle.

Why ? - It is key to being able to write the equations of equilibrium—which are used to solve for the unknowns (usually forces or angles).

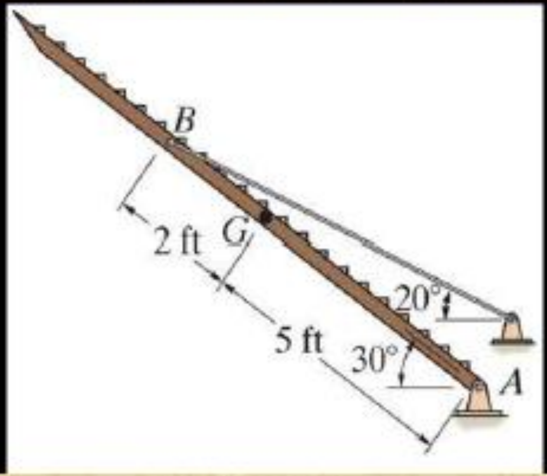
Steps of Drawing a FBD

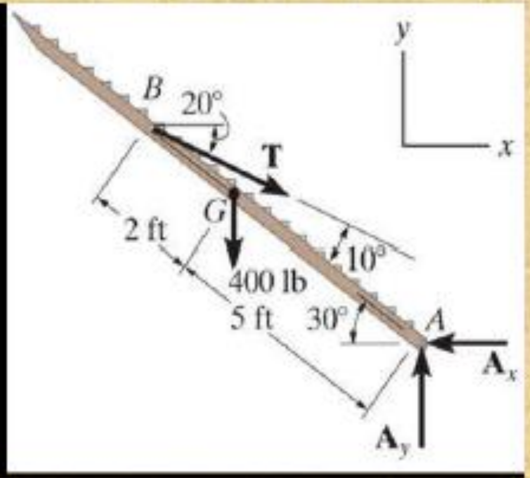
1. Imagine the particle to be isolated or cut free from its surroundings.
2. Show all the forces that act on the particle.

   Active forces: They want to move the particle.

   Reactive forces: They tend to resist the motion.
3. Identify each force and show all known magnitudes and directions.

Steps of Drawing a FBD





Real world example → Idealized model → FBD

THE PROCESS OF SOLVING RIGID BODY EQUILIBRIUM PROBLEMS

1) Draw a **free-body diagram (FBD)** showing all the external forces.
2) Apply the equations of equilibrium to solve for any unknowns.

Note: If there are more unknowns than the number of independent equations, then we have a statically **indeterminate situation.** We cannot solve these problems using just statics.

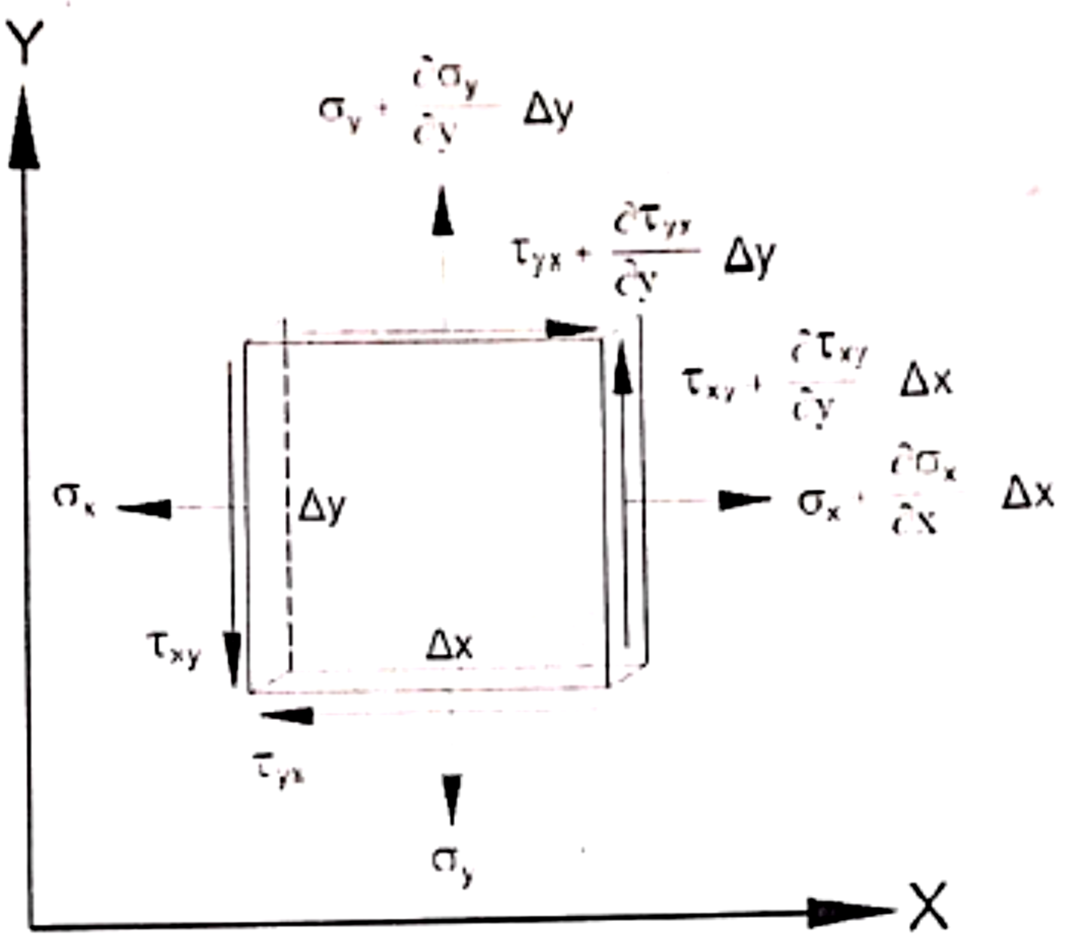
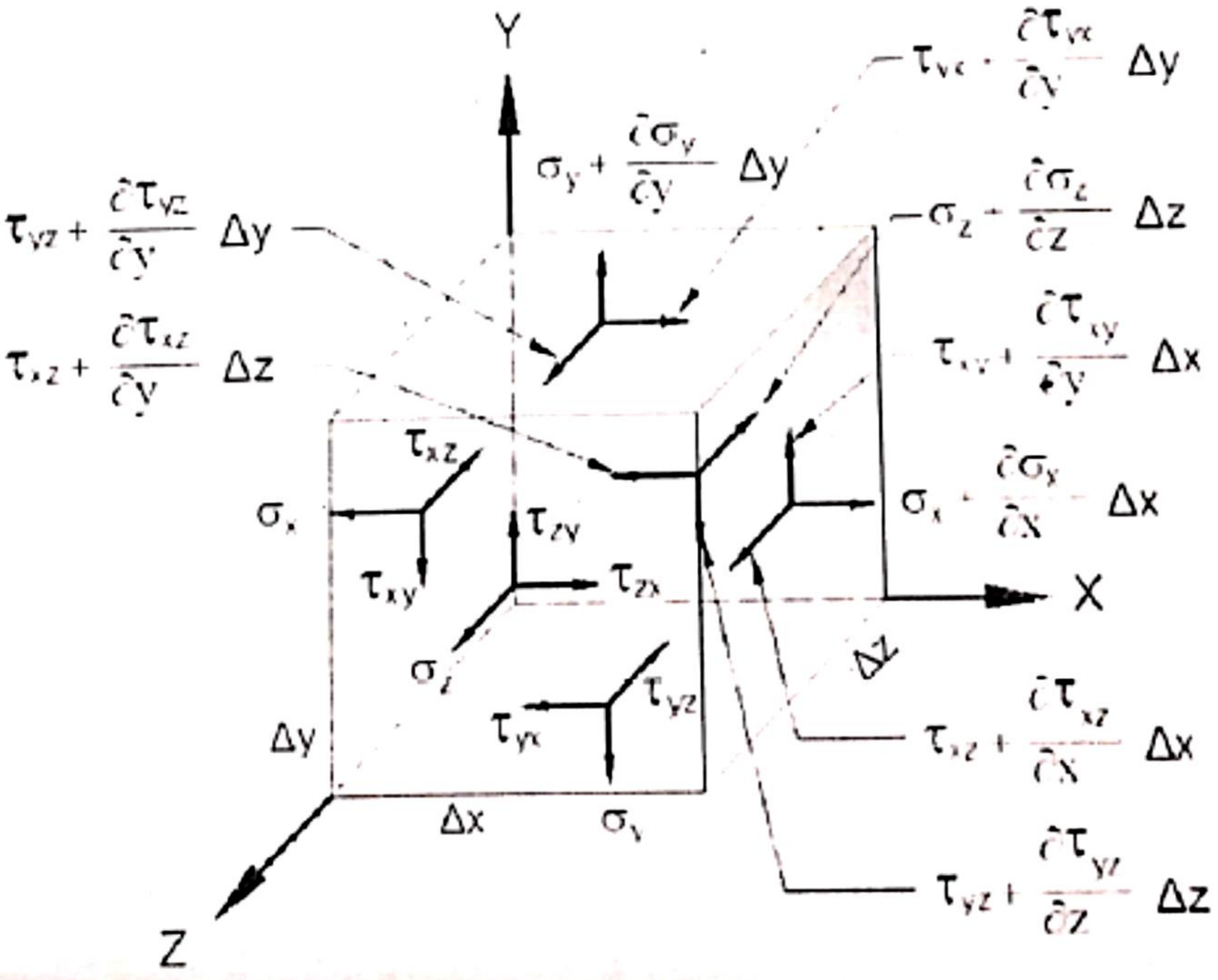
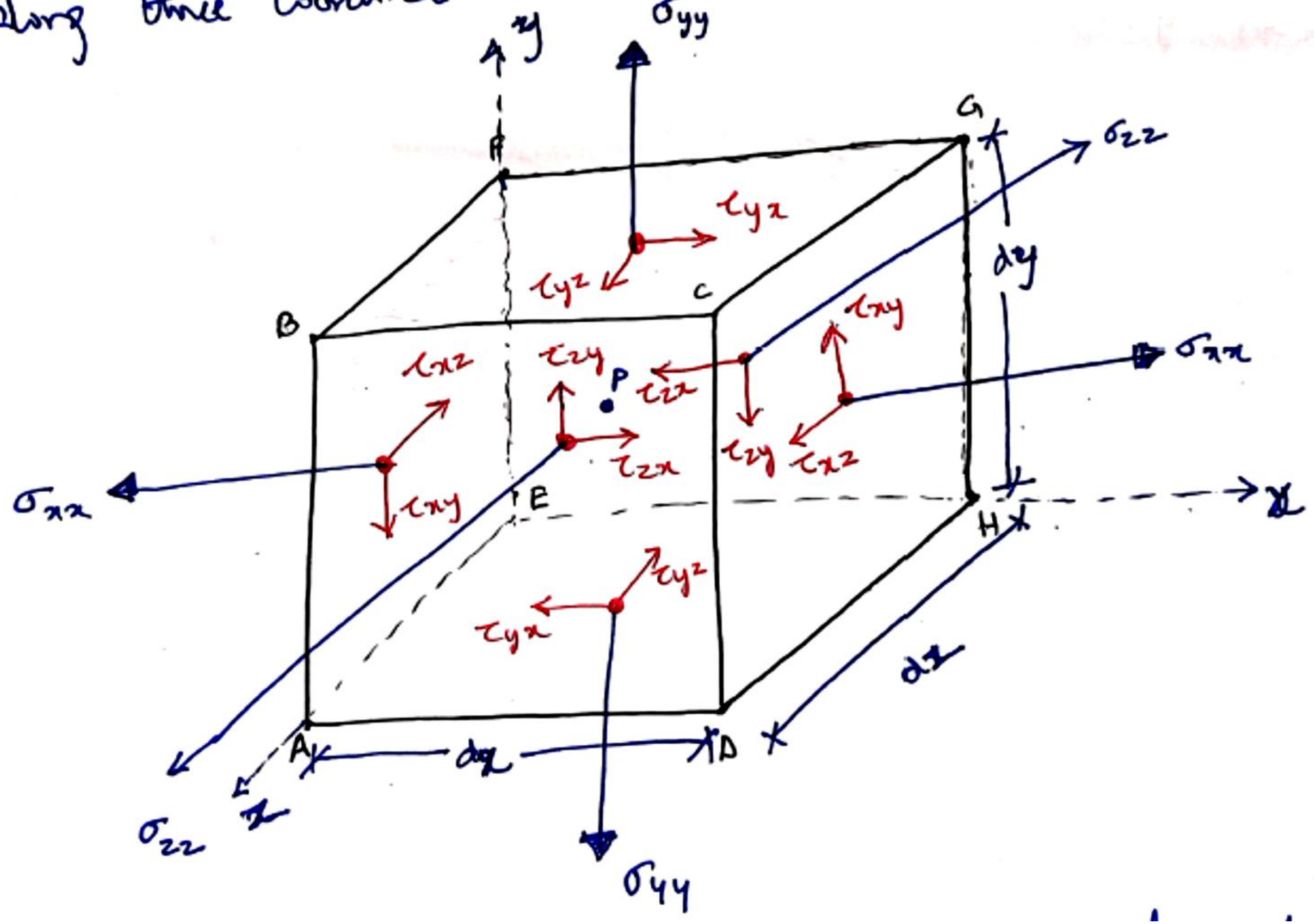
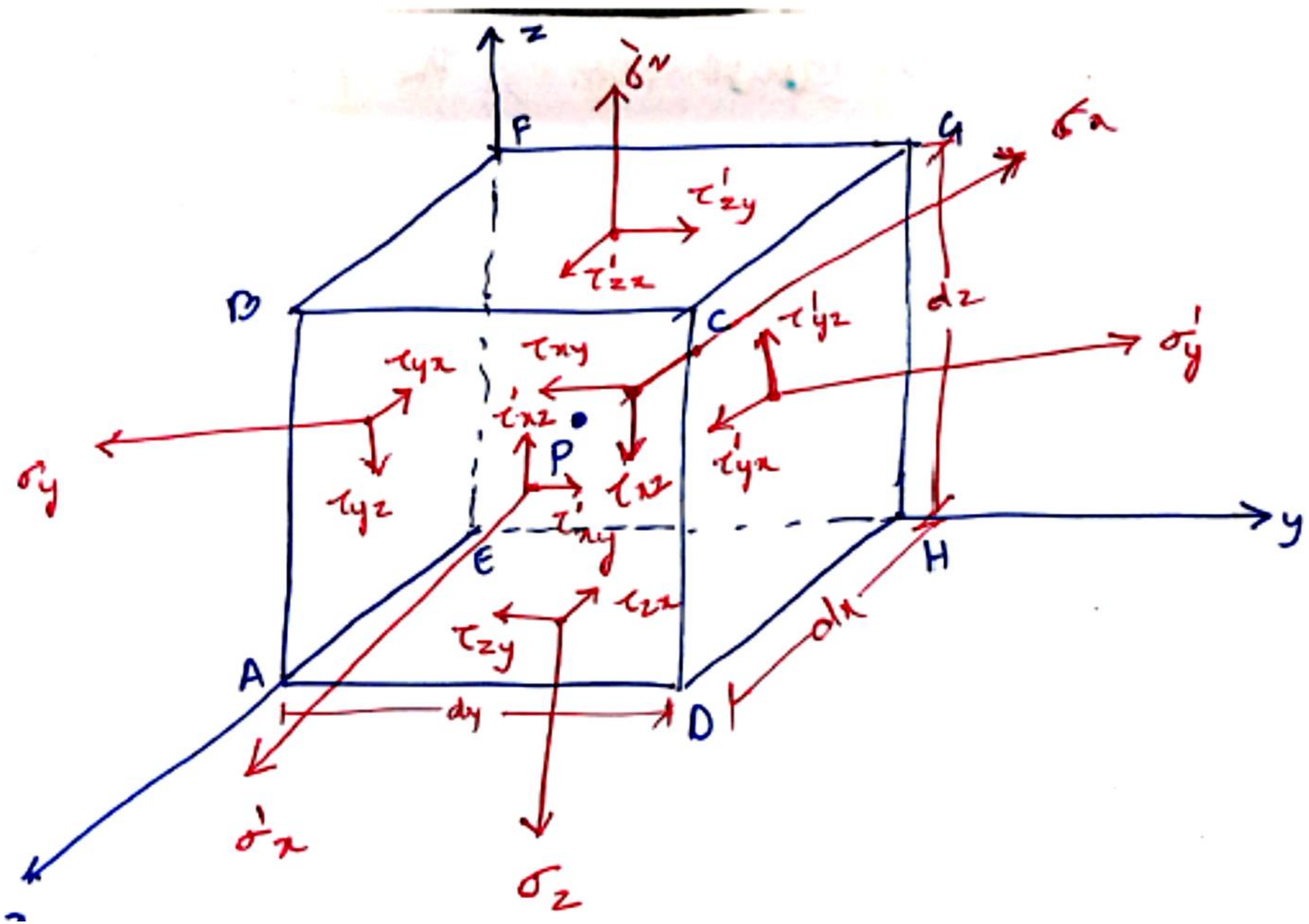


Figure 2.11(a) Stress components acting on a plane element (2D)


Y
X
Z
$\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y$
$\sigma_y + \frac{\partial \sigma_y}{\partial y} \Delta y$
$\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} \Delta y$
$\sigma_z + \frac{\partial \sigma_z}{\partial z} \Delta z$
$\tau_{xz} + \frac{\partial \tau_{xz}}{\partial y} \Delta z$
$\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} \Delta x$
$\sigma_x + \frac{\partial \sigma_y}{\partial x} \Delta x$
τ_{xz}
σ_x
τ_{zy}
τ_{xy}
τ_{zx}
σ_z
τ_{yz}
τ_{yx}
Δy
Δx
σ_y
Δz
$\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} \Delta x$
$\tau_{yz} + \frac{\partial \tau_{yz}}{\partial z} \Delta z$

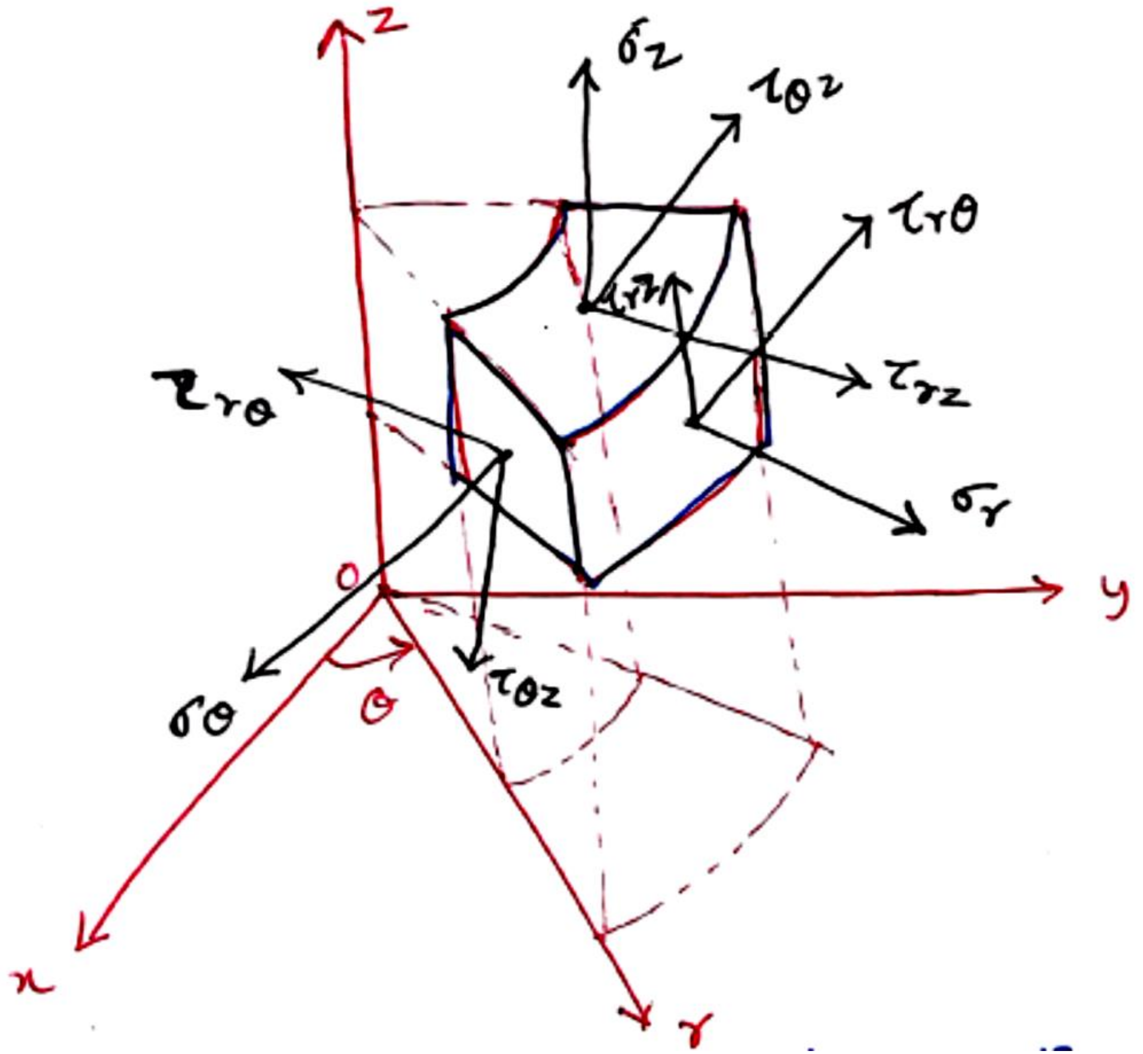
along three coordinate axes



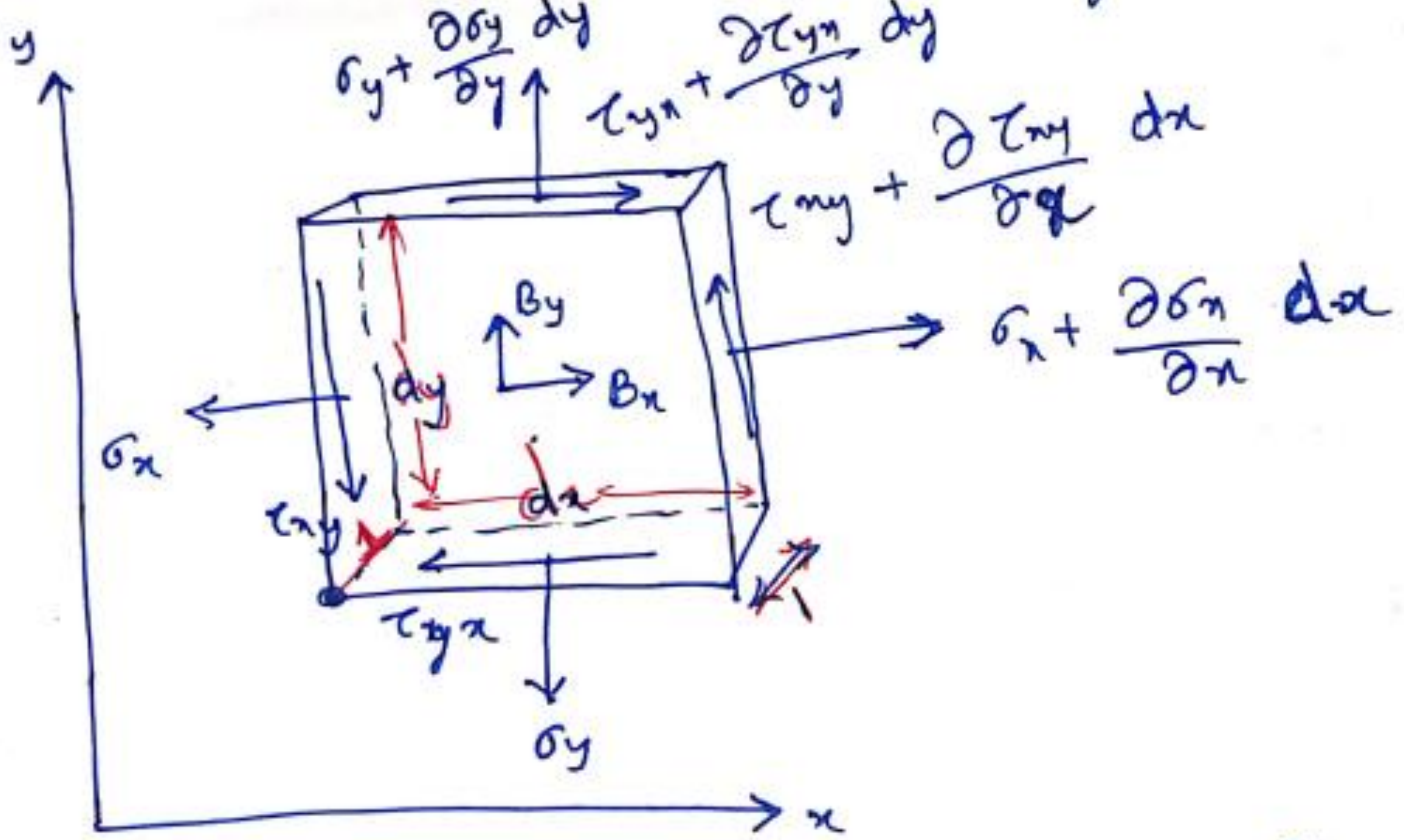


z
y
A
B
C
D
E
F
G
H
P
dx
dy
dz
σ_x
σ'_x
σ_y
σ'_y
σ_z
τ_{xy}
τ'_{xy}
τ_{xz}
τ'_{xz}
τ_{yx}
τ'_{yx}
τ_{yz}
τ'_{yz}
τ_{zx}
τ'_{zx}
τ_{zy}
τ'_{zy}

Stress at a point (Cylindrical Coordinates) (Polar Coord

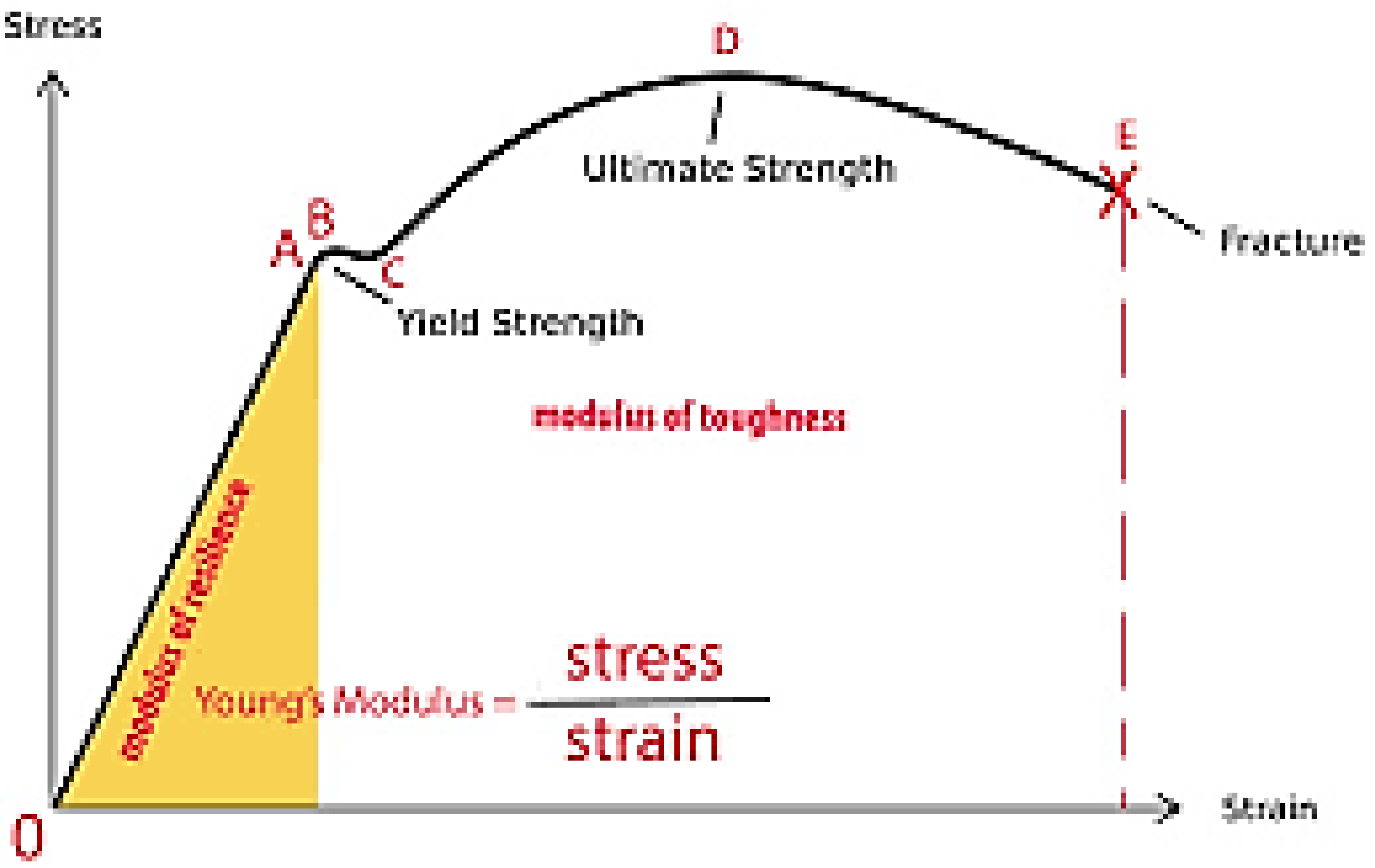


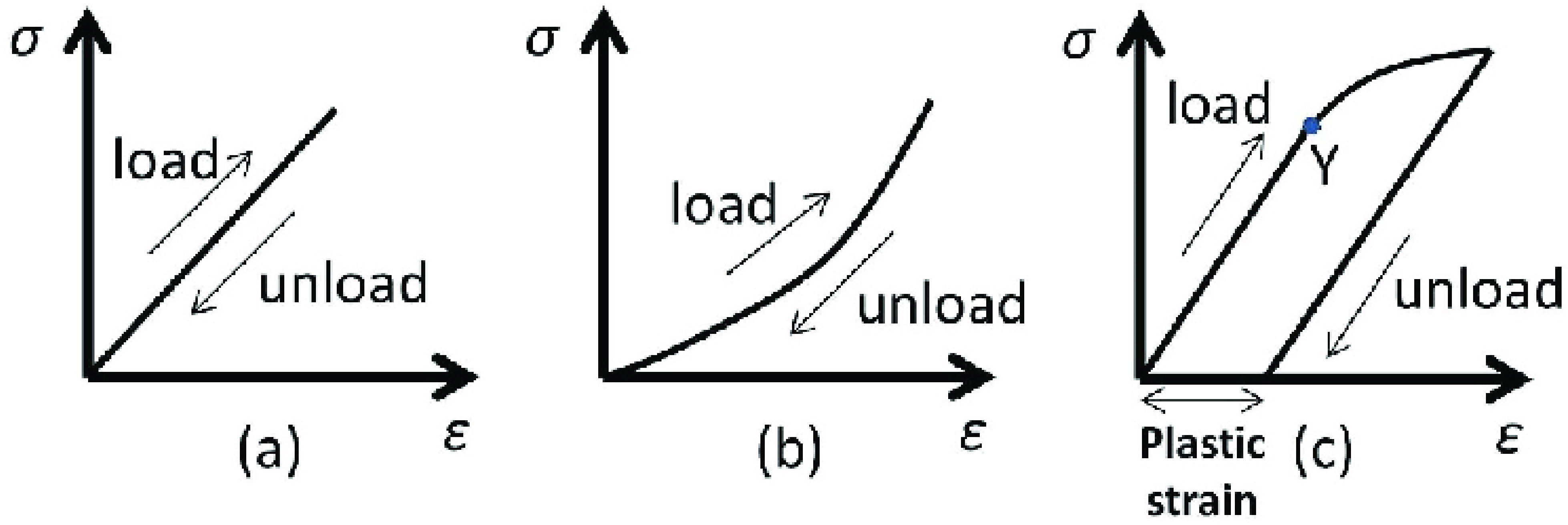
Derive equations for equilibrium of a differential Elem

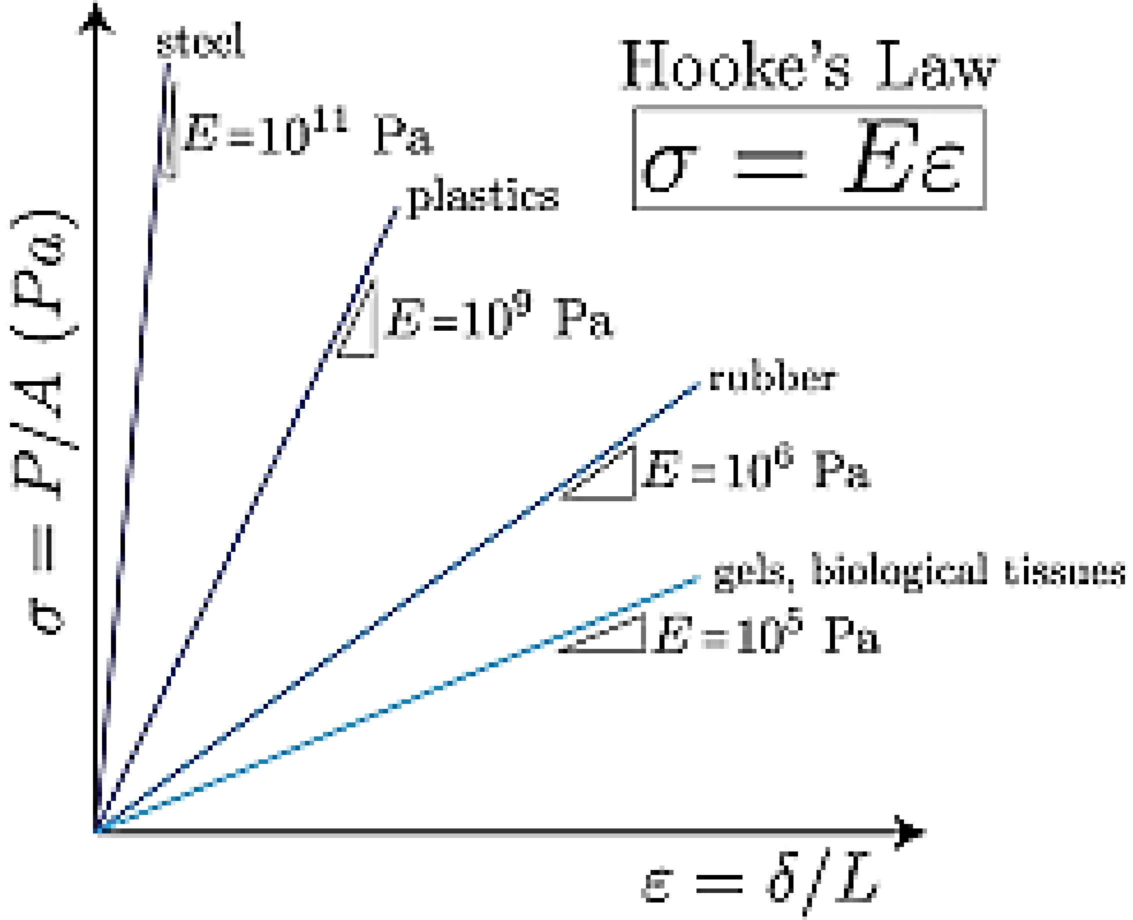


Theory of Elasticity

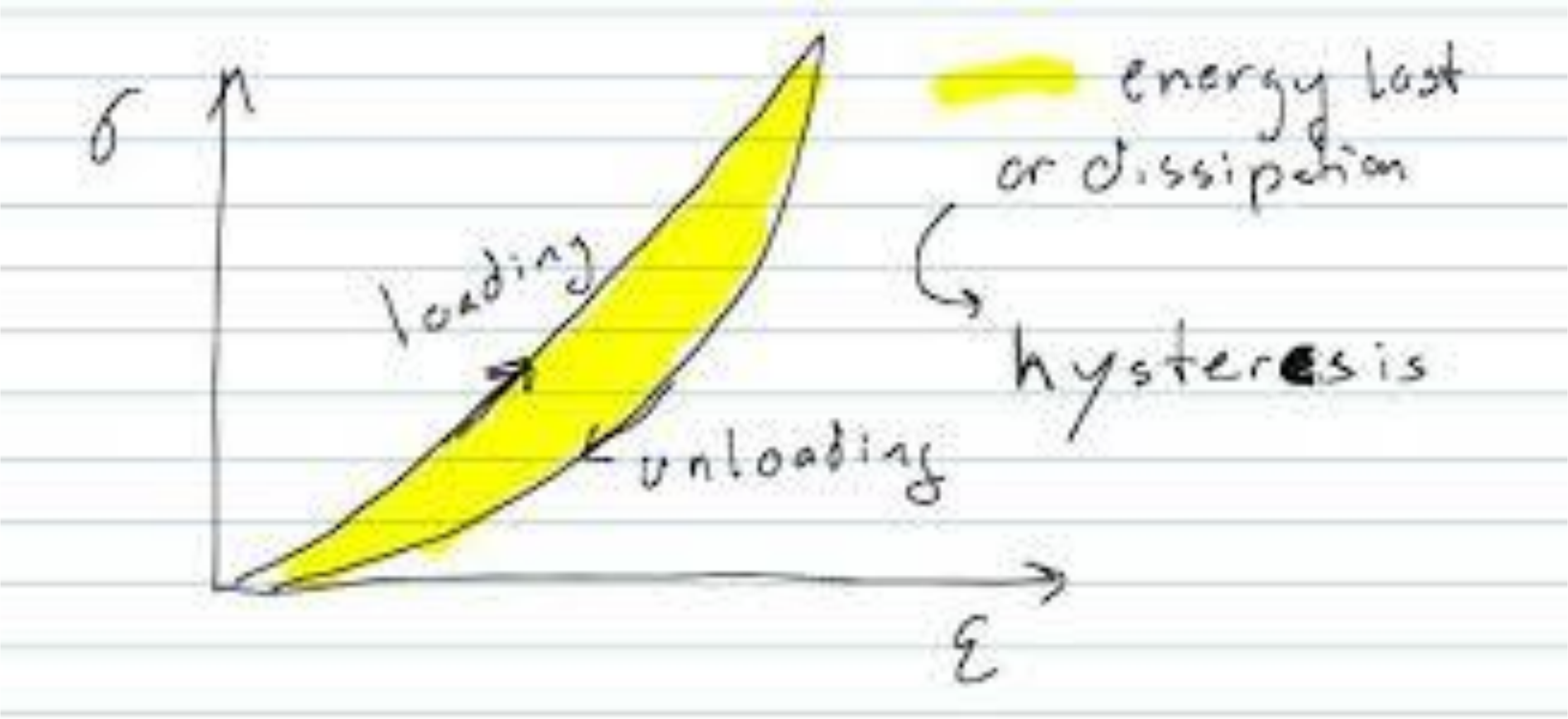
Dr V Srinivasa Reddy


Stress
D
Ultimate Strength
E
Fracture
B
A
C
Yield Strength
modulus of toughness
modulus of resilience
Young's Modulus = stress / strain
0
Strain

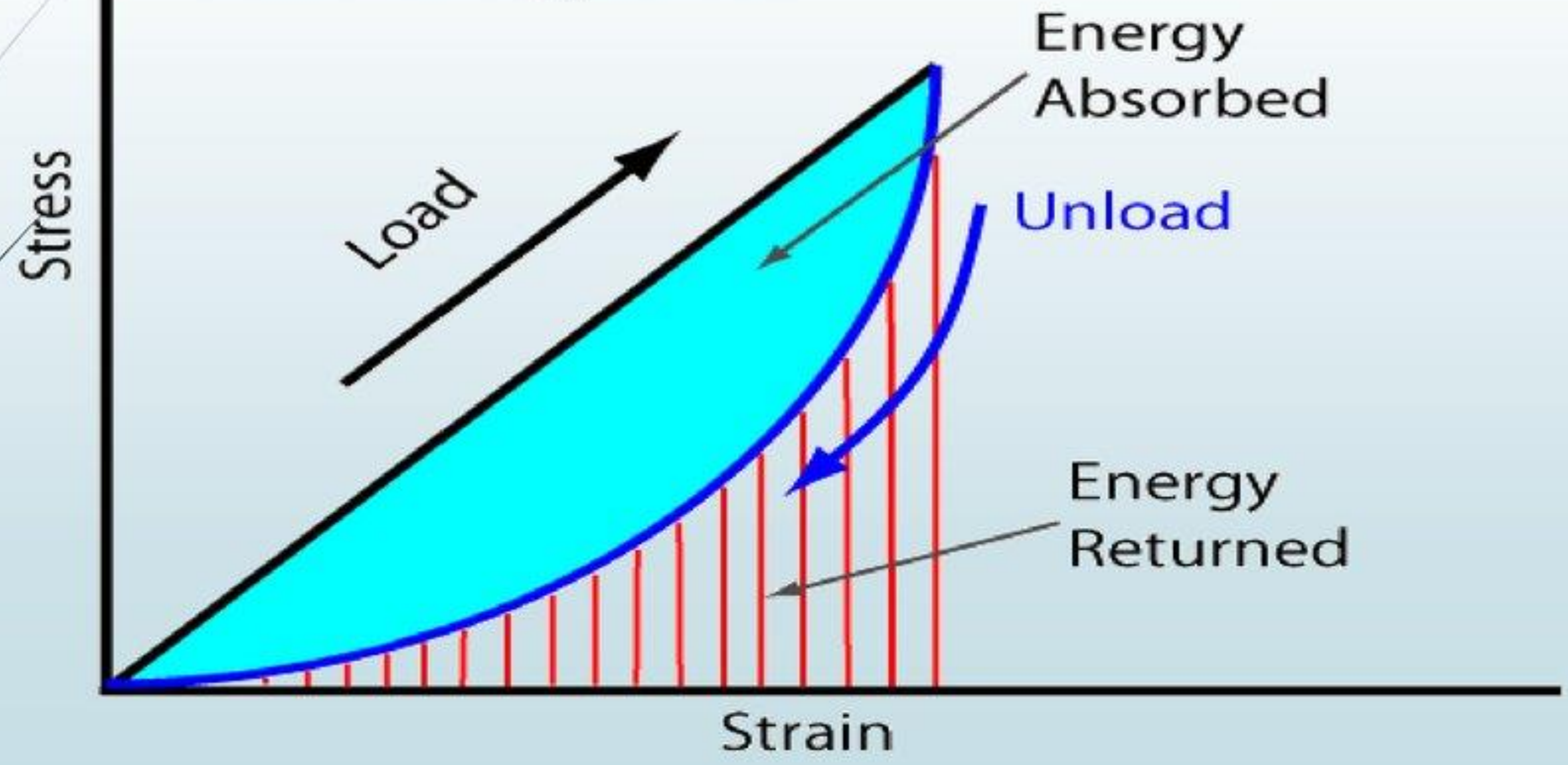

σ
load
unload
ε
(a)
σ
load
unload
ε
(b)
σ
load
Y
unload
ε
Plastic
strain
(c)



Hooke's Law
$\sigma = E\varepsilon$
steel
$E = 10^{11}$ Pa
plastics
$E = 10^{9}$ Pa
rubber
$E = 10^{6}$ Pa
gels, biological tissues
$E = 10^{5}$ Pa
$\sigma = P/A\ (Pa)$
$\varepsilon = \delta/L$


σ
loading
unloading
ε
energy lost
or dissipation
hysteresis

Viscoelastic materials exhibit a time delay in returning the material to original shape. Some energy is lost.



Α	α	alpha	Ν	ν	nu
Β	β	beta	Ξ	ξ	ksi
Γ	γ	gamma	Ο	ο	omicron
Δ	δ	delta	Π	π	pi
Ε	ε	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σς	sigma
Η	η	eta	Τ	τ	tau
Θ	θ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	φ	phi
Κ	κ	kappa	Χ	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
Μ	μ	mu	Ω	ω	omega

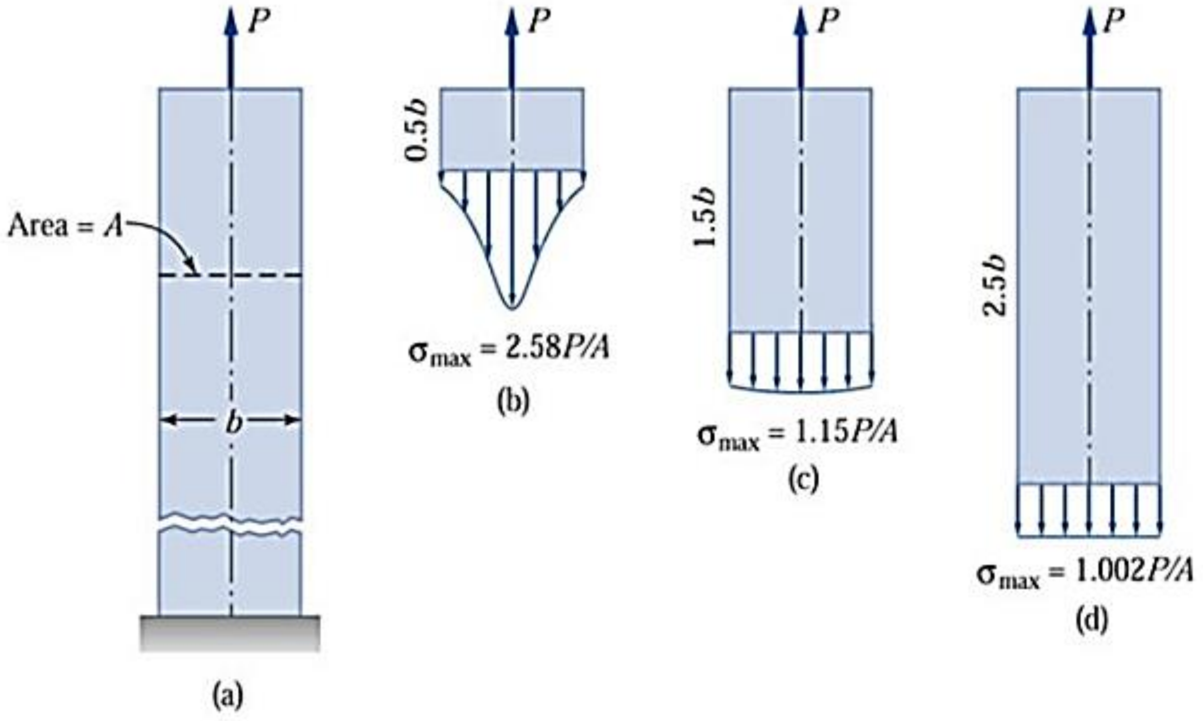
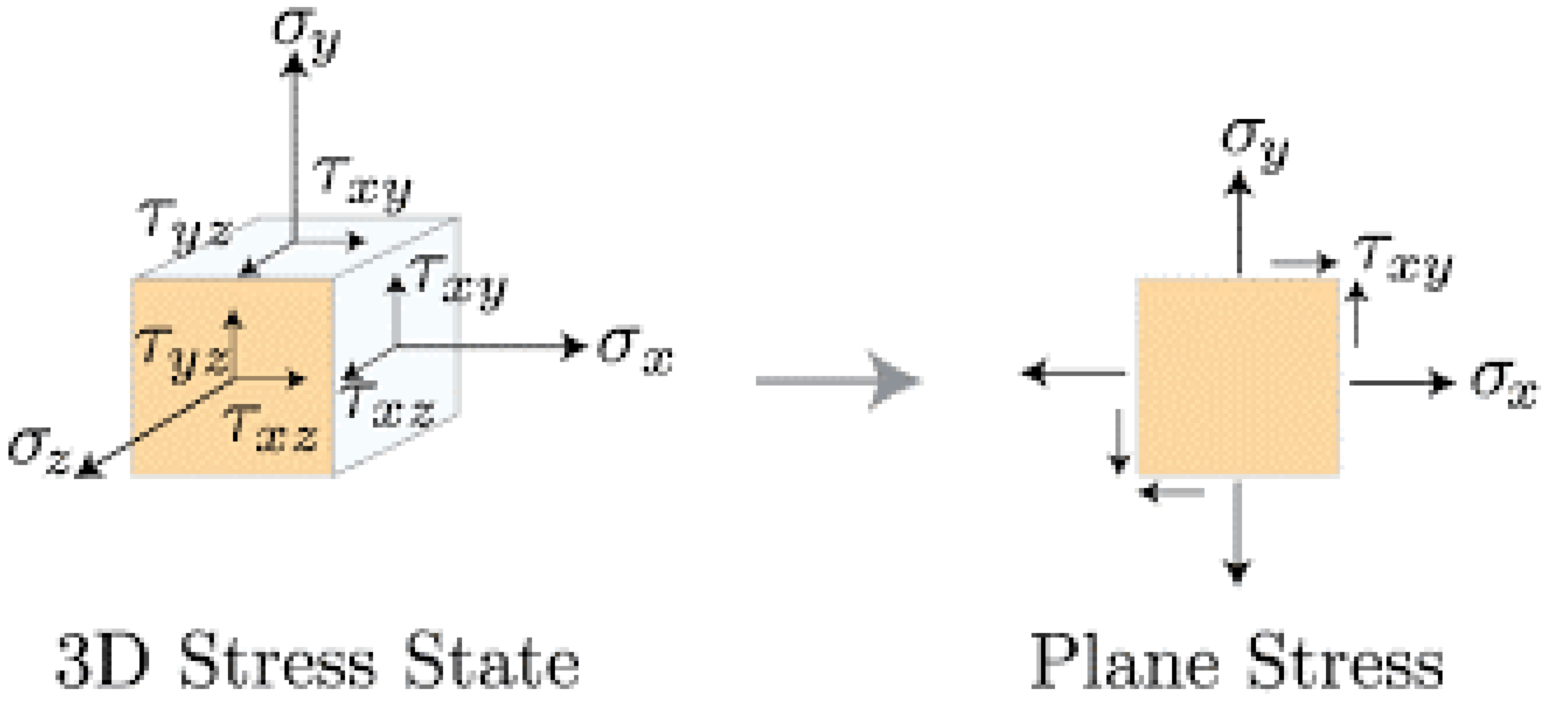


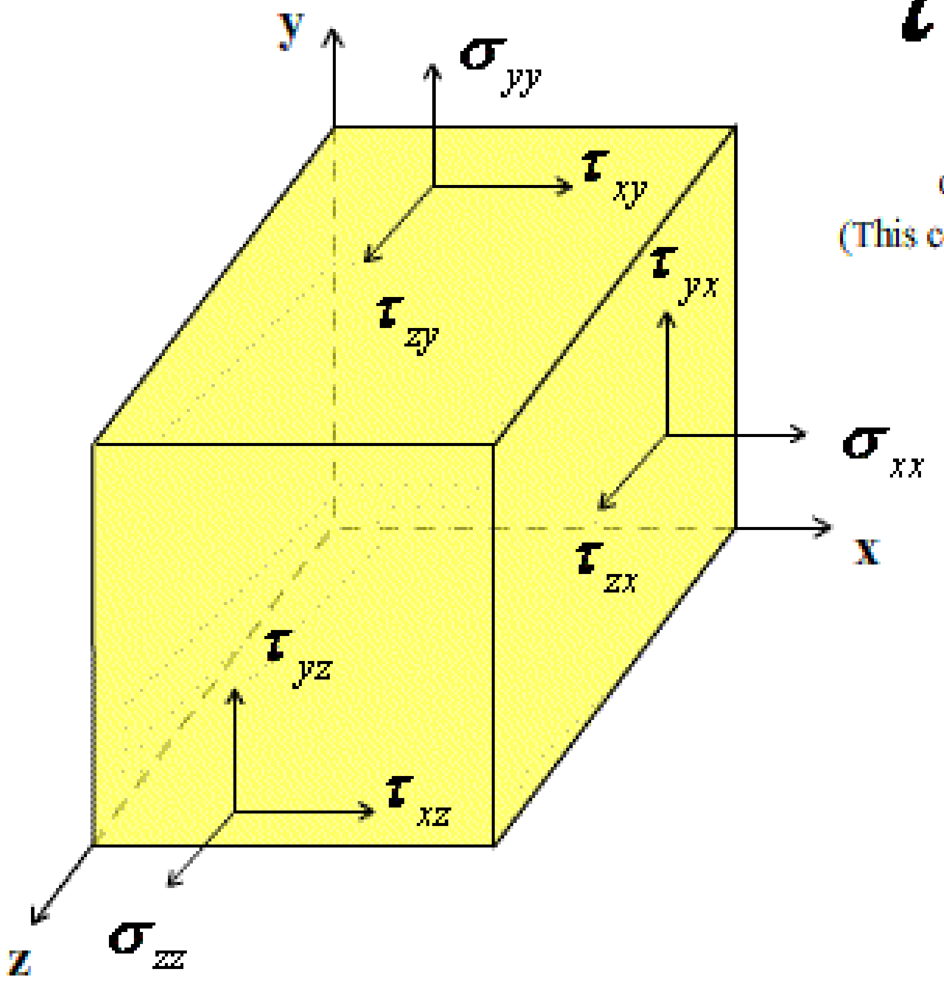
FIG. 1.7 Normal stress distribution in a strip caused by a concentrated load

ILLUSTRATING ST. VENANT'S PRINCIPLE



σ_y
τ_{xy}
τ_{yz}
τ_{xy}
σ_x
τ_{yz}
τ_{xz}
σ_z
τ_{xz}
3D Stress State
σ_y
τ_{xy}
σ_x
Plane Stress

The 9 components of a stress tensor:



The stress acts in the x-direction

$$\tau_{xy}$$

on the plane with a normal in the y direction

(This convention maybe vice versa in some books.)

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

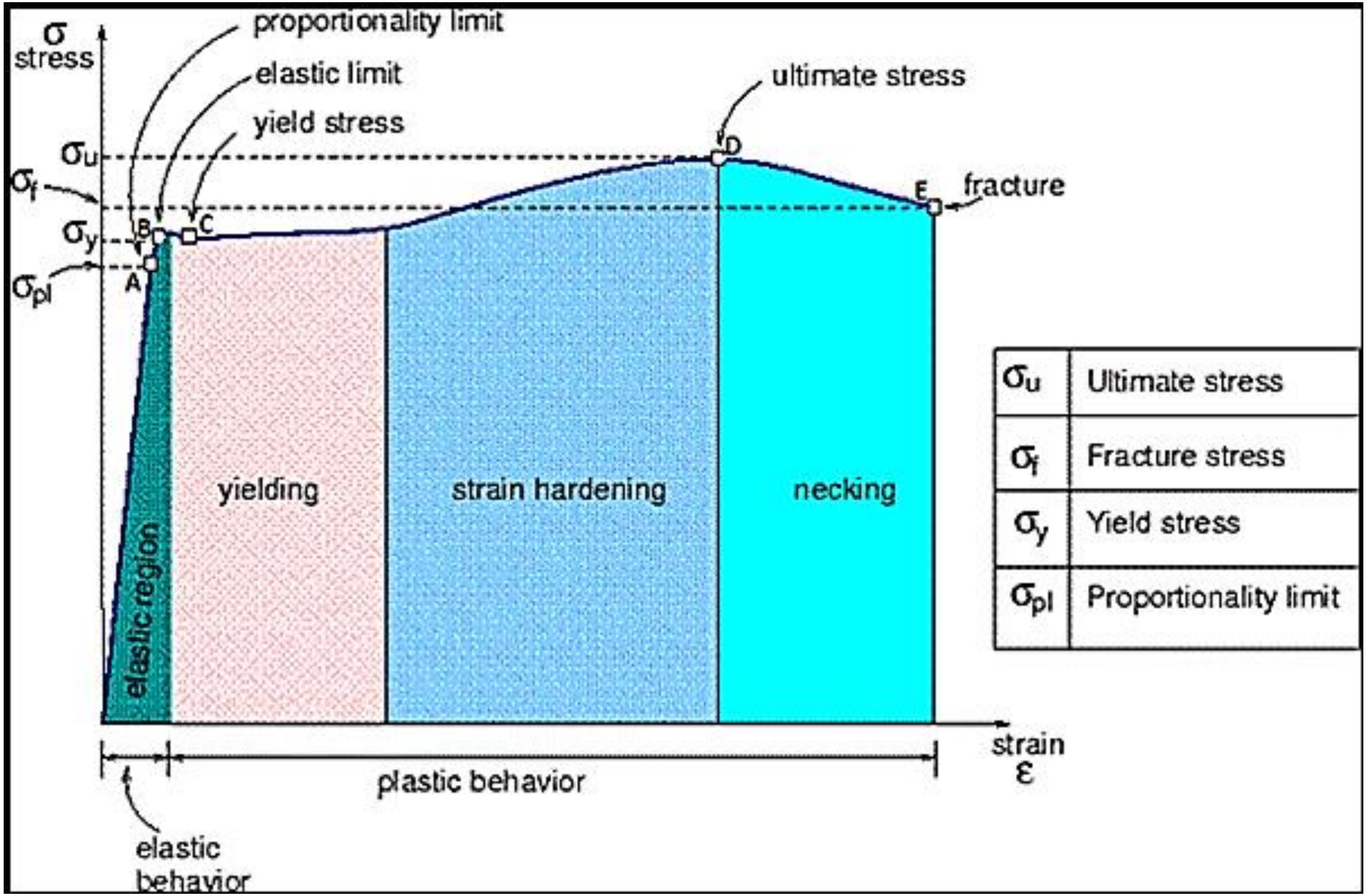
Tensor Equation: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Matrix Equation: $\sigma_{p} = C_{pq}\varepsilon_{q}$

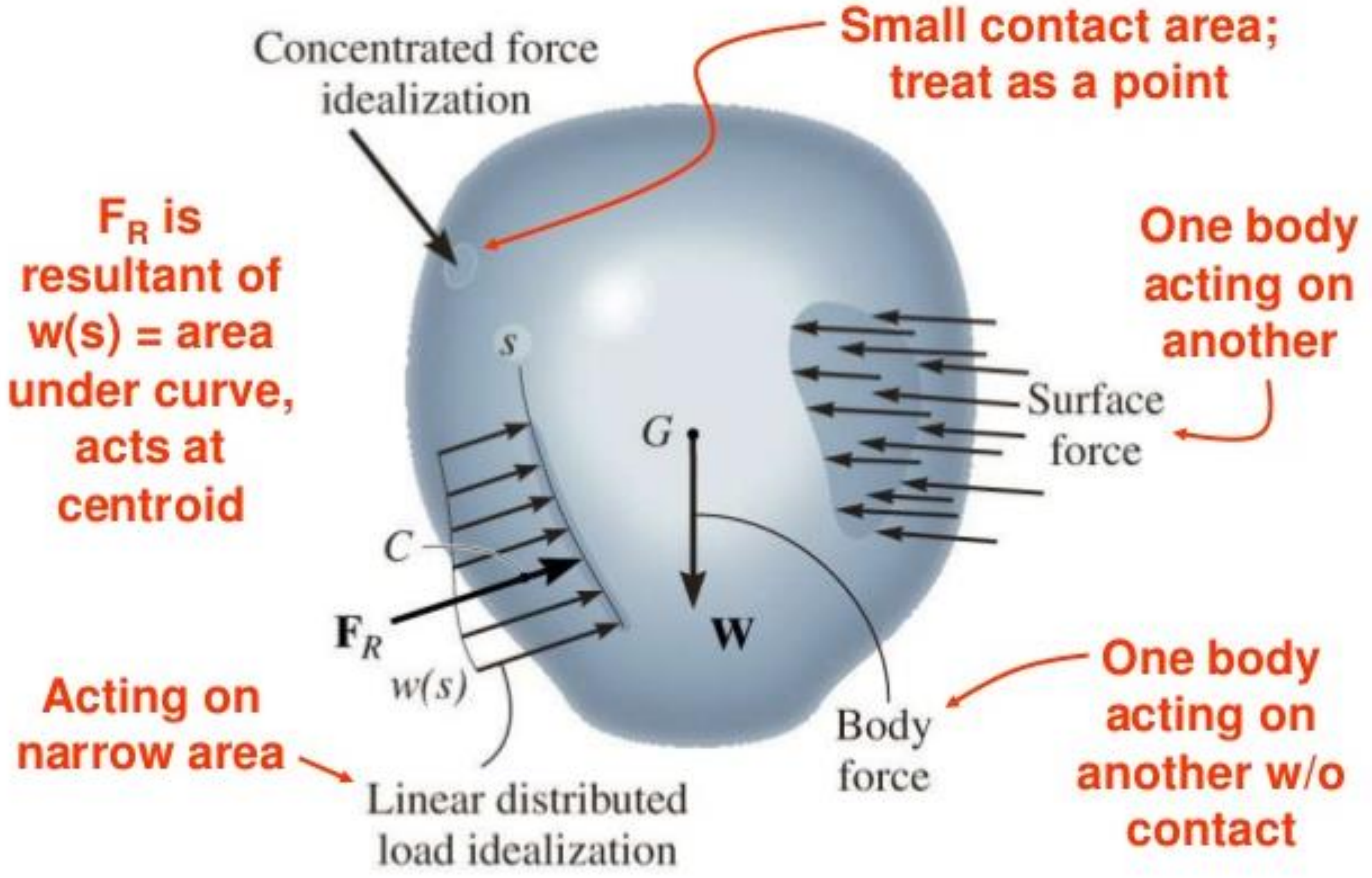
BASIC ASSUMPTIONS IN THEORY OF ELASTICITY

- The body is continuous
- The body is perfectly elastic
- The body is homogeneous
- The body is isotropic

  example: polycrystalline ceramics and steel
  wood and fiber reinforced composite
- The displacements and strains are small


σ
stress
proportionality limit
elastic limit
yield stress
ultimate stress
fracture
σu
σf
σy
σpl
A
B
C
D
E
elastic region
yielding
strain hardening
necking
strain
ε
elastic behavior
plastic behavior
σu Ultimate stress
σf Fracture stress
σy Yield stress
σpl Proportionality limit

Statics Review: External Loads



Static Equilibrium

- Vectors: $\Sigma \mathbf{F} = 0$ $\Sigma \mathbf{M} = 0$

- Coplanar (2D) force systems:

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

$$\Sigma M_o = 0$$

← **Perpendicular to the plane containing the forces**

- Draw a FBD to account for ALL loads acting on the body.

STATICS: You need to be able to…

- Draw free-body diagrams,
- Know support types and their corresponding reactions,
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- Solve for appropriate internal loads by taking cuts of inspection,
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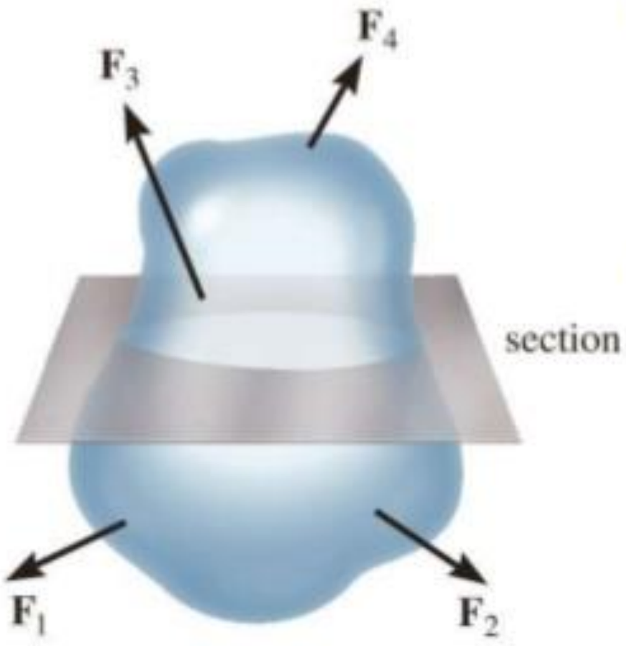
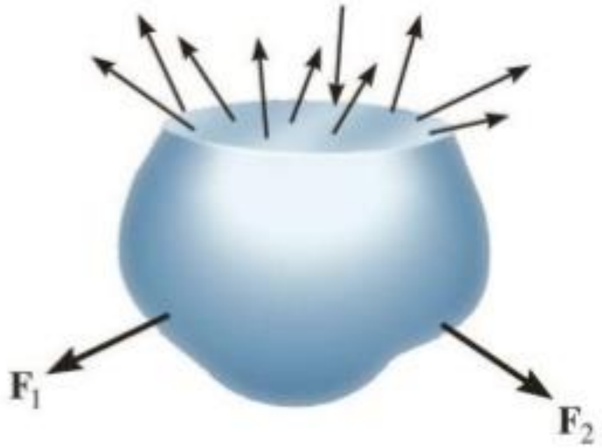
- Determine the moment of inertia about an axis through the centroid of an area.

Internal Reactions



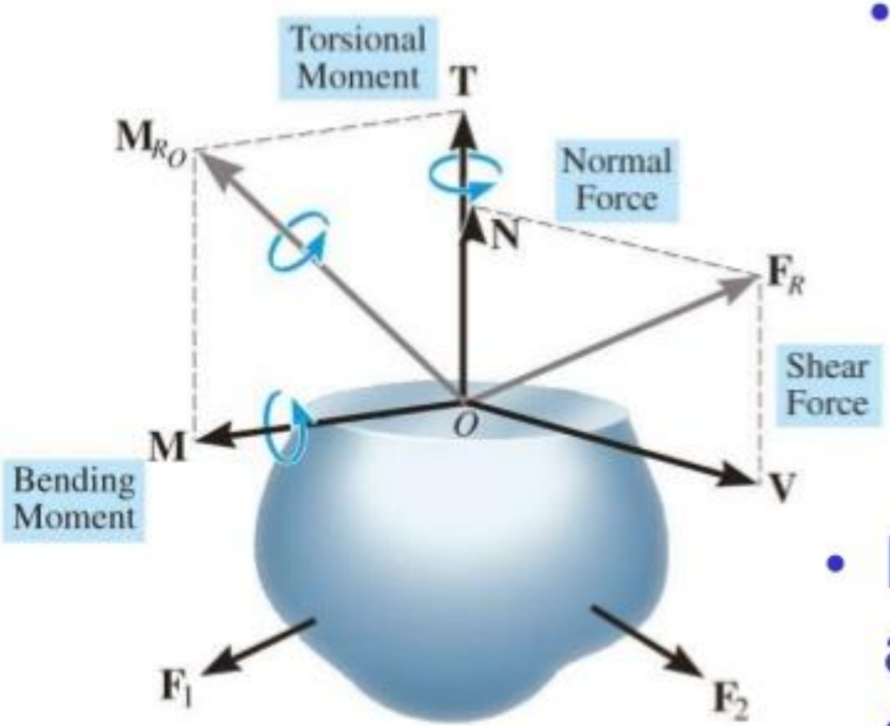
- Internal reactions are necessary to hold body together under loading.
- Method of sections - make a cut through body to find internal reactions at the point of the cut.

FBD After Cut



- Separate the two parts and draw a FBD of either side
- Use equations of equilibrium to relate the external loading to the internal reactions.

Components of Resultant



- Components are found perpendicular & parallel to the section plane.

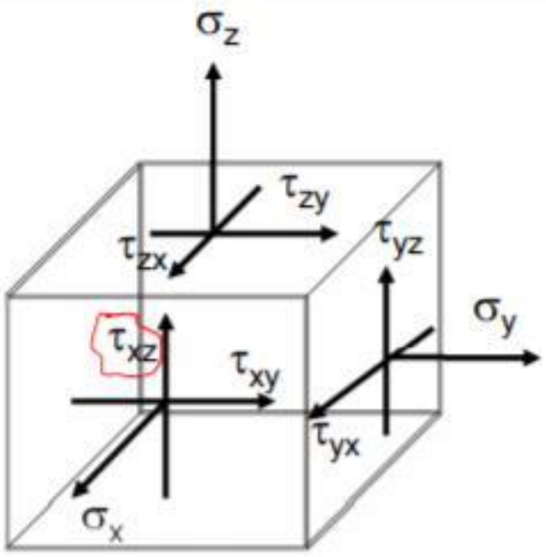
- Internal reactions are used to determine stresses.

3D Differential Equations of Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$$



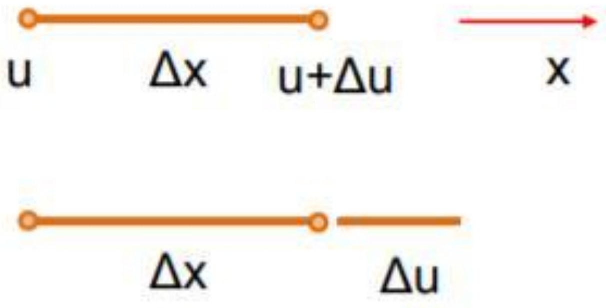
Governing equations for 3D elasticity

3 equations, 6 unknown functions

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{yx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{yx} \end{bmatrix}$$

Cornell Engineering

Strain-Displacement Relations: Normal Strain



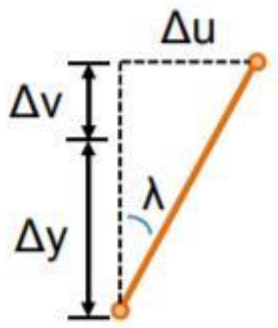
$$u + \Delta u = u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} + H.O.T.$$

$$\varepsilon_x = \frac{\Delta u}{\Delta x}$$

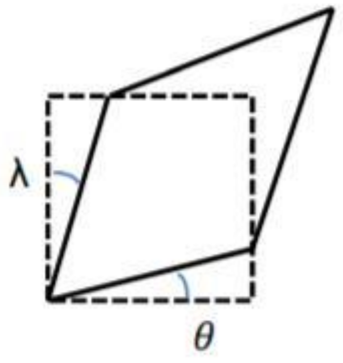
$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \Delta x \to 0$$

Similarly: $\varepsilon_y = \frac{\partial v}{\partial y}$ $\qquad \varepsilon_z = \frac{\partial w}{\partial z}$

Strain-Displacement Relations: Shear Strain



$$\lambda \approx \frac{\partial u}{\partial y}$$



$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

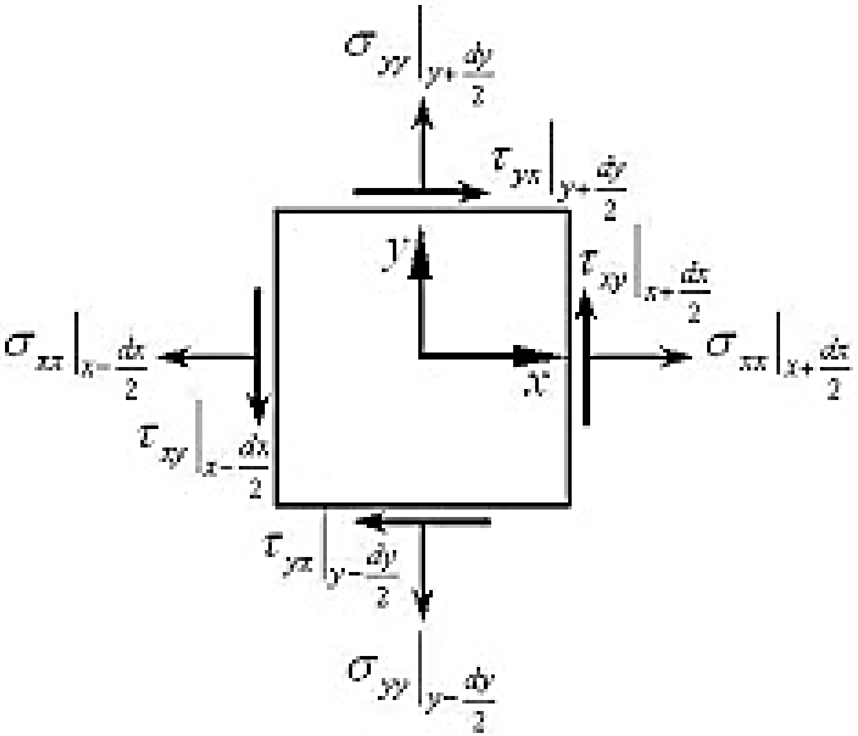
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

EQUILIBRIUM EQUATIONS

- Equilibrium Relation (2D)

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} = 0 \end{cases}$$



- Equilibrium Relation (3D)

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} = 0 \end{cases}$$

There are two basic conditions of equilibrium.

- ✓ Translational equilibrium.
- ✓ Rotational equilibrium.

- The term "translational equilibrium" describes an object that experiences no linear acceleration. (First condition of equilibrium)
- An object experiencing no rotational acceleration (a component of torque) is said to be in rotational equilibrium. (Second condition of equilibrium)
- Typically, an object at rest in a stable situation experiences both linear and rotational equilibrium.

There are two kinds of mechanical equilibrium:

- ✓ static equilibrium and
- ✓ dynamic equilibrium.

- Any object which is in static equilibrium has zero net force acting on it and is at rest.
- Any object which is in dynamic equilibrium has zero net force acting on it and is moving at a constant velocity.

Newton's First Law of Physics:

- *A body at rest will stay at rest and a body in motion will stay in motion unless acted upon by an unbalanced force.*
- Therefore, sum of all forces must be zero.

$$\Sigma F = 0$$

Resultant of all forces acting on a particle is zero.

Equilibrium

Equilibrium Equation from Newton's Law

- If an object is in equilibrium, then the resultant force acting on an object equals zero. This is expressed as follows:

$$\overline{F}_R = \sum \overline{F} = 0 \quad \text{(vectorequation)}$$

Some problems can be analyzed using only 2D, while others require 3D.

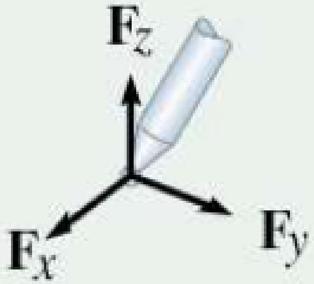
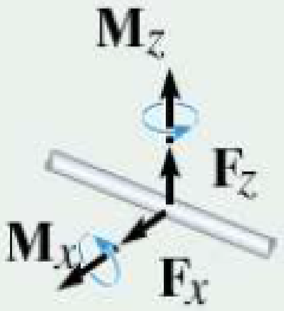
Necessary Condition for Equilibrium

- The necessary conditions for **equilibrium** are:

(i) the vector sum of all *external forces* is zero.

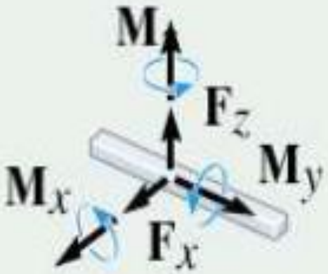
(ii) the sum of the moments of all *external forces* about any line is zero.

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a forc which acts away from the member i
(2)  smooth surface support		One unknown. The reaction is a forc which acts perpendicular to the surfac

Types of Connection	Reaction	Number of Unknowns
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface
(4)  ball and socket		Three unknowns. The reactions are three rectangular force components.
(5)  single journal bearing		Four unknowns. The reactions are two force and two couple-moment

(6)

single journal bearing with square shaft



Five unknowns. The reactions are two force and three couple-moment components.

(7)

single thrust bearing

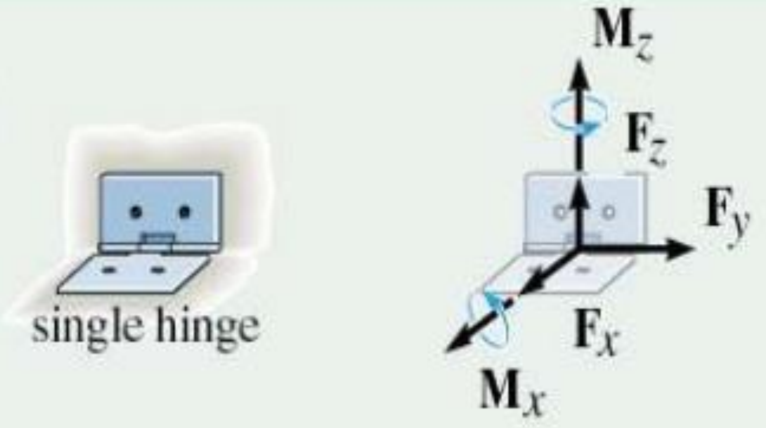
Five unknowns. The reactions are three force and two couple-moment components.

(8)

single smooth pin

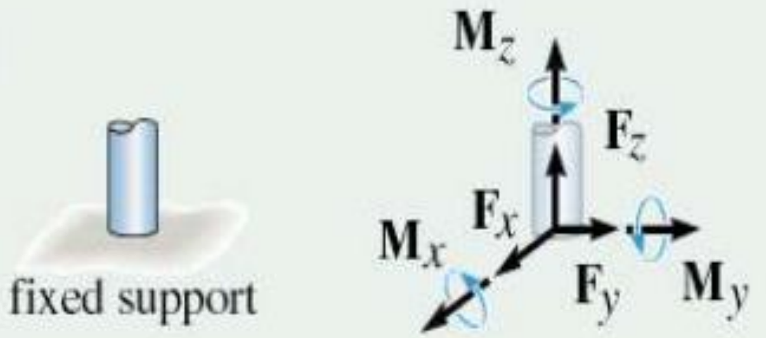
Five unknowns. The reactions are three force and two couple-moment components.

(9)



Five unknowns. The reactions are three force and two couple-moment components.

(10)



Six unknowns. The reactions are three force and three couple-moment components.

THE WHAT, WHY AND HOW OF A FREE BODY DIAGRAM (FBD)

Free Body Diagrams are one of the most important things for you to know how to draw and use.

What ? - It is a drawing that shows all external forces acting on the particle.

Why ? - It is key to being able to write the equations of equilibrium—which are used to solve for the unknowns (usually forces or angles).

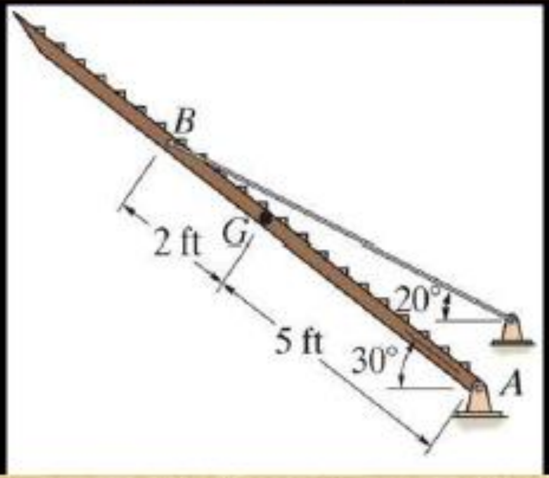
Steps of Drawing a FBD

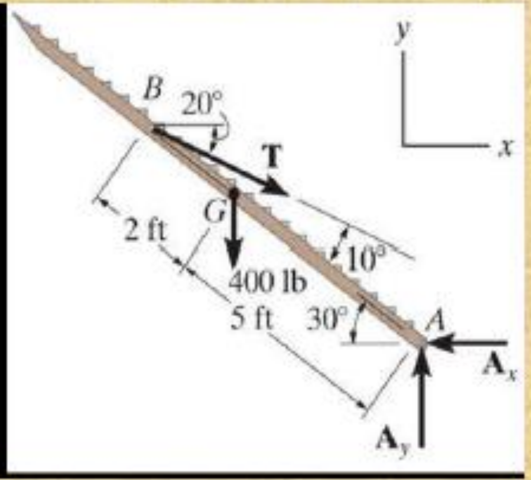
1. Imagine the particle to be isolated or cut free from its surroundings.
2. Show all the forces that act on the particle.

   Active forces: They want to move the particle.

   Reactive forces: They tend to resist the motion.
3. Identify each force and show all known magnitudes and directions.

Steps of Drawing a FBD





Real world example → Idealized model → FBD

THE PROCESS OF SOLVING RIGID BODY EQUILIBRIUM PROBLEMS

1) Draw a **free-body diagram (FBD)** showing all the external forces.
2) Apply the equations of equilibrium to solve for any unknowns.

Note: If there are more unknowns than the number of independent equations, then we have a statically **indeterminate situation.** We cannot solve these problems using just statics.

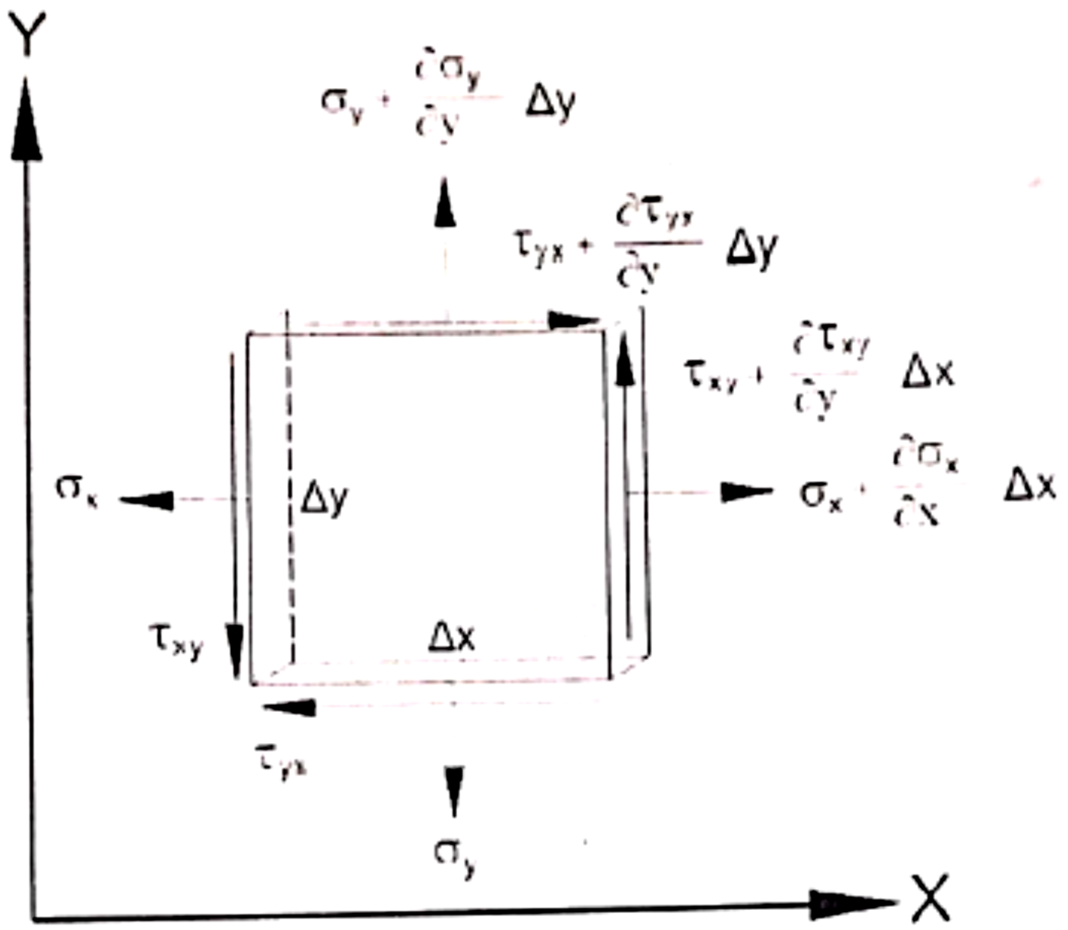
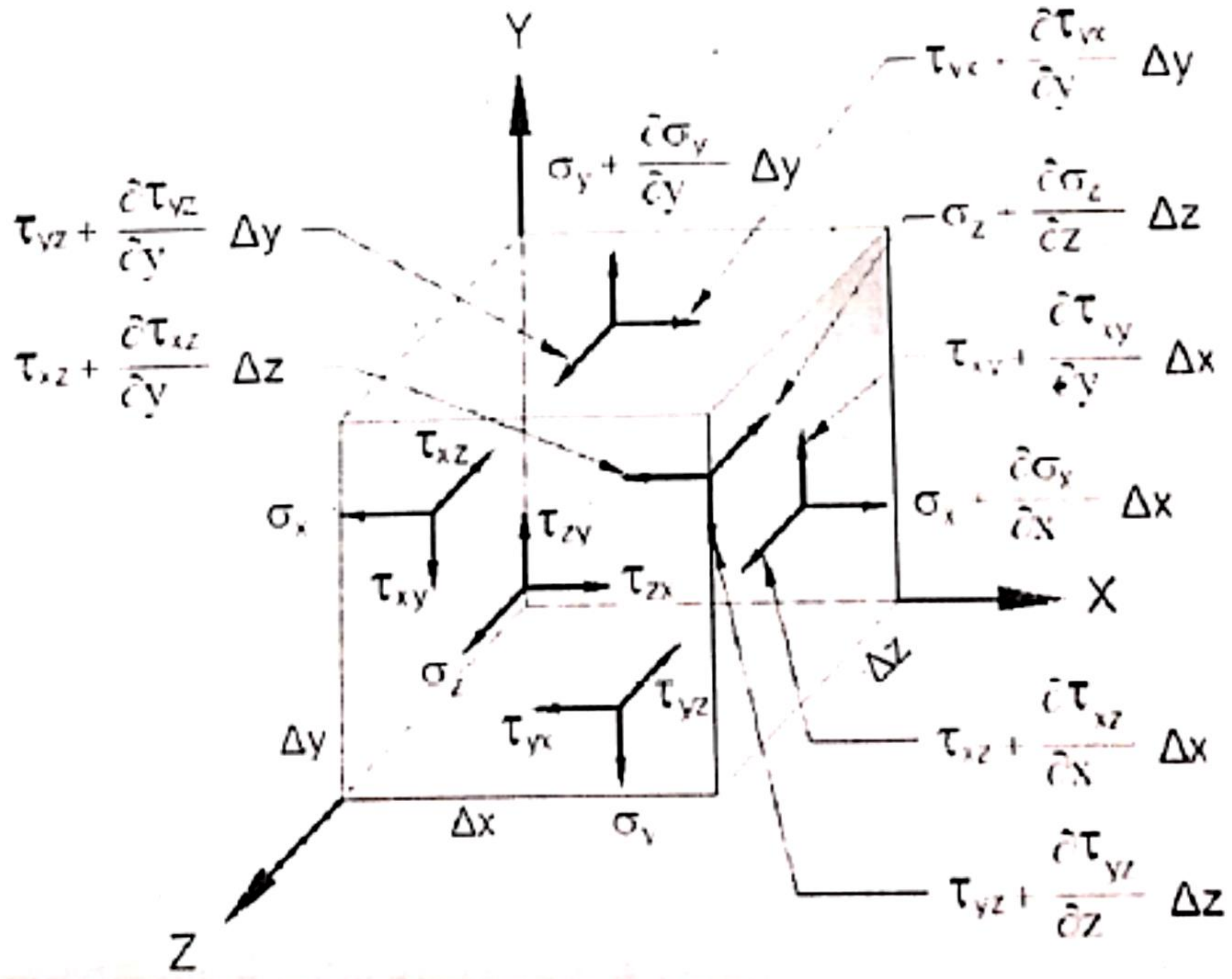
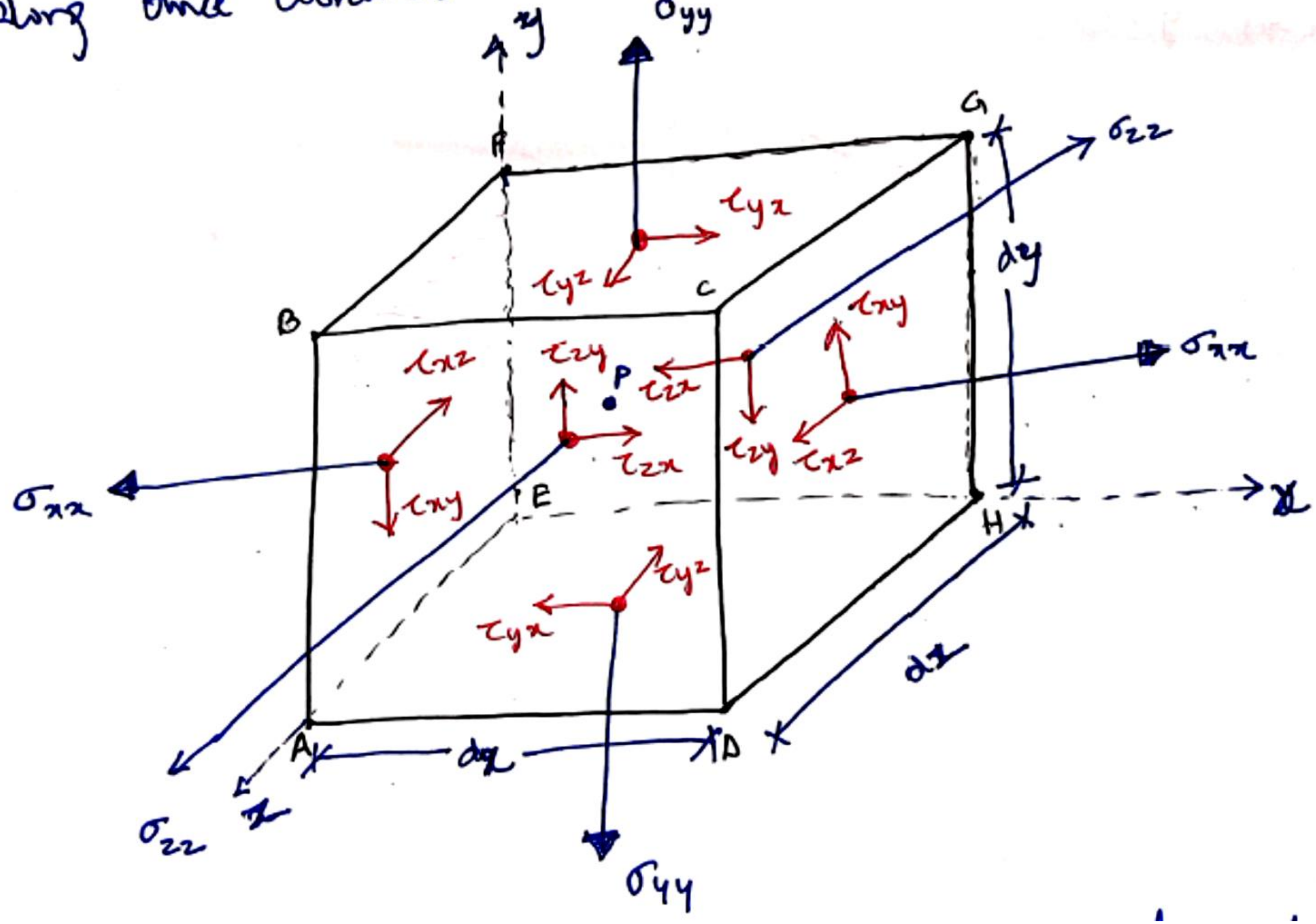


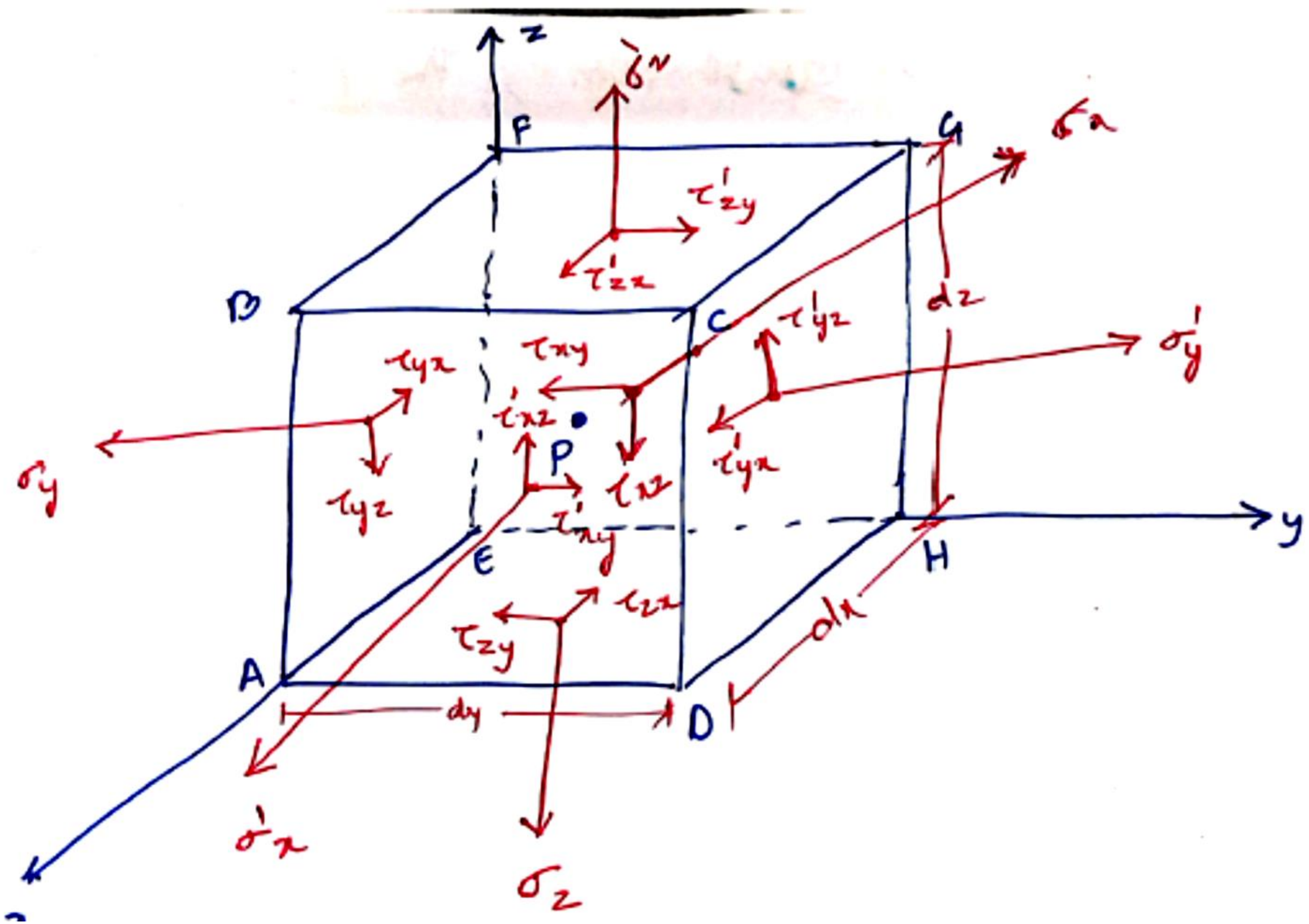
Figure 2.11(a) Stress components acting on a plane element (2D)



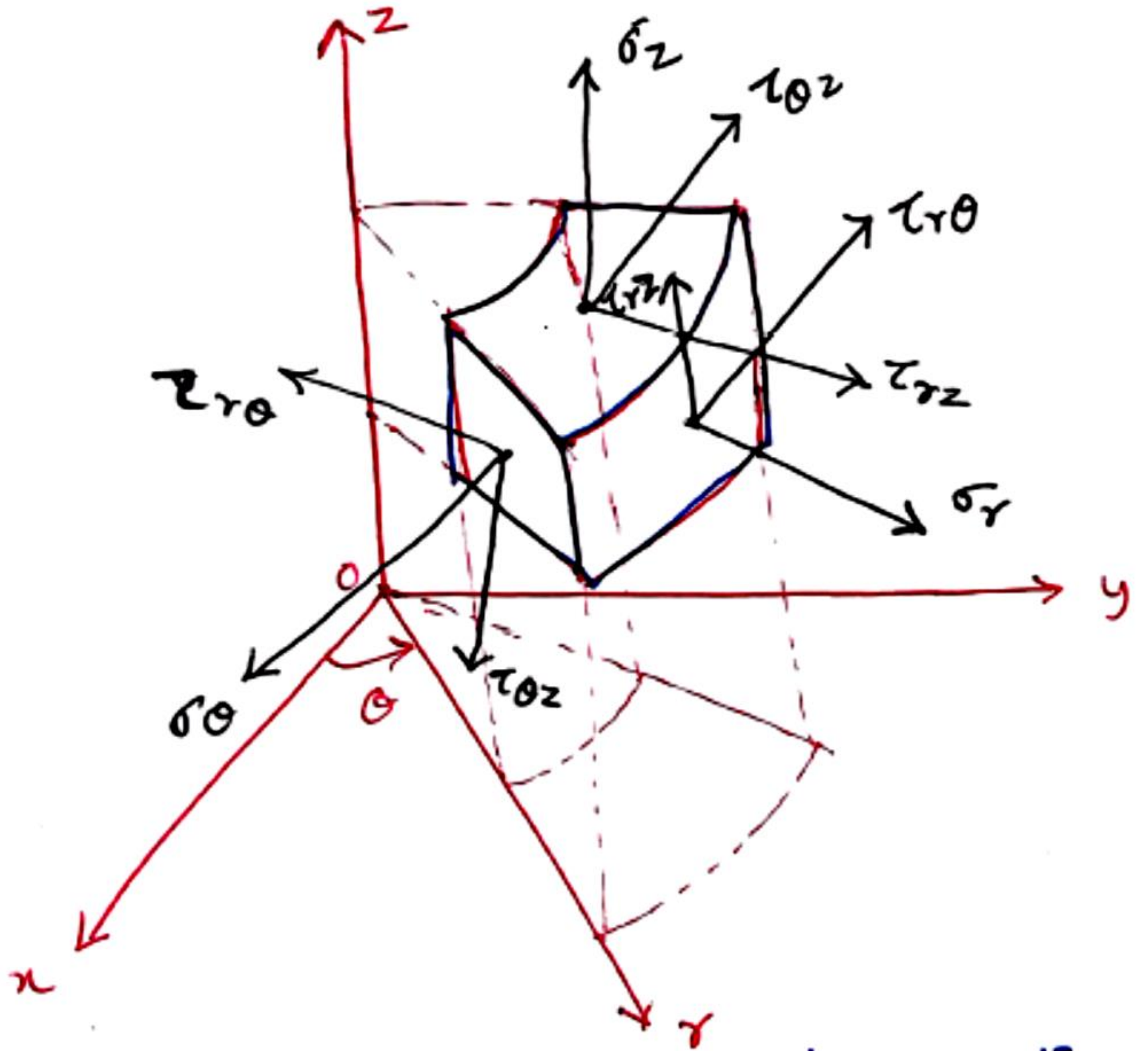
Y
X
Z
τyx + ∂τyx/∂y Δy
σy + ∂σy/∂y Δy
σz + ∂σz/∂z Δz
τyz + ∂τyz/∂y Δy
τxz + ∂τxz/∂y Δz
τxy + ∂τxy/∂y Δx
σx + ∂σy/∂x Δx
τxz
σx
τzy
τxy
τzx
σz
τyz
τyx
Δy
Δx
σy
Δz
τxz + ∂τxz/∂x Δx
τyz + ∂τyz/∂z Δz

along three coordinate axes

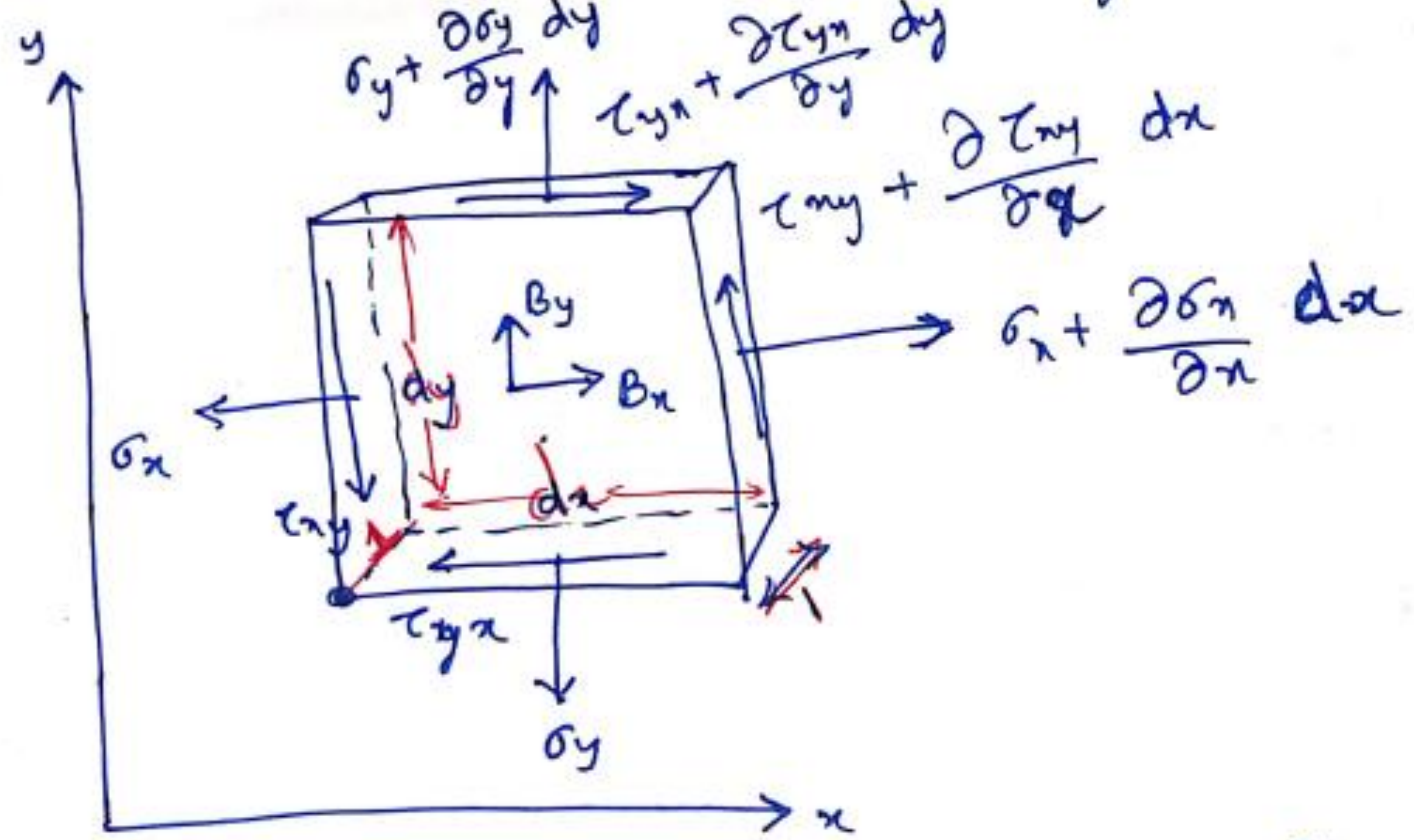



z
F
G
B
C
A
D
E
H
P
y
dx
dy
dz
σx
σ'x
σy
σ'y
σz
τ'zy
τ'zx
τ'yz
τ'yx
τxy
τyx
τyz
τxz
τ'xz
τ'xy
τzy
τzx

Stress at a point (Cylindrical Coordinates) (Polar Coord



Derive equations for equilibrium of a Differential Elem



Gokaraju Rangaraju Institute of Engineering & Technology

(Autonomous)

Results

Year: M.Tech I Year - I Sem Academic Year : 2021-22

Structural Engg

S.No	Roll No	GR20D5001	GR20D5002	GR20D5004	GR20D5006	GR20D5009	GR20D5010	GR20D5011	GR20D5152	SGPA	Credits
1	21241D2007	9	10	10	10	10	9	8	9	9.50	18
2	21241D2002	9	10	10	10	10	8	8	8	9.39	18
3	21241D2014	9	10	10	8	9	10	7	9	9.06	18
4	21241D2006	10	9	9	9	10	8	7	8	8.94	18
5	21241D2001	9	9	9	9	9	8	8	9	8.78	18
6	21241D2005	9	9	9	9	10	7	7	8	8.67	18
7	21241D2009	8	9	9	10	10	8	6	8	8.67	18
8	21241D2016	9	9	9	8	10	8	7	8	8.61	18
9	21241D2004	7	9	9	9	9	7	8	8	8.33	18
10	21241D2008	9	9	9	7	9	7	7	8	8.22	18
11	21241D2012	8	8	9	9	9	7	7	9	8.22	18
12	21241D2003	8	9	9	7	9	8	7	8	8.17	18
13	21241D2015	8	8	9	7	9	8	7	9	8.00	18
14	21241D2011	8	7	9	7	8	7	6	8	7.50	18
15	21241D2020	6	8	9	7	7	6	7	8	7.22	18
16	21241D2010	7	6	8	7	8	7	6	8	7.00	18
17	21241D2021	6	7	8	7	7	6	6	8	6.78	18
18	21241D2017	6	6	7	7	8	7	6	7	6.67	18
19	21241D2013	6	8	8	0	10	8	6	6	6.33	15
20	21241D2018	0	0	8	7	8	7	6	8	4.83	12
21	21241D2019	0	0	0	0	0	0	0	0	0.00	0

GR20D5001 Matrix Methods in Structural Analysis
GR20D5002 Advanced Solid Mechanics
GR20D5004 Advanced Concrete Technology
GR20D5006 Analytical and Numerical Methods for Structural Engineering
GR20D5009 Structural Design Lab
GR20D5010 Advanced Concrete Lab
GR20D5011 Research Methodology and IPR
GR20D5152 English for Research Paper Writing

GOKARAJU RANGARAJU INSTITUTE OF ENGINEERING & TECHNOLOGY

Department of Civil Engineering

Year: M.Tech I Year - I Sem **Academic Year : 2021-22**

Structural Engineering

Total Strength of the Class:21 **Student's Batch :2021-2023**

S.No	Name of the Subject	Subject Code	No. of students appeared	No. of students Passed	No. of students Failed	GP 10	GP 9	GP 8	GP 7	GP 6	Pass %
	Theory										
1	Matrix Methods in Structural Analysis	GR20D5001	21	19	2	1	7	5	2	4	90.48
2	Advanced Solid Mechanics	GR20D5002	21	19	2	3	8	4	2	2	90.48
3	Advanced Concrete Technology	GR20D5004	21	20	1	3	12	4	1	-	95.24
4	Analytical and Numerical Methods for Structural Engineering	GR20D5006	21	19	2	3	5	2	9	-	90.48
5	Research Methodology and IPR	GR20D5011	21	20	1	-	-	4	9	7	95.24
6	English for Research Paper Writing	GR20D5152	21	20	1	-	5	13	1	1	95.24
-											
6	Structural Design Lab	GR20D5009	21	20	1	-	-	4	9	7	95.24
7	Advanced Concrete Lab	GR20D5010	21	20	1	-	5	13	1	1	95.24

Grade	Grade Point
O	10
A+	9
A	8
B+	7
B	6
F	Fail

Subjects & Faculty Details

S.No	Name of the Subject	Faculty
1	Matrix Methods in Structural Analysis	Dr. G V V Satyanarayana (842)
2	Advanced Solid Mechanics	Mr.Kusuma Veera Babu (1650)
3	Advanced Concrete Technology	Dr. K.Sriknath (1594)
4	Analytical and Numerical Methods for Structural Engineering	Mr.V.Naresh Kumar Varma (1359)
5	Research Methodology and IPR	Dr.Mohammed Hussain (861)
6	English for Research Paper Writing	Mr. M. Aravind Kumar(708)
7	Structural Design Lab	Mr.C.Vanadeep(1645)/Mr.C.Vivek Kumar(1500)
8	Advanced Concrete Lab	Dr.V.Srinivas Reddy (Dr.VSR-1117)/Mr.Y.Kamala Raj (929)

Arrear Position - First Year First Semester

Arrear Details					
Description	**All Pass**	**One Arrear**	**Two Arrears**	**Three Arrears**	**>Three Arrears**
No. of Students	18	1	1	-	1

Performance

Class Toppers (Three Positions)			
S.No	**Name of the Student**	**Hall Ticket No.**	**SGPA**
1	MARIYALA VAISHNAVI	21241D2007	9.50
2	BANDI SRI RAM GOPAL	21241D2002	9.39
3	SK SAI CHANDRA	21241D2014	9.06

Overall Pass :86% **Passed in First class : 80.95 %**

HOD

Gokaraju Rangaraju Institute of Engineering and Technology

(Autonomous)

Bachupally, Kukatpally, Hyderabad – 500 090

Direct Internal CO Attainments

Academic Year	2021-22	Department	Civil Engineering	Name of the Programme	M.Tech		
Year - Semester	I-I	Course Name :	Advanced Solid Mechanics	Course Code	GR20D5002	Section	A

	Mid -I									Mid -II									Assignment Marks					Assessment
	Q.No 1(a)	Q.No 2(a)	Q.No 3(a)	Q.No 4(a)	Q.No 5(a)	Q.No 6(a)			Objective Marks	Q.No 1(a)	Q.No 2(a)	Q.No 3(a)	Q.No 4(a)	Q.No 5(a)	Q.No 6(a)			Objective Marks	I	II	III	IV	V	Marks
Enter CO Number → 1,2,3,4,5,6,7	1	2	1	3	2	2			1,2,3	3	3	4	4	5	5			3,4,5	1	2	3	4	5	1,2,3,4,5
Marks →	5	5	5	5	5	5			5	5	5	5	5	5	5			5	5	5	5	5	5	5
S.No/Roll No.	***Note :*** Enter Marks Between Two Green rows. ***Another Note :*** Additional Columns if Required should be inserted after column H and appropriately rename the Q. Nos. For Calculations consult Departments CO-PO Incharge																							
21241D2001		4		5	5				5		3		5		4			5	5	5	5	5	5	4
21241D2002		5		5	5				4	5			5		4			4	5	5	5	5	5	5
21241D2003	3			4	4				4		4		3		4			3	5	5	5	5	5	5
21241D2004		4		4	3				3	4		4			5			4	5	5	5	5	5	3
21241D2005		5		4	4				3	5			5		4			5	5	5	5	5	5	5
21241D2006		3		4	4				3		5		5		5			4	5	5	5	5	5	5
21241D2007		5		5	5				5		5		5		5			4	5	5	5	5	5	5
21241D2008				4	4	3			4		3		4		4			4	5	5	5	5	5	4
21241D2009		4		5	5				4	2	5		5		4			4	5	5	5	5	5	5
21241D2010	1			2	5				4	4			4		2			4	5	5	5	5	5	4
21241D2011		3		4	5				4		4		4		4			3	5	5	5	5	5	5
21241D2012	4	2		5	5				5		4		5		4			4	5	5	5	5	5	3
21241D2013		3		4	4				4		4.5		4.5					4	5	5	5	5	5	5
21241D2014	5			5	5				4	5			5		5			4	5	5	5	5	5	5
21241D2015		4		5	4				3		5		5					4	5	5	5	5	5	5
21241D2016	4	5	4		5	1			3		5	5		4				3	5	5	5	5	5	5
21241D2017		3		4	5				4	3	3		5					4	5	5	5	5	5	4
21241D2018		4			5				4		4		5	3				3	5	5	5	5	5	3
21241D2019	4				4				3									0	5	5	5	5	5	3
21241D2020	2	3			3				4	0	3				3			4	5	5	5	5	5	3
21241D2021		4		4	3				3		4		5		0			4	5	5	5	5	5	3
if your class strength is > 60 then *insert rows* *above the green row(last record)* , Similarly *delete the empty rows above green row* if the class strenght is < 60)																								
Total number of students appeared for the examination (NST)	21	21	21	21	21	21			21	21	21	21	21	21	21			21	21	21	21	21	21	21
Total number of students attempted the question (NSA)	7	16	1	17	21	2			21	8	15	2	17	2	15			21	21	21	21	21	21	21
Attempt % (TAP) = (NSA/NST)*100	33.33	76.19	4.76	80.95	100.00	9.52			100.00	38.10	71.43	9.52	80.95	9.52	71.43			100.00	100.00	100.00	100.00	100.00	100.00	100.00
Total number of Students who got more than 60% marks (NSM)	5	15	1	16	21	1			21	6	15	2	17	2	13			20	21	21	21	21	21	21
Attainment % (TMP) = (NSM/NSA)*100	71.43	93.75	100.00	94.12	100.00	50.00			100.00	75.00	100.00	100.00	100.00	100.00	86.67			95.24	100.00	100.00	100.00	100.00	100.00	100.00
Score(S)	3	3	3	3	3	2			3	3	3	3	3	3	3			3	3	3	3	3	3	3
Note : CO attainment is considered to be zero if the attempt % is less than 30%																								
CO Validation	1	2	1	3	2	2			1,2,3	3	3	4	4	5	5			3,4,5	1	2	3	4	5	1,2,3,4,5
Course Outcome	CO1	CO2	CO1	CO3	CO2	CO2			CO1,CO2, CO3	CO3	CO3	CO4	CO4	CO5	CO5			CO3,CO4, CO5	CO1	CO2	CO3	CO4	CO5	CO1,CO2,CO3,CO4,CO5
Marks (Y)	5	5	5	5	5	5			5	5	5	5	5	5	5			5	5	5	5	5	5	5
No. of COs Shared (Z)	1	1	1	1	1	1			3	1	1	1	1	1	1			3	1	1	1	1	1	5
Y/Z	5	5	5	5	5	5			1.66667	5	5	5	5	5	5			1.66667	5	5	5	5	5	1
S*Y/Z	15	15	15	15	15	10			5	15	15	15	15	15	15			5	15	15	15	15	15	3
CO1	1	0	1	0	0	0			1	0	0	0	0	0	0			0	1	0	0	0	0	1
CO2	0	1	0	0	1	1			1	0	0	0	0	0	0			0	0	1	0	0	0	1
CO3	0	0	0	1	0	0			1	1	1	0	0	0	0			1	0	0	1	0	0	1
CO4	0	0	0	0	0	0			0	0	0	1	1	0	0			1	0	0	0	1	0	1
CO5	0	0	0	0	0	0			0	0	0	0	0	1	1			1	0	0	0	0	1	1
CO6	0	0	0	0	0	0			0	0	0	0	0	0	0			0	0	0	0	0	0	0
CO7	0	0	0	0	0	0			0	0	0	0	0	0	0			0	0	0	0	0	0	0

Weighted Average for Attainment relevance (Internal CODn)	CO1	CO2	CO3	CO4	CO5	CO6	CO7
	3.00	2.78	3.00	3.00	3.00		

!!

!! Caution !! For CO Values < 2.1 should be justified with Remidial Action Report.

Gokaraju Rangaraju Institute of Engineering and Technology

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Indirect CO Attainments

Academic Year	2021-22	Department	Civil Engineering	Name of the Programme	M.Tech			
Year - Semester	I-I	Course Name :	Advanced Solid Mechanics	Course Code	GR20D5002	Section	A	

Course Outcomes survey on Scale 1 (Low) to 5 (High)

Enter Course Outcomes →	1	2	3	4	5		
CO Number → 1,2,3,4,5,6,7	1	2	3	4	5		
Marks →	5	5	5	5	5		
S.No/Roll No.	Note : Enter Marks Between Two Green rows.						
21241A2001	5	4	5	5	5		
21241A2002	5	4	4	5	5		
21241A2003	5	5	4	5	5		
21241A2004	4	5	5	5	5		
21241A2005	4	5	5	5	5		
21241A2006	5	5	4	4	5		
21241A2007	5	4	4	5	5		
21241A2008	5	4	4	5	5		
21241A2009	5	4	5	5	5		
21241A2010	4	5	5	5	5		
21241A2011	4	5	5	5	5		
21241A2012	5	5	4	5	5		
21241A2013	4	5	5	5	4		
21241A2014	4	5	4	5	5		
21241A2015	5	4	4	5	5		
21241A2016	5	4	4	5	5		
21241A2017	5	5	5	4	4		
21241A2018	5	5	4	5	5		
21241A2019	5	5	5	4	5		
21241A2020	5	4	5	5	5		
21241A2021	4	5	5	5	5		
if your class strength is > 60 then *insert rows above the green row(Last Record)* , Similarly *delete the empty rows above green row* if the class strenght is < 60)							
Total number of students appeared for the examination (NST)	21	21	21	21	21		
Total number of students attempted the question (NSA)	21	21	21	21	21		
Attempt % (TAP) = (NSA/NST)*100	100.00	100.00	100.00	100.00	100.00		
Total number of Students who got more than 60% marks (NSM)	21	21	21	21	21		
Attainment % (TMP) = (NSM/NSA)*100	100.00	100.00	100.00	100.00	100.00		
Score(S)	3	3	3	3	3		

CO attainment is considered zero if the attempt % is less than 30%

Indirect CO (COIn)	CO1	CO2	CO3	CO4	CO5	CO6	CO7
	3	3	3	3	3		

!! Caution !! For CO Values < 2.1 should be justified with Remidial Action Report.

Gokaraju Rangaraju Institute of Engineering and Technology

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Bachupally, Kukatpally, Hyderabad – 500 090

Direct External CO Attainment

Academic Year	2021-22	Department	Civil Engineering	Name of the Programme	M.Tech		
Year - Semester	I-I	Course Name :	Advanced Solid Mechanics	Course Code	GR20D5002	Section	A

	Part A										Part B										
	Q.No 1 (a) Marks	Q.No 1 (b) Marks	Q.No 1 (c) Marks	Q.No 1 (d) Marks	Q.No 1 (e) Marks	Q.No 1 (f) Marks	Q.No 1 (g) Marks	Q.No 1 (h) Marks	Q.No 1 (i) Marks	Q.No 1 (j) Marks	Q.No 2A Marks	Q.No 3 Marks	Q.No 4 Marks	Q.No 5 Marks	Q.No 6A Marks	Q.No 7A Marks	Q.No 8 Marks	Q.No 9A Marks	Q.No 9B Marks	Q.No 10A Marks	Q.No 11A Marks
Enter CO Number → 1,2,3,4,5,6,7	1	1	2	2	3	3	4	4	5	5	1	1	2	2	3	3	3	4	4	5	5
Marks →	2	2	2	2	2	2	2	2	2	2	10	10	10	10	10	10	5	5	5	10	10
S.No/Roll No.	***Note :*** Enter Marks Between Two Green rows. ***Another Note :*** Additional Columns if Required should be inserted after column H and appropriately rename the Q. Nos. For Calculations consult Departments CO-PO Incharge																				
1	2	2	1	1	1	2	2	2	2	2	9			7	8		8			8	
2	2	2	1	2	2	1	0		2	2	7			7	8			3	1	8	
3	1	1	1	1	1	1	1	0	2	2	2		7		8			2			7
4	0	0	1	1	0	0	0	0	0	0	7		6		6		7			7	
5	2	2	2	1	1	2	2	2	2	2	8			7	6		8			7	
6	2	2	2	2	2	2	2	2	2	2	9			10	10			5	4	10	
7	2	2	2	2	2	2	2	2	2	2	7			9	10		10			9	
8	2	2	2	2	2	2		2	2	2	7		8		8			2	3		9
9	0	0	1	1	1	1	1	1	0	1	8		8			4		2	3	6	
10	2	2	2	1	2	2	2	2	2	2	8			7	8		9			9	
11	2	2	1	1	1	2	2	2	2	2	9		8		8			4	5	9	
12	2	2	1	1	1	2	2	2	2	2	9		8		8			4	4	9	
13	2	2	2	0	2	2			2	2		9		7	8			4	4	5	
14	2	2	2	2	2	2	2	1	2	2		8		10	10			5	5		10
15	2	2	2	0	2	2	2		2	2	6		7		8		2			7	
16	2	2	2	2	2	2	2	2	2	2	7		10		7			4	4		8
17	2	2	2	1	2	2	2	1	1	2	8		9		7			5	5		9
18	2	2	2	2	2	2	2	2	2	2	6			10		8	7				8
19	0	1	2	0					2		6			7	7		8			8	
if your class strength is > 60 then *insert rows above the green row* , Similarly *delete the empty rows above green row* if the class strenght is < 60)																					
Total number of students appeared for the examination (NST)	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19
Total number of students attempted the question (NSA)	19	19	19	19	18	18	16	15	19	18	17	2	9	10	17	2	8	11	10	13	6
Attempt % (TAP) = (NSA/NST)*100	100.00	100.00	100.00	100.00	94.74	94.74	84.21	78.95	100.00	94.74	89.47	10.53	47.37	52.63	89.47	10.53	42.11	57.89	52.63	68.42	31.58
Total number of Students who got more than 60% marks (NSM)	16	17	19	16	17	17	14	13	17	17	16	2	9	10	17	1	7	8	9	12	6
Attainment % (TMP) = (NSM/NSA)*100	84.21	89.47	100.00	84.21	94.44	94.44	87.50	86.67	89.47	94.44	94.12	100.00	100.00	100.00	100.00	50.00	87.50	72.73	90.00	92.31	100.00
Score(S)	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	3	3	3	3	3
CO attainment is considered zero if the attempt % is less than 30%																					
CO Validation	1	1	2	2	3	3	4	4	5	5	1	1	2	2	3	3	3	4	4	5	5
Course Outcome	CO1	CO1	CO2	CO2	CO3	CO3	CO4	CO4	CO5	CO5	CO1	CO1	CO2	CO2	CO3	CO3	CO3	CO4	CO4	CO5	CO5
Marks (Y)	2	2	2	2	2	2	2	2	2	2	10	10	10	10	10	10	5	5	5	10	10
No. of COs Shared (Z)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Y/Z	2	2	2	2	2	2	2	2	2	2	10	10	10	10	10	10	5	5	5	10	10
S*Y/Z	6	6	6	6	6	6	6	6	6	6	30	30	30	30	30	20	15	15	15	30	30
CO1	1	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
CO2	0	0	1	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
CO3	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
CO4	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	1	0	0
CO5	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	1
CO6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CO7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Weighted Average for Attainment relevance	CO1	CO2	CO3	CO4	CO5	CO6	CO7
	3.00	3.00	2.66	3.00	3.00		

!! Caution !! For CO Values < 2.1 should be justified with Remidial Action Report.

Gokaraju Rangaraju Institute of Engineering and Technology
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Summary Sheet CO Attainments

Academic Year:	2021-22		Name of the Program:	M.Tech
Course/Subject:	Advanced Solid Mechanics		Course Code:	GR20D5002
Department:	Civil Engineering		Year - Semester :	I-I
Section	A			

Attainment/CO	**CO1**	**CO2**	**CO3**	**CO4**	**CO5**	**CO6**	**CO7**
Attainment for Direct Internal CO (Mid I & II, Assignments, Tutorials, Assessments, etc.)	3.00	2.78	3.00	3.00	3.00		
Attainment for Direct External CO (End Semester Exam)	3.00	3.00	2.66	3.00	3.00		
Direct CO (0.3*Internal + 0.7*External)	3.00	2.93	2.76	3.00	3.00		
Indirect CO	3.00	3.00	3.00	3.00	3.00		
Final CO (COFn) = (0.9 x Direct CO + 0.1 x Indirect CO)	**3.00**	**2.94**	**2.78**	**3.00**	**3.00**		

CO	**Course Outcome**	**Remedial Action for COs Less than 70% (2.10)**
CO1	Express algorithms in a language independent manner (as pseudo codes).	
CO2	Analyze the efficiency of the algorithms.	
CO3	Apply various searching and sorting algorithms for different applications.	
CO4	Illustrate various techniques like divide and conquer, greedy and dynamic approach in solving problems.	
CO5	Identify the appropriate algorithm design techniques for real world problems.	
CO6	Compare and Contrast various problem solving techniques and select the best suitable approach.	
CO7	Differentiate between deterministic and non-deterministic problems	

ID No.	Name of the Faculty	Department	Signature
1650	Mr. K.Veera Babu	Civil Engineering	

HOD **DAA**

Copy to: IQAC

Gokaraju Rangaraju Institute of Engineering and Technology

(Autonomous)

Bachupally, Kukatpally, Hyderabad – 500 090

Direct Internal CO Attainments

Academic Year	2021-22
Year - Semester	I-I

Department	Civil Engineering
Course Name :	Advanced Solid Mechanics

Name of the Programme	M.Tech
Course Code	GR20D5002

P-Outcomes / C-Outcomes	A	B	C	D	E	F	G	H	I	J	K	L
1	H	M		M	M	M						
2	H	M		M	M	M						
3	H	H	H	H	H	M						
4	H	H	H	H	H	M						
5	H	H	H	H	H	M						

Mapping Matrix in blue shaded rows 12 - 18 for seven CO s automatically PO Attainments are Calculated

Convert above mappings to scale 1-3

P-Outcomes / C-Outcomes	A	B	C	D	E	F	G	H	I	J	K	L
CO1	3	2		2	2	2						
CO2	3	2		2	2	2						
CO3	3	3	3	3	3	2						
CO4	3	3	3	3	3	2						
CO5	3	3	3	3	3	2						
Expected Attainment	3.00	2.60	3.00	2.60	2.60	2.00	0.00	0.00	0.00	0.00	0.00	0.00

Fill the below table with obtained attainments in mids, external and Tutorial/Attendence

	CO1	CO2	CO3	CO4	CO5	CO6	CO7
Final Cos CoF	3.00	2.94	2.78	3.00	3.00		

	Attained PO A	Attained PO B	Attained PO C	Attained PO D	Attained PO E	Attained PO F	Attained PO G	Attained PO H	Attained PO I	Attained PO J	Attained PO K	Attained PO L
CO1	3.00	2.00		2.00	2.00	2.00						
CO2	2.94	1.96		1.96	1.96	1.96						
CO3	2.78	2.78	2.78	2.78	2.78	1.86						
CO4	3.00	3.00	3.00	3.00	3.00	2.00						
CO5	3.00	3.00	3.00	3.00	3.00	2.00						
Attained	2.94	2.55	2.93	2.55	2.55	1.96	0.00	0.00	0.00	0.00	0.00	0.00

Note : If Average Attainment of a PO is #Div/0! Relace the corresponding PO with blank.

	A	B	C	D	E	F	G	H	I	J	K	L
	PO 1	**PO 2**	**PO 3**	**PO 4**	**PO 5**	**PO 6**	**PO 7**	**PO 8**	**PO 9**	**PO 10**	**PO 11**	**PO 12**
Expected	3.00	2.60	3.00	2.60	2.60	2.00						
Attained	2.94	2.55	2.93	2.55	2.55	1.96						
	98.15	**98.02**	**97.59**	**98.02**	**98.02**	**98.15**						

Note : PO is Satisfied if attained PO > 70, U indicates PO Unsatisfied

Faculty Co-Ordinator

HOD

Gokaraju Rangaraju Institute of Engineering and Technology (Autonomous)
Bachupally, Kukatpally, Hyderabad – 500 090. (040) 6686 4440

COURSE COMPLETION STATUS

Academic **Y**ear : 2022-23

Semester : I

Name of the Program: M.TECH. STRUCTURAL ENGINEERING

Course/Subject: ADVANCED SOLID MECHANICS Course Code: GR225002

Name of the Faculty: DR. V SRINIVASA REDDY Dept.: CIVIL ENGINEERING

Designation: PROFESSOR.

Actual Date of Completion & Remarks, if any

Units	**Remarks**	**No. of Objectives Achieved**	**No. of Outcomes Achieved**
UNIT 1	Have a good understanding of the theory, concepts, principles and governing equations of Elasticity principles	COB1	CO1
UNIT 2	Develop equations of equilibrium and draw relations among stress, strain and displacement and utilize the equilibrium equations, compatibility equations and various boundary conditions to analyze elastic problems.	COB2	CO2
UNIT 3	Gain the understating of three-dimensional problems of elasticity in Cartesian coordinates system ad able to determine principal stresses and planes of 3D problems	COB3	CO3
UNIT 4	Apply the principles of elasticity to solve torsional problems in prismatic bars and tubes	COB4	CO4
UNIT 5	Use the concepts of stresses and strains for plastic deformation to comprehend the yield criteria of materials	COB5	CO5

Signature of HOD

Date:

Signature of faculty

Date:

Note : After the completion of each unit mention the number of objectives and outcomes achieved